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SPECTRAL DOMAINS IN SEVERAL COMPLEX VARIABLES

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ABSTRACT. In this paper we study the concepts of spectral domain and complete spectral domain in several complex variables. For a domain Ω in $\mathbf{C}^n$ and an n-tuple T of commuting operators on a Hilbert space $\mathcal{H}$ such that the Taylor spectrum of T is a subset of Ω, we introduce the quantities $K_\Omega(T)$ and $M_\Omega(T)$. These quantities are related to the quantities $K_X(T)$ and $M_X(T)$ introduced by Paulsen for a compact subset X. When T is an n-tuple of 2×2 matrices, $K_\Omega(T)$ and $M_\Omega(T)$ are expressed in terms of the Carathéodory metric and the Möbius distance. This in turn answers a question by Paulsen for tuples of 2×2 matrices. We also establish von Neumann's inequality for an n-tuple of upper triangular Toeplitz matrices. We study the regularity of $K_\Omega(T)$ and $M_\Omega(T)$ and obtain various comparisons of these two quantities when T is an n-tuple of Jordan blocks.

1. Introduction. This work is motivated primarily by two papers, namely, [**1**] and [**23**]. The former shows a strong connection between operator theory and complex geometry by giving an operator theoretic proof of a fundamental result on invariant metrics for convex domains in $\mathbf{C}^n$. For an infinitesimal version of that result, see [**25**]. The second paper is a survey of results concerning spectral sets and centering around von Neumann's inequality.

In this paper we study the concepts of spectral domain and complete spectral domain in several complex variables. We use some ideas from complex geometry to obtain some results in multi-variable operator theory.

Our first group of results consists of improvements of results of several authors concerning n-tuples of 2×2 matrices. This is summarized in Theorem 1. As a consequence, we answer, in this case, a question of Paulsen, and we give a new proof of von Neumann's inequality for any n-tuple of 2×2 matrices.

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Our second group of results is concerned with n-tuples of finite dimensional operators. We establish two estimates for the quantity $M_\Omega(T)$ in terms of $K_\Omega(T)$, one which is general (Proposition 4.1) and one which is valid for Jordan blocks (Theorem 3). We also prove some regularity properties for these quantities for Jordan blocks (Proposition 4.2) and establish the von Neumann inequality for n-tuples of upper triangular Toeplitz matrices (Theorem 2).

We introduce now some basic definitions. Other definitions will be stated as needed. Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. Let $T = (T_1, T_2, \ldots, T_n)$ be a commuting n-tuple of operators in $\mathcal{L}(\mathcal{H})$. Let $\sigma(T)$, or σ_T, be the Taylor spectrum of T. It follows from the work of Taylor [**28**] that

1) $\sigma(T)$ is a compact nonempty subset of $\mathbf{C}^n$.

2) If $H(\sigma(T))$ is the algebra of functions holomorphic in a neighborhood of $\sigma(T)$, then there is a continuous homomorphism $\phi : H(\sigma(T)) \to \mathcal{L}(\mathcal{H})$ such that $\phi(1) = I$ and $\phi(z_i) = T_i$. $\phi(f)$ is denoted by $f(T)$.

3) If $f = (f_1, \ldots, f_m) : U \to \mathbf{C}^m$ is a holomorphic mapping from a neighborhood U of $\sigma(T)$ to $\mathbf{C}^m$, then $f(\sigma(T)) = \sigma(f(T))$, where $f(T) = (f_1(T), \ldots, f_m(T))$.

The following definitions of spectral domain and complete spectral domain were first introduced by Agler [**1**]. They are variations of the concepts of spectral set and complete spectral set, which were introduced by von Neumann [**18**] and Arveson [**3**], respectively.

Let Ω be a domain in $\mathbf{C}^n$ containing $\sigma(T)$, and let $H(\Omega, \overline{D})$ be the set of holomorphic mappings from Ω to the closed unit disk $\overline{D}$.

Define

$$K_\Omega(T) = \sup\{\|f(T)\|; f \in H(\Omega, \overline{D})\}. \tag{1.1}$$

We say that a domain Ω is a *spectral domain* of T if $\sigma_T \subset \Omega$ and $K_\Omega(T) \le 1$. The domain Ω is called K-*spectral domain* if $K_\Omega(T) \le K < +\infty$.

Let $\mathcal{M}_m$ be the algebra of $m \times m$ matrices and $B_{m\times m}$ be the unit ball in $\mathcal{M}_m$ (under the matrix norm). For $f(\cdot) = (f_{ij}(\cdot)) \in H^\infty(\Omega) \otimes \mathcal{M}_m$, let

$$\|f(\cdot)\| = \sup\{\|(f_{ij})(z)\|_{\mathcal{M}_m}; z \in \Omega\}. \tag{1.2}$$

Let $\|f(T)\|$ be the norm of the operator $f(T) = [f_{ij}(T)]$ acting on m copies of $\mathcal{H}$. Let $H(\Omega, \overline{B}_{m\times m})$ be the set of holomorphic mappings from Ω to $\overline{B}_{m\times m}$. Define

$$M_\Omega^m(T) = \sup\{\|f(T)\|; f \in H(\Omega, \overline{B}_{m\times m})\} \tag{1.3}$$

and

$$M_\Omega(T) = \sup\{M_\Omega^m(T); m \geq 1\}. \tag{1.4}$$

A domain Ω is said to be a *complete spectral domain* of T if $\sigma_T \subset \Omega$ and $M_\Omega(T) \leq 1$. It is called *complete K-spectral domain* if $M_\Omega(T) \leq K < +\infty$.

This paper is organized as follows. In Section 2 we study the basic properties of the quantities $K_\Omega(T)$ and $M_\Omega(T)$ for a general n-tuple T and a domain $\Omega \supset\supset \sigma(T)$. We prove that if $\overline{\Omega}$ is rationally convex, then Ω is a spectral domain, respectively complete spectral domain, if and only if $\overline{\Omega}$ is a spectral set, respectively complete spectral set. In Section 3 we study the case when T is an n-tuple of 2×2 matrices. We prove that in this case $K_\Omega(T) = M_\Omega(T)$. We express $K_\Omega(T)$ in terms of the Carathéodory metric and the Möbius distance. This enables us to solve (for 2×2 matrices) a problem raised by Paulsen, see [**23**, Problem 13]: If T_i, $i = 1, 2$, are commuting 2×2 matrices and X_i is a K_i-spectral set for T_i, then $X_1 \times X_2$ is a K-spectral set for $T = (T_1, T_2)$ where $K = \max\{K_1, K_2\}$. In Section 4 we study the properties of $K_\Omega(T)$ and $M_\Omega(T)$ when T is an n-tuple of commuting finite matrices. We are especially interested when T is an n-tuple of Jordan blocks. Since most of the recent work on this subject is concerned with finite dimensional operators, Jordan blocks are natural objects on which to study the relation between the quantities $K_X(T)$ and $M_X(T)$.

