ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 27, Number 4, Fall 1997

FINITE HOMOGENEOUS SPACES

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ABSTRACT. We characterize homogeneous topological spaces which contain a nonempty discrete open subset. As a consequence, we characterize and enumerate finite homogeneous topological spaces.

1. Introduction. Several authors studied finite topological spaces with some topological structures, for example, Warren [**6**] gave a formula for the number of nonhomeomorphic topologies on a finite set by means of T_0 identification spaces.

Mashhour, Abd El-Monsef and Farrag [**4**] studied the maximum cardinality for topologies on a finite set, and in [**3**] they studied the number of topologies on a finite set.

A topological space X is called a homogeneous space provided that if p and q are two points of X, then there exists a homeomorphism $h : X \to X$ such that $h(p) = q$. The concept of homogeneity was introduced by W. Sierpinski [**5**]. Several mathematicians have studied homogeneous spaces. One may consult [**2**] for this study.

Let X be any set. By τ_{dis} and τ_{ind} we mean the discrete and indiscrete topologies on X, respectively. $|X|$ denotes the cardinality of the set X. If n is a natural number, $\tau(n)$ will denote the number of positive divisors of n.

2. Finite homogeneous spaces. In this section we shall give a characterization of finite homogeneous spaces, and we shall study the number of finite homogeneous topological spaces (up to homeomorphism).

Let us start with the following definition, see [**1**, p. 70].

Definition 2.1. Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ be a collection of topological spaces such that $X_\alpha \cap X_\beta = \varnothing$ for all $\alpha \neq \beta$. Let $X = \cup_{\alpha\in\Lambda} X_\alpha$ be

Received by the editors on March 28, 1995, and in revised form on November 21, 1995.

topologized by $\tau = \{G \subseteq X : G \cap X_\alpha \in \tau_\alpha \text{ for all } \alpha \in \Lambda\}$. Then (X, τ) is called the *sum* of the spaces $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$.

Now we can state our main result.

Theorem 2.2. *Let (X, τ) be a topological space which contains a nonempty open indiscrete subset* (*in the induced topology*). *Then the following are equivalent*:

(a) *(X, τ) is a homogeneous space.*

(b) *(X, τ) is a disjoint union of indiscrete topological spaces all of which are homeomorphic to one another.*

Proof. Let G denote the group of homeomorphisms from (X, τ) to itself, and let Y denote a nonempty open subset of X which is indiscrete.

First we show $(a) \Rightarrow (b)$. Notice that, for any $g, g' \in G$, we have that gY is either equal to $g'Y$ or disjoint from $g'Y$. This is true since $gY \cap g'Y$ will be open in the induced topology of gY, which is the indiscrete topology. This means that X will be the disjoint union of the gY since G acts transitively on X. Moreover, since gY for $g \in G$ is an open cover for X, a set will be open in X if and only if its intersection with each of the gY is open. This completes the first implication.

Next we show $(b) \Rightarrow (a)$. Let X be the disjoint union of Y_i for $i \in I$, each of which is an indiscrete space in the induced topology. Because each Y_i is indiscrete, every open set in X either contains Y_i or has empty intersection with it. Hence, every open set in X is of the form $\cup\{Y_i : i \in J\}$ for some $J \subset I$. Thus, a permutation of the set X will be a homeomorphism if and only if it takes each Y_i to another one of the Y_i. Now, assume x and x' are in X. We show that there is a homeomorphism taking x to x'. Assume $x \in Y_i$ and $x' \in Y_j$. There is a bijection $f : Y_i \to Y_j$ since Y_i is assumed to be homeomorphic to Y_j. Define F, a homeomorphism from X to X, as follows: F restricted to Y_i is f, F restricted to Y_j is f^{-1}, and F fixes every other element of X. Next, let g be any permutation of Y_j that takes $f(x)$ to x'. Extend g to G, a permutation of X, by letting G fix every point not in Y_j. Then $G \circ F$ is a homeomorphism that takes x to x'. The proof is complete. It should be noted that G is the wreath product of S_Y with S_I where

Y is any one of the Y_i and S_Y denotes the group of permutations on the set Y. □

It is clear that (X, τ) has a nontrivial indiscrete open subset if and only if $\tau - \{\phi\}$ has a minimal element in the partial order given by set inclusion. For this reason we have two corollaries. The first one studies the homogeneous spaces whenever the topology on the set X has only finitely many members.

Corollary 2.3. *Let (X, τ) be a topological space with $|\tau| < \aleph_0$, then (X, τ) is homogeneous if and only if (X, τ) is the sum of mutually homeomorphic indiscrete spaces.*

Proof. Since τ is finite, then $\tau \backslash \{\phi\}$ has a minimal element (with set inclusion as the partial order). This element would be indiscrete and open, and hence Theorem 2.2 is applicable. □

Corollary 2.4. *Let (X, τ) be a finite topological space. (X, τ) is homogeneous if and only if (X, τ) is the sum of indiscrete topological spaces all of which are homeomorphic to one another.*

It is clear that the discrete and the indiscrete topologies on any set are the sum of mutually homeomorphic indiscrete spaces.

In the next results we shall consider only finite spaces. We shall give the precise number of topologies on a finite homogeneous space (up to homeomorphism). In fact, we have the following result.

Corollary 2.5. *If X is a finite nonempty set with $|X| = n$, then there are precisely $\tau(n)$ homogeneous spaces on X (up to homeomorphism), where $\tau(n)$ is the number of positive divisors of n.*

Proof. If $n = 1$, then there is one and only one topology on X. So we may assume that $n > 1$ and $X = \{x_1, \dots, x_n\}$. For each divisor k of n, $1 \le k \le n$, consider the following homogeneous space (X, τ_k) where τ_k is the topology generated by the base

$$\mathcal{B}_k = \{\{x_1, \dots, x_k\}, \{x_{k+1}, \dots, x_{2k}\}, \dots, \{x_{n-k+1}, \dots, x_n\}\}.$$

It is obvious that (X,τ_k) is not homeomorphic to (X,τ_j) for $k \neq j$. So there are at least $\tau(n)$ nonhomeomorphic topologies on X each of which is homogeneous. If τ is any topology on X such that (X,τ) is not homeomorphic to any topological space (X,τ_k) for all positive divisors k of n, then its base $\mathcal{B}$ is not of the form $\mathcal{B}_k$ (up to homeomorphism) for all divisors k of n, hence $\mathcal{B}$ has two members U and V such that $|U| \neq |V|$. So, by Corollary 2.4, (X,τ) is not homogeneous. □

As applications of Corollary 2.4 and Corollary 2.5, we have the following results:

Corollary 2.6. *If X is a finite set and (X,τ) is a connected homogeneous topological space, then τ is the indiscrete topology.*

Proof. Since (X,τ) is homogeneous, then (X,τ) is the sum of mutually homeomorphic indiscrete spaces, and since (X,τ) is connected, then the only nonempty closed and open set is X, hence $\mathcal{B} = \{X\}$. Therefore, $\tau = \tau_{\text{ind}}$