2. General properties of $K_\Omega(T)$ and $M_\Omega(T)$. In this section we study the basic properties of the quantities $K_\Omega(T)$ and $M_\Omega(T)$. The following proposition follows directly from the definitions.

Proposition 2.1. *Let T be an n-tuple of commuting operators in $\mathcal{L}(\mathcal{H})$. Then*

1) *for each m, there exists $f \in H(\Omega, \overline{B}_{m\times m})$ so that $\|f(T)\| = M_\Omega^m(T)$.*

2) $K_\Omega(T) = M^1_\Omega(T)$ *and* $M^m_\Omega(T) \le M^{m+1}_\Omega(T)$.

3) *For any domain* $\Omega \supset\supset \sigma(T)$, $K_\Omega(T) < +\infty$.

4) $M^m_\Omega(T) \le mK_\Omega(T)$.

5) K_Ω, M^m_Ω *and* M_Ω *satisfy the decreasing property, i.e., if* $\phi : \Omega_1 \to \Omega_2$ *is a holomorphic mapping from domain* Ω_1 *to domain* Ω_2 *and* $\sigma(T) \subset\subset \Omega_1$, *then*

$$G_{\Omega_1}(T) \ge G_{\Omega_2}(\phi(T)).$$

Here and in what follows, G *denotes either* K, M^m *or* M. *In particular, if* $\Omega_1 \subset \Omega_2$, *then* $G_{\Omega_1}(T) \ge G_{\Omega_2}(T)$.

We now recall the concepts of spectral set and complete spectral set.

Let $X \supset \sigma(T)$ be a compact subset of $\mathbf{C}^n$. Define

$$\begin{aligned} K_X(T) &= \sup\{\|f(T)\|; f \in R(X, \overline{D})\} \\ M^m_X(T) &= \sup\{\|f(T)\|; f \in R(X, \overline{B}_{m\times m})\} \\ M_X(T) &= \sup\{M^m_X(T); m \ge 1\}, \end{aligned}$$

where $R(X, \overline{D})$, respectively $R(X, \overline{B}_{m\times m})$, is the set of rational mappings r with poles off X such that $r(X) \subset \overline{D}$, respectively $r(X) \subset \overline{B}_{m\times m}$. A compact set X is called a *spectral set* if $X \supset \sigma(T)$ and $K_X(T) \le 1$. It is called a K*-spectral set* if $K_X(T) \le K < +\infty$. The definitions of *complete spectral set* and *complete* K*-spectral set* are obtained by replacing K_X by M_X. It follows from a theorem of Arveson [**3**] that X is a complete spectral set if and only if T has a normal ∂X-dilation. We list some important results in this language:

1) (von Neumann [**18**]). If $T \in \mathcal{L}(\mathcal{H})$ and $\|T\| \le 1$, then $K_{\overline{D}}(T) \le 1$.

2) (Sz-Nagy [**17**]). If $T \in \mathcal{L}(\mathcal{H})$ and $\|T\| \le 1$, then $M_{\overline{D}}(T) \le 1$.

3) (Ando [**2**]). If $T = (T_1, T_2)$ is a commuting two tuple of operators in $\mathcal{L}(\mathcal{H})$ such that $\|T_i\| \le 1$, $i = 1, 2$, then $M_{\overline{D}^2}(T) \le 1$. In particular, von Neumann's inequality holds for a 2-tuple of commuting contractions.

von Neumann's inequality does not extend to a tuple of more than two operators. Such counterexamples have been found by Varopoulos [**29**] and others. By the theorem of Arveson mentioned above, and the well-known example of Parrott [**19**], Ando's theorem (as stated above) does not extend to n-tuples, for $n \geq 3$.

Proposition 2.2. *If* $X \supset \sigma(T)$ *is rationally convex, then*

$$G_X(T) = \sup\{G_\Omega(T), \text{ all domains } \Omega \supset\supset X\} \tag{2.1}$$

where G *is either* K, M^m *or* M. *Furthermore, if* Ω *is a bounded domain in* $\mathbf{C}^n$ *such that* $\overline{\Omega}$ *is rationally convex, then*

$$G_{\overline{\Omega}}(T) = G_\Omega(T). \tag{2.2}$$

Proof. We need only to prove that (2.1) is true for M^m. The proof is based on a classical theorem of Oka-Weil, see, for example, [**10**], which says that if X is a rationally convex set, then any holomorphic function in a neighborhood of X can be uniformly approximated by rational functions r_j with poles off X.

Let $\Omega \supset\supset X$ be a domain in $\mathbf{C}^n$, and let $f \in H(\Omega, \overline{B}_{m\times m})$ such that $\|f(T)\| = M^m_\Omega(T)$. For any $\varepsilon, \delta > 0$, by Oka-Weil's theorem, there are (matrix value) rational functions $r(z)$ such that $\|r(z) - f(z)\| < \delta$. Choosing δ sufficiently small, then $\|f(T) - r(T)\| < \varepsilon$. On the other hand, $\|r(z)\|_X < 1 + \delta$. Thus

$$M^m_X(T) \geq (1+\delta)^{-1}(M^m_\Omega(T) - \varepsilon). \tag{2.3}$$

Letting $\delta, \varepsilon \to 0$, we have $M^m_\Omega(T) \leq M^m_X(T)$.

For the inequality in the other direction, let $r(z)$ be a rational function with poles off X such that $\|r(T)\| \geq M^m_X(T) - \varepsilon$ (here we assume that $M^m_X(T) < \infty$, the situation for $M^m_X(T) = \infty$ is similar, we omit the details). By choosing domain $\Omega \supset\supset X$ sufficiently close to X, we have $\|r(z)\|_\Omega < 1 + \varepsilon$. Thus,

$$M^m_\Omega(T) \geq (1+\varepsilon)^{-1}(M^m_X(T) - \varepsilon).$$

Therefore,

$$\sup\{M^m_\Omega(T), \Omega \supset X\} \geq (1+\varepsilon)^{-1}(M^m_X(T) - \varepsilon).$$

Letting $\varepsilon \to 0$, we then obtain the desired inequality. $\square$

Remark. 1) Since every planar compact subset is rationally convex, the equality (2.2) is true for every planar domain $\Omega \supset \sigma(T)$.

2) A set $S \subset \mathbf{C}^n$ is called a *Reinhardt set* if $(e^{i\theta_1}z_1, e^{i\theta_2}z_2, \dots, e^{i\theta_n}z_n) \in S$ whenever $(z_1, z_2, \dots, z_n) \in S$ and $(\theta_1, \theta_2, \dots, \theta_n) \in \mathbf{R}^n$. It follows from Theorem 3.3 in [**10**] that if Ω is a pseudoconvex Reinhardt domain in $\mathbf{C}^n$ such that

$$\Omega \cap \{(z_1, z_2, \dots, z_n); z_i = 0\} \neq \varnothing, \quad i = 1, 2, \dots, n,$$

whenever $\overline{\Omega} \cap \{(z_1, z_2, \dots, z_n); z_i = 0\} \neq \varnothing$, then $\overline{\Omega}$ is rationally convex. Thus equality (2.2) holds for any pseudoconvex Reinhardt domain with a differentiable boundary, see [**9**] for more details.

Proposition 2.3. *If* $\Omega_1 \subset \Omega_2 \subset \Omega_3 \cdots$, $\cup \Omega_j = \Omega$ *and* $\Omega \supset\supset \sigma(T)$, *then* $\lim_{j\to\infty} M^m_{\Omega_j}(T) = M^m_{\Omega}(T)$.

Proof. First, by the decreasing property, $M^m_{\Omega_j}(T) \geq M^m_{\Omega_{j+1}}(T) \geq M^m_{\Omega}(T)$. Let $g_j \in H(\Omega_j, \overline{B}_{m\times m})$ be such that $M^m_{\Omega_j}(T) = \|g_j(T)\|$. Since $\{g_j\}$ is a normal family, there exists a subsequence $\{g_j\}$, for simplicity we use the same notation, which locally uniformly converges to some $f \in H(\Omega, \overline{B}_{m\times m})$. Thus

$$\lim_{j\to\infty} M^m_{\Omega_j}(T) = \|f(T)\| \leq M^m_{\Omega}(T). \qquad \square$$

For an invertible $S \in \mathcal{L}(\mathcal{H})$, let $c(S) = \|S\| \cdot \|S^{-1}\|$ be its condition number. The proof of the following property follows from the proof of the main theorem in [**22**].

Proposition 2.4. *For any domain* $\Omega \supset\supset \sigma(T)$, $G_\Omega(T) = \min\{c(S), G_\Omega(S^{-1}TS) = 1\}$.

3. Two-dimensional case. In this section we study the case when $\mathcal{H}$ is two-dimensional. We show the relationship among the quantities

K_Ω, M_Ω and the invariant metric and distance. We answer positively the following question asked by Paulsen (in slightly different language) for the two-dimensional case: If $\Omega_i \subset \mathbf{C}$ is a K_i-spectral domain of commuting operators T_i, $i = 1, 2$, is $\Omega_1 \times \Omega_2$ a K_1K_2-spectral domain for $T = (T_1, T_2)$?

First, we recall some definitions. For $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbf{C}^n$, let $L_\xi = \sum_{k=1}^n \xi_k \partial/\partial z_k$. The Carathéodory metric of $\Omega \subset \mathbf{C}^n$ is defined by

$$F_\Omega(z, \xi) = \sup\{|L_\xi f(z)|; f \in H(\Omega, \overline{D}), f(z) = 0\} \tag{3.1}$$

for $z \in \Omega$ and $\xi \in \mathbf{C}^n$. For $z^1, z^2 \in \Omega$, the Möbius distance is defined by

$$\rho_\Omega(z^1, z^2) = \sup\left\{\left|\frac{f(z^1) - f(z^2)}{1 - \overline{f(z^1)}f(z^2)}\right|; f \in H(\Omega, \overline{D})\right\}. \tag{3.2}$$

Both F_Ω and ρ_Ω are decreasing under holomorphic mappings. In particular, they are biholomorphic invariants. It was shown by Pflug and Jarnicki [**24**] that both F_Ω and ρ_Ω satisfy the product property, i.e., if $\Omega_i \subset \mathbf{C}^{n_i}$, $i = 1, 2$, then

$$F_{\Omega_1 \times \Omega_2}((z_1, z_2), (\xi_1, \xi_2)) = \max\{F_{\Omega_1}(z_1, \xi_1), F_{\Omega_2}(z_2, \xi_2)\} \tag{3.3}$$

$$\rho_{\Omega_1 \times \Omega_2}((z_1, z_2), (w_1, w_2)) = \max\{\rho_{\Omega_1}(z_1, w_1), \rho_{\Omega_2}(z_2, w_2)\} \tag{3.4}$$

for all $z_i, w_i \in \Omega_i$ and $\xi_i \in \mathbf{C}^{n_i}$, $i = 1, 2$.

Let $T = (T_1, T_2, \ldots, T_n)$ be an n-tuple of 2×2 matrices. Then $\sigma(T)$ consists of either a single point or two distinct points. We discuss these two cases separately:

I. *The case when* $\sigma(T) = \{z\}$. For this case it is easy to see that T is unitarily equivalent to $T(z, \xi) = zI + \xi J$ for some $\xi \in \mathbf{C}^n$. Here I is the identity matrix and J is the restriction of the backward shift operator to $\mathbf{C}^2$. Thus we may assume that $T = T(z, \xi)$, and for any ($\mathcal{M}_m$ valued) holomorphic function f near z, $f(T) = f(z)I + L_\xi f(z)J = T(f(z), L_\xi f(z))$.

II. *The case when* $\sigma(T) = \{z^1, z^2\}$. It follows from Proposition 2.1 in [**1**] that T is unitarily equivalent to

$$T(z^1, z^2, c) = \left(\begin{pmatrix} z_1^1 & (z_1^2 - z_1^1)c \\ 0 & z_1^2 \end{pmatrix}, \dots , \begin{pmatrix} z_n^1 & (z_n^2 - z_n^1)c \\ 0 & z_n^2 \end{pmatrix} \right),$$

where c is some nonnegative constant; and for any ($\mathcal{M}_m$ valued) holomorphic function on a domain $\Omega \supset \{z^1, z^1\}$, we have $f(T(z^1, z^2, c)) = T(f(z^1), f(z^2), c)$.

The following appeared in a different form in [**7**]. Our method comes naturally from properties of invariant metrics.

Theorem 1. *Let* $T = (T_1, \dots , T_n)$ *be a commuting* n*-tuple of* 2×2 *matrices, and let* Ω *be a domain in* $\mathbf{C}^n$ *which contains* $\sigma(T)$*. Then* $M_\Omega(T) = K_\Omega(T)$*. In particular,* Ω *is a spectral domain of* T *if and only if it is a complete spectral domain of* T*. Furthermore, if* $\sigma(T)$ *consists of a single point* $\{z\}$*, then* T *is unitarily equivalent to* $T(z, \xi)$*, and*

$$K_\Omega(T) = \max\{1, F_\Omega(z, \xi)\}; \tag{3.5}$$

if $\sigma(T)$ *consists of two points* z^1, z^2*, then* T *is unitarily equivalent to* $T(z^1, z^2, c)$ *and*

$$K_\Omega(T) = \max\left\{1; \frac{1 - \sqrt{1 - \rho_\Omega^2(z^1, z^2)}}{\rho_\Omega(z^1, z^2)}((1 + c^2)^{1/2} + c)\right\}. \tag{3.6}$$

Proof. First we consider Case I. By definition, we have $K_\Omega(T(z, \xi)) \geq \max\{1, F_\Omega(z, \xi)\}$. Let $f \in H(\Omega, \overline{D})$. Since $f(T(z, \xi)) = T(f(z), L_\xi f(z))$, it follows from direct computation that

$$\|f(T(z, \xi))\| = \frac{1}{2}(|L_\xi f(z)| + (4|f(z)|^2 + |L_\xi f(z)|^2)^{1/2}).$$

Therefore,

$$\|f(T(z, \xi)\| \leq \frac{|L_\xi f(z)|}{1 - |f(z)|^2} \quad \text{if and only if} \quad \frac{|L_\xi f(z)|}{1 - |f(z)|^2} \geq 1.$$

However, the second inequality of the preceding line is equivalent to $F_\Omega(z,\xi) \geq 1$. Thus, it follows from the first inequality that $K_\Omega(T(z,\xi)) \leq \max\{1, F_\Omega(z,\xi)\}$.

We reduce the proof of $M_\Omega(T) = K_\Omega(T)$ in this case into two steps.

Step 1. If $K_\Omega(T) = 1$, then $M_\Omega(T) = 1$. Let $g \in H(\Omega, \overline{D})$ be the extremal function for the Carathéodory metric at (z,ξ), i.e., $F_\Omega(z,\xi) = |L_\xi g(z)|$. For any holomorphic mapping $\Phi \in H(\Omega, B_{m\times m})$, we have

$$F_\Omega(z,\xi) \geq F_{B_{m\times m}}(\Phi(z), \Phi_*(\xi)).$$

It is easy to see that there is a holomorphic mapping $h \in H(D, B_{m\times m})$ such that

$$h(0) = \Phi(z) \quad \text{and} \quad (h \circ g)_*(\xi) = \Phi_*(\xi).$$

Thus $h \circ g(T) = \Phi(T)$. Step 1 now follows from Sz-Nagy's dilation theorem.

Step 2. $M_\Omega(T) = K_\Omega(T)$. Suppose $K_\Omega(T(z,\xi)) \geq 1$. Let $c = K_\Omega(T(z,\xi))$. It follows from (3.5) that $c = F_\Omega(z,\xi)$. Thus, by homogeneity of $F_\Omega(z,\xi)$, we have

$$F_\Omega(z, \xi/c) = 1.$$

By Step 1, $M_\Omega(T(z,\xi/c)) = 1$. It follows from the following claim that $M_\Omega(T(z,\xi)) = c$.

Claim. $M_\Omega(T(z,\xi)) \leq cM_\Omega(T(z,\xi/c))$.

Proof of the Claim. By the definition of M_Ω^m,

$$cM_\Omega^m(T(z,\xi/c)) = \sup_\Phi \sup_{X,Y} \left\{ \left| c\sum_{i,j=1}^m \langle f_{ij}(T(z,\xi/c))X_i, Y_j\rangle \right| ; \|X\| \leq 1, \|Y\| \leq 1 \right\} \tag{3.7}$$

where the first sup is taken over all $\Phi = (f_{ij}) \in H(\Omega, \overline{B}_{m\times m})$ and the second sup is taken over $\|X\| \leq 1$ and $\|Y\| \leq 1$. Here we use notations

$X = (X_1, X_2, \dots, X_m)$ and $Y = (Y_1, Y_2, \dots, Y_m)$, where

$$X_i = \begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix} \quad \text{and} \quad Y_j = \begin{pmatrix} y_{ij} \\ y_{2j} \end{pmatrix}.$$

Note that

$$\begin{aligned} c\sum \langle f_{ij}(T(z,\xi/c))X_i, Y_j\rangle &= \sum \left\langle \begin{pmatrix} cf_{ij}(z) & L_\xi f_{ij}(z) \\ 0 & cf_{ij}(z) \end{pmatrix} X_i, Y_j \right\rangle \\ &= \sum \left\langle \begin{pmatrix} f_{ij}(z) & L_\xi f_{ij}(z) \\ 0 & f_{ij}(z) \end{pmatrix} \begin{pmatrix} cx_{1i} \\ x_{2i} \end{pmatrix}, \begin{pmatrix} y_{1j} \\ cy_{2j} \end{pmatrix} \right\rangle. \end{aligned} \tag{3.8}$$

Now let $\hat{X} = (\hat{X}_1, \hat{X}_2, \dots, \hat{X}_m)$ and $\hat{Y} = (\hat{Y}_1, \hat{Y}_2, \dots, \hat{Y}_m)$ with

$$\hat{X}_i = \begin{pmatrix} cx_{1i} \\ x_{2i} \end{pmatrix} \quad \text{and} \quad \hat{Y}_j = \begin{pmatrix} y_{1j} \\ cy_{2j} \end{pmatrix}.$$

Since $\{X; \|X\| \le 1\} \supset \{X; \|\hat{X}\| \le 1\}$ and $\{Y; \|Y\| \le 1\} \supset \{Y; \|\hat{Y}\| \le 1\}$, it follows from (3.7) and (3.8) that

$$\begin{aligned} &cM_\Omega^m(T(z,\xi/c)) \\ &\qquad \ge \sup\sup \left\{ \left| \sum \langle f_{ij}(T(z,\xi))\hat{X}_i, \hat{Y}_j\rangle \right|; \|\hat{X}\| \le 1, \|\hat{Y}\| \le 1 \right\} \\ &\qquad = M_\Omega^m(T(z,\xi)). \end{aligned}$$

Thus, we conclude the proof of the claim. □

Now we consider Case II. We provide a proof using the decreasing property of K_Ω and the results of Holbrook [**12**] and Paulsen [**21**]. Let

$$A_\Omega(z^1, z^2) = \sup\{|f(z^1)|; f \in H(\Omega, \overline{D}), f(z^1) = -f(z^2)\}.$$

By using a Möbius transformation, one obtains that

$$A_\Omega(z^1, z^2) = \frac{1 - (1 - \rho_\Omega^2(z^1, z^2))^{1/2}}{\rho_\Omega(z^1, z^2)}. \tag{3.9}$$

However,

$$K_\Omega(T) \geq \sup\{\|f(T)\|; f \in H(\Omega, \overline{D}), f(z^1) = -f(z^2)\} \tag{3.10}$$
$$= A_\Omega(z^1, z^2)((1+c^2)^{1/2} + c)$$

Combining (3.9) and (3.10), we obtain the inequality "$\geq$" in (3.6).

Now let $f \in H(\Omega, \overline{D})$ be the extremal mapping for $K_\Omega(T)$, i.e., $K_\Omega(T) = \|f(T)\|$. Then it follows from the result of Holbrook [**12**] (see also Paulsen [**21**]) that

$$\begin{aligned}\|f(T)\| &= K_{\overline{D}}(T(f(z^1), f(z^2), c)) \\ &= \max\left\{\frac{1-(1-\rho_D^2(f(z^1), f(z^2)))^{1/2}}{\rho_D(f(z^1), f(z^2))}; 1\right\} \\ &\leq \max\left\{\frac{1-(1-\rho_\Omega^2(z^1, z^2))^{1/2}}{\rho_\Omega(z^1, z^2)}; 1\right\}.\end{aligned}$$

The last inequality follows from the facts that $\rho_\Omega(z^1, z^2) \geq \rho_D(f(z^1), f(z^2))$ and the function $(1-(1-t^2)^{1/2})/t$ is increasing for $t \in (0,1)$.

Finally, let $\phi = (\phi_{ij}) \in H(\Omega, \overline{B}_{m\times m})$ be an extremal mapping for $M_\Omega^m(T)$ and $f \in H(\Omega, \overline{D})$ be an extremal mapping for $\rho_\Omega(z^1, z^2)$. Since

$$\begin{aligned}\rho_D(f(z^1), f(z^2)) &= \rho_\Omega(z^1, z^2) \\ &\geq \rho_{B_{m\times m}}(\phi(z^1), \phi(z^2)),\end{aligned}$$

it follows that, see page 493 in [**1**], that there exists $h \in H(D, B_{m\times m})$ such that $h \circ f(z^i) = \phi(z^i)$, $i = 1, 2$. Therefore, it follows from the result of Holbrook [**12**] that

$$\begin{aligned}\|\phi(T)\| &= \|h \circ f(T)\| = \|h(f(T))\| \leq M_{\overline{D}}(f(T)) \\ &= K_{\overline{D}}(f(T)) \leq K_\Omega(T). \qquad \square\end{aligned}$$

Remark 1. It follows from Theorem 1 that the asymptotic behavior of $K_\Omega(T(z,\xi))$ and $M_\Omega(T(z,\xi))$ is the same as that of the Carathéodory metric. When Ω is a strictly pseudoconvex domain in $\mathbf{C}^n$ or a pseudoconvex domain of finite type in $\mathbf{C}^2$, the asymptotic behavior of the

Carathéodory metric is well known by the work of Graham [**11**] and Catlin [**6**].

2) We have given direct proofs for Case I of Theorem 1 without using any planar domain results and relied on some planar domain results in Case II. It is also possible to give a direct proof for Case II, and to provide a proof for case I by reducing Ω to the unit disk as in the proof of Case II.

The following answers a spectral domain version of Paulsen's question in the case when $\dim \mathcal{H} = 2$.

Corollary 1. *Let T_i, $i = 1, 2$, be an n_i-tuple of commuting 2×2 matrices. If $\Omega_i \subset \mathbf{C}^{n_i}$ is a K_i-spectral domain for T_i, then $\Omega_1 \times \Omega_2 \subset \mathbf{C}^{n_1+n_2}$ is a K-spectral domain for the commuting $(n_1 + n_2)$-tuple (T_1, T_2) where $K = \max\{K_1, K_2\}$.*

This follows easily from the product properties (3.3) and (3.4) of the Carathéodory metric and the Möbius distance [**24**] and the formulas (3.5) and (3.6) that express K_Ω in terms of F_Ω and ρ_Ω.

We mention the following corollary as a generalization of von Neumann's inequality to n-tuples of 2×2 matrices. See Theorem 2 below for another case of the validity of von Neumann's inequality.

Corollary 2. *If T is an n-tuple of commuting contractive operators in $\mathcal{L}(\mathbf{C}^2)$, then*

$$\|p(T)\| \leq \|p\|_{H^\infty(D^n)}$$

for all polynomials p in n complex variables.

Remark. It was first proved by Holbrook [**12**] that $K_{\overline{D}}(T) = M_{\overline{D}}(T)$ for a 2×2 matrix. For any compact set $X \subset\subset \mathbf{C}$ and any 2×2 matrix T with a single eigenvalue, Misra [**14**] proved that $K_X(T) \leq 1$ implies $M_X(T) \leq 1$. This result was generalized by Paulsen [**21**], who proved that $M_X(T) = K_X(T)$ for any compact set $X \subset\subset \mathbf{C}$ and any 2×2 matrix T. For an n-tuple T of commuting 2×2 matrices and a domain $\Omega \subset \mathbf{C}^n$, Agler [**1**] proved that $K_\Omega(T) \leq 1$ implies $M_\Omega(T) \leq 1$, and Chu [**7**] proves $M_X(T) = K_X(T)$. In the case when $K_i = 1$ and $\sigma(T)$

consists of a single point, Corollary 1 was proved in [**25**]. Drury [**8**] first proved von Neumann's inequality for tuples of 2×2 matrices. A generalization of Drury's result appears in [**13**].

4. Finite dimensional cases. In this section we study the properties of K_Ω and M_Ω when $\dim \mathcal{H}$ is finite. We are especially interested in the case when $T = T(z, \xi) = zI_p + \xi J_p$ is the n-tuple of Jordan blocks, where I_p is the $p \times p$ identical matrix and J_p is the $p \times p$ matrix with 1 for all super-diagonal entries and 0 for remaining entries.

Let

$$C_p = \max \left\{ \sum_{k \le l} |a_l b_k|; a, b \in \mathbf{C}^p, \|a\| = \|b\| = 1 \right\}. \tag{4.1}$$

It is easy to see that $(p+1)/2 < C_p < p$ and that C_p is the norm of the $p \times p$ upper triangular matrix, all of whose nonzero entries are 1. The exact value of C_p for $p = 2, 3, 4$ can be easily calculated.

The following theorem sharpens a result of Smith [**20**, Exercise 3.11], [**23**, Proposition 4.5], and extends it to n-tuples. This result is one of the few known results which are valid for arbitrary matrices. The only other result that we know of which is valid for arbitrary matrices is [**5**, Theorem 2]: $M_X(T) \lesssim \log p (K_X(T))^4$.

Proposition 4.1. *Let T be an n-tuple of commuting $p \times p$ matrices, and let $\Omega \supset \sigma(T)$ be a domain in $\mathbf{C}^n$. then*

$$M_\Omega(T) \le C_p K_\Omega(T). \tag{4.2}$$

Proof. By a generalized version of Schur's theorem, there is a unitary matrix P such that P^*TP is an n-tuple of upper triangular matrices. Replacing T by P^*TP, we may assume that T is an n-tuple of upper triangular matrices.

Denote $f(T) = (L_{kl}(f))_{1 \le k,l \le p}$ for $f \in H^\infty(\Omega)$. Then each L_{kl} is a bounded linear functional on $H^\infty(\Omega)$ and $L_{kl} = 0$ when $k > l$. Furthermore,

$$|L_{kl}(f)| \le K\|f\| \tag{4.3}$$

for all $f \in H^\infty(\Omega)$, where $K = K_\Omega(T)$.

Let $(f_{ij}) \in H(\Omega, \overline{B}_{m\times m})$, and let

$$X_i = \begin{pmatrix} x_{1i} \\ \vdots \\ x_{pi} \end{pmatrix} \in \mathbf{C}^p \quad \text{and} \quad Y_j = \begin{pmatrix} y_{1j} \\ \vdots \\ y_{pj} \end{pmatrix} \in \mathbf{C}^p$$

such that $\sum_{i=1}^m \|X_i\|^2 = \sum_{j=1}^m \|Y_j\|^2 = 1$. Then

$$\begin{aligned} \sum_{i,j=1}^m \langle f_{ij}(T)X_i, Y_j\rangle &= \sum_{i,j=1}^m \sum_{k\le l} L_{kl}(f_{ij})x_{li}\bar{y}_{kj} \\ &= \sum_{k\le l} L_{kl}\bigg(\sum_{i,j=1}^m f_{ij}x_{li}\bar{y}_{kj}\bigg). \end{aligned} \tag{4.4}$$

Denote

$$a_l = \bigg\{\sum_{i=1}^m |x_{li}|^2\bigg\}^{1/2} \quad \text{and} \quad b_k = \bigg\{\sum_{j=1}^m |y_{kj}|^2\bigg\}^{1/2}.$$

Let $g_{kl} = \sum_{i,j=1}^m f_{ij}x_{li}\bar{y}_{kj}$. Then

$$\|g_{kl}\| \le a_l b_k. \tag{4.5}$$

It follows from (4.3), (4.4), (4.5) and (4.1) that

$$\begin{aligned} \bigg|\sum_{i,j=1}^m \langle f_{ij}(T)X_i, Y_j\rangle\bigg| &= \bigg|\sum_{k\le l} L_{kl}(g_{kl})\bigg| \\ &\le K\sum_{k\le l}|a_l b_k| \\ &\le KC_p. \qquad \square \end{aligned}$$

It has been shown by the examples of Varopoulus [**29**] that von Neumann's inequality is not true for 3-tuples of $p\times p$ matrices, $p \ge 5$. Here we shall show that von Neumann's inequality is true for n-tuples of upper triangular Toeplitz matrices. Recall that a $p \times p$ matrix is *Toeplitz* if it has constant diagonals.

Theorem 2. *Let $T = (T_1, T_2, \dots, T_n)$ be an n-tuple of commuting $p \times p$ upper triangular Toeplitz matrices T_i such that $\|T_i\| \le 1$, $1 \le i \le n$. Then*

$$\|p(T)\| \le \|p\|_{H^\infty(D^n)}$$

for any polynomial of n variables. Furthermore, $M_\Omega(T) = 1$.

Proof. It follows from a theorem of Carathéodory, see page 186 in [**26**], that there exist holomorphic functions $f_i : D \to \overline{D}$ such that, for $1 \le i \le n$,

$$f_i(J) = \sum_{j=0}^{p-1} \frac{1}{j!} f_i^{(j)}(0) J^j = T_i.$$

Let $F(\zeta) = (\zeta, f_1(\zeta), \dots, f_n(\zeta))$. Then, by definition, $F(J) = (J, T)$. By the decreasing property of M_Ω,

$$\begin{aligned} M_D(J) &\ge M_{D\times D^n}(I, T) \\ &\ge M_{D^n}(T). \end{aligned}$$

The last inequality is obtained by using the projection mapping from $D \times D^n$ to D^n. Also, we used the fact that $M_{D\times D^n} = M_{\overline{D}\times\overline{D}^n}$ (Proposition 2.2). Now, since $M_D(J) = 1$, we then have $M_{D^n}(T) = 1$. □

We now turn our attention to the case when $T = T(z, \xi)$ is an n-tuple of $p \times p$ Jordan blocks. For $f \in H^\infty(\Omega)$, it is easy to see that, see, for example, [**25**],

$$f(T(z,\xi)) = \sum_{k=0}^{p-1} \frac{1}{k!} L_\xi^k f(z) J^k. \tag{4.6}$$

In the case when $p \ge 3$, the relationship between $K_\Omega(T(z,\xi))$ and the (higher order) Carathéodory metric is much more complicated than the case when $p = 2$ (Theorem 1). It would be of interest to know the boundary asymptotic behavior of $K_\Omega(T(z,\xi))$ and $M_\Omega(T(z,\xi))$ when Ω is a strictly pseudoconvex domain. However, by regarding $T(z,\xi)$ as a compression of an n-tuple of commuting normal operators on $H^2(\partial D)$ (as in [**25**]), one obtains that

$$M_\Omega(T(z,\xi)) \leq \left(\frac{t}{d(z)}\right)^n$$

where $t = \max\{|\xi_i|; 1 \leq i \leq n\}$ and $d(z)$ is the Euclidean distance of $z \in \Omega$ to the boundary $\partial\Omega$ of Ω.

Proposition 4.2. *Let Ω be a domain in $\mathbf{C}^n$. Then*

1) $G_\Omega(T(z,\xi))$ *is a continuous function for* $(z,\xi) \in \Omega \times \mathbf{C}^n$.

2) $\log G_\Omega(T(z,\xi))$ *is a plurisubharmonic function for* $(z,\xi) \in \Omega \times \mathbf{C}^n$.

Proof. Again, the proof is based on a normal family argument and the special formula for $f(T(z,\xi))$. By Dini's theorem, we need only to prove the case when $G_\Omega = M_\Omega^m$.

1) Fix $(z^0,\xi^0) \in \Omega \times \mathbf{C}^n$. There exists $f \in H(\Omega, \overline{B}_{m\times m})$ such that

$$\|f(T(z^0,\xi^0))\| = M_\Omega^m(T(z^0,\xi^0)).$$

Since $\|f(T(z,\xi))\| \to \|f(T(z,\xi))\|$ as $(z,\xi) \to (z^0,\xi^0)$, for $\varepsilon > 0$,

$$M_\Omega^m(T(z,\xi)) \geq \|f(T(z,\xi))\| \geq M_\Omega^m(T(z^0,\xi^0)) - \varepsilon$$

when $|(z,\xi)-(z^0,\xi^0)| < \delta$ for sufficiently small δ. On the other hand, since the set $H(\Omega, \overline{B}_{m\times m})$ is a normal family, after possible shrinking of δ, we have

$$\|f(T(z,\xi)) - f(T(z^0,\xi^0))\| < \varepsilon$$

for all $|(z,\xi)-(z^0,\xi^0)| < \delta$ and $f \in H(\Omega, \overline{B}_{n\times n})$. Thus, $M_\Omega^m(T(z,\xi)) = \sup_{f\in H(\Omega,B_{m,m})} \|f(T)\| \leq M_\Omega^m(T(z^0,\xi^0)) + \varepsilon$.

2) Fix $(z^0,\xi^0) \in \Omega \times \mathbf{C}^n$. We only need to prove that, for any $(z,\xi) \in \Omega \times \mathbf{C}^n$,

$$\log M_\Omega^m(T(z^0,\xi^0)) \leq \frac{1}{2\pi r}\int_{|\lambda|=r} \log M_\Omega^m(T(z^0+\lambda z, \xi^0+\lambda\xi))|d\lambda|$$

holds for all sufficiently small r.

Let $f = (f_{ij}) \in H(\Omega, \overline{B}_{m\times m})$ be the extremal mapping for $M_\Omega^m(T(z^0,\xi^0))$, i.e.,

$$M_\Omega^m(T(z^0,\xi^0)) = \|f(T(z^0,\xi^0))\|.$$

It is easy to see that $\log\|f(T(z,\xi))\|$ is a plurisubharmonic function for $(z,\xi) \in \Omega \times \mathbf{C}^n$. Thus,

$$\begin{aligned}\log M_\Omega^m(T(z^0,\xi^0)) &= \log\|f(T(z^0,\xi^0))\| \\ &\leq \frac{1}{2\pi r}\int_{|\lambda|=r} \log\|f(T(z^0+\lambda z, \xi^0+\lambda\xi))\|\,|d\lambda| \\ &\leq \frac{1}{2\pi r}\int_{|\lambda|=r} \log M_\Omega^m(T(z^0+\lambda z, \xi^0+\lambda\xi))|d\lambda|. \qquad \square\end{aligned}$$

Recall that an n-tuple $T = (T_1, T_2, \dots, T_n)$ of bounded operators is *power bounded* by A if $\|T^\alpha\| \leq A$ for any n-tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of nonnegative integers. Here $T^\alpha = T_1^{\alpha_1} \circ T_2^{\alpha_2} \cdots T_n^{\alpha_n}$. The following proposition is an application of Grothendieck's inequality, for which see [**30**].

Proposition 4.3. *Let $T = (T_1, T_2, \dots, T_n)$ be an n-tuple of commuting bounded operators on a Hilbert space $\mathcal{H}$. If T is power bounded by A, then there exists a universal constant C such that, for any $\Phi = (\phi_{jk}) \in H(D^n, B_{m\times m})$ where ϕ_{jk} are polynomials with degree $\leq d$,*

$$\|\Phi(T)\| \leq C(\log d)^n A^2. \tag{4.7}$$

Proof. The proof of this proposition is similar to that of Lemma 2 in [**5**]. We include it here for the reader's convenience. Let $D(\zeta)$ be the (analytic part of the) Dirichlet kernel, i.e.,

$$D(\zeta) = \sum_{k=0}^{p} \zeta^k \tag{4.8}$$

and let $\mathcal{D}(z) = D(z_1)D(z_2)\cdots D(z_n)$. It follows that $\mathcal{D}(z) = f(z)g(z)$ for some $f, g \in H^2(D^n)$ such that $\|\mathcal{D}\|_{L^1} = \|f\|_{L^2}\|g\|_{L^2}$. Suppose that the Fourier expansions of f and g are

$$f(e^{i\theta_1}, \ldots, e^{i\theta_n}) = \sum_{\substack{|\alpha|=0 \\ \alpha\geq 0}}^{\infty} a_\alpha e^{i\alpha\cdot\theta};$$
$$g(e^{i\theta_1}, \ldots, e^{i\theta_n}) = \sum_{\substack{|\alpha|=0 \\ \alpha\geq 0}}^{\infty} b_\alpha e^{i\alpha\cdot\theta}.$$

Let $I = [0, 2\pi]$. For $x_j, y_k \in \mathcal{H}$, we have

$$\begin{aligned}
&\Big|\sum_{j,k=1}^{m} \langle \phi_{jk}(T)x_j, y_k\rangle\Big| \\
&= \Big|\sum_{j,k=1}^{m} \int_{I^n} \phi_{jk}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \\
&\qquad\qquad \cdot \langle \mathcal{D}(e^{-i\theta_1}T_1, \ldots, e^{-i\theta_n}T_n)x_j, y_k\rangle \frac{d\theta}{(2\pi)^n}\Big| \\
&= \Big|\sum_{j,k=1}^{m} \int_{I^n} \phi_{jk}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \\
&\qquad \cdot \langle f(e^{-i\theta_1}T_1, \ldots, e^{-i\theta_n}T_n)x_j, \\
&\qquad\qquad \cdot \bar{g}(e^{-i\theta_1}T_1^*, \ldots, e^{-i\theta_n}T_n^*)y_k\rangle \frac{d\theta}{(2\pi)^n}\Big| \\
&= \Big|\sum_{j,k=1}^{m} \sum_{\alpha,\beta} \hat{\phi}_{jk}(\alpha+\beta)a_\alpha b_\beta \langle T^\alpha x_j, T^{*\beta} y_k\rangle\Big| \\
&\lesssim A^2 \sup\Big\{\Big|\sum_{j,k=1}^{m} \sum_{\alpha,\beta} \hat{\phi}_{jk}(\alpha+\beta)a_\alpha b_\beta \|x_j\|\,\|y_k\| s_{j\beta} t_{\alpha k}\Big|\Big\}
\end{aligned}$$

where the sup is taken over all scalars $s_{j\beta}, t_{\alpha k} \in D$. The last inequality follows from Grothendieck's theorem. However, the term inside the sup sign is

$$\Big| \sum_{j,k=1}^{m} \int_{I^n} \phi_{jk}(e^{i\theta_1}, \dots , e^{i\theta_n}) \|x_j\| \Big(\sum_{\beta} s_{j\beta} b_\beta e^{-i\beta\cdot\theta} \Big) \cdot \|y_k\| \Big(\sum_{\alpha} t_{\alpha k} a_\alpha e^{-i\alpha\cdot\theta} \Big) \frac{d\theta}{(2\pi)^n} \Big|$$

$$\leq \|\Phi\| \Big\{ \int_{I^n} \sum_{j=1}^{m} \|x_j\|^2 \Big| \sum_{\beta} s_{j\beta} b_\beta e^{-i\beta\cdot\theta} \Big|^2 d\theta \Big\}^{1/2} \cdot \Big\{ \int_{I^n} \sum_{k=1}^{m} \|y_k\|^2 \Big| \sum_{\alpha} t_{\alpha k} a_\alpha e^{-i\alpha\cdot\theta} \Big|^2 d\theta \Big\}^{1/2}$$

$$\leq \|f\|_{L^2} \|g\|_{L^2} \Big\{ \sum_{j=1}^{m} \|x_j\|^2 \Big\}^{1/2} \Big\{ \sum_{j=1}^{m} \|y_k\|^2 \Big\}^{1/2}.$$

We obtain (4.7) by combining the above inequalities and recalling the known fact that $\|D\|_{L^1} \lesssim \log d$. □

Remark. If we use the following kernel

$$B_p(z) = \sum_{\substack{\|\alpha\| \leq p \\ \alpha \geq 0}} \Big(1 - \frac{\|\alpha\|^2}{p^2} \Big)^{(n-1)/2} z^\alpha \tag{4.9}$$

instead of the Dirichlet kernel $\mathcal{D}(z)$, then the term $(\log p)^n$ in the inequality (4.7) can be replaced by the L^1-norm of B_p. If the sum in (4.9) is taken over all $\|\alpha\| \leq p$, then the resulting function is the Bochner-Riesz kernel. The L^1-norm of the Bochner-Riesz kernel is $\sim \log p$ [**27**, Theorem 4].

In the case when T is n-tuple of Jordan blocks, we have the following as an application of Proposition 4.3.

Theorem 3. *Let $z \in D^n$, and let $T = T(z, \xi)$ be an n-tuple of commuting $p \times p$ Jordan blocks. Then*

$$M_{D^n}(T) \leq C \cdot (\log p)^n K_{D^n}^2(T),$$

where C is a constant independent of p.

Proof. Let

$$\psi_i(\zeta) = \frac{\zeta - z_i}{1 - \bar{z}_i\zeta}$$

and $\Psi = (\psi_1, \dots, \psi_n)$. Then $M_{D^n}(T(z,\xi)) = M_{D^n}(\Psi(T(z,\xi)))$. If $f, g \in H^\infty(D^n)$ have the same $(p-1)$th order Taylor polynomial at the origin, then $f(\Psi(T(z,\xi))) = g(\Psi(T(z,\xi)))$.

For $\phi \in H^\infty(D^n)$, define

$$\begin{aligned} L(\phi)(z) &= \phi * \mathcal{D}(z) \\ &= \int_{I^n} \phi(z_1 e^{-i\theta_1}, \dots, z_n e^{-i\theta_n}) \mathcal{D}(e^{i\theta_1}, \dots, e^{i\theta_n}) \frac{d\theta}{(2\pi)^n}. \end{aligned}$$

Then $L(\phi)$ is a polynomial of degree p^n which agrees with the Taylor expansion of ϕ up to pth order. Thus, from the Cauchy estimates,

$$\|L(\phi)\|_{H^\infty(D^n)} \lesssim 2^{np} \sup\{|\phi(z)|; |z_j| = 1/2\}.$$

Now let $N = 2np$, and let $h \in L^1(\partial D)$ be a function such that

1) $\|h\|_{L^2} \leq 2$, $\hat{h}(j) \geq 0$ for all j;

2) $\hat{h}(j) = 1$ if $|j| < N/2$; $\hat{h}(j) = 0$, if $|j| > N$.

Let $H(e^{-i\theta_1}, \dots, e^{-i\theta_n}) = h(e^{i\theta_1}) \cdots h(e^{i\theta_n})$. For $\phi \in H^\infty(D^n)$, define

$$E(\phi) = \phi * F + L(\phi - \phi * H).$$

Then $E(\phi)$ is a polynomial of degree N^n and agrees with ϕ up to pth order. Furthermore, we have

$$\begin{aligned} \|E(\phi)\|_{H^\infty(D^n)} &\lesssim 2^n \|\phi\|_{H^\infty(D^n)} \\ &\quad + 2^{np} \sup\Big\{ \sum_{\alpha_i \geq N/2} |\hat{\phi}(\alpha_1 \cdots \alpha_n)|\,|z^\alpha|; |z_i| = 1/2 \Big\} \\ &\lesssim \|\phi\|_{H^\infty(D^n)}. \end{aligned}$$

Therefore, for $\Phi = (\phi_{ij}) \in H(D^n, B_{m\times m})$, it follows from (4.7) that

$$\begin{aligned} \|\Phi(\Psi(T(z,\xi)))\| &= \|(E \circ \Phi)(\Psi(T(z,\xi)))\| \\ &\lesssim (\log p)^n \|E \circ \Phi\| K^2_{D^n}(\Psi(T(z,\xi))) \\ &\lesssim (\log p)^n K^2_{D^n}(T(z,\xi)). \end{aligned}$$

This concluses the proof of Theorem 3. □

Remarks. Blower, in [**4**], proves a result similar to Theorem 3 for a nilpotent matrix. Misra, in [**15**], respectively, [**16**], finds necessary and sufficient conditions for $K_{D^n}(T) \leq 1$, respectively $M_{D^n}(T) \leq 1$, where

$$T = (T_1, \ldots , T_n), \qquad T_j = \begin{pmatrix} \lambda_j & \mathbf{v_j} \\ 0 & I_{p-1} \end{pmatrix}.$$

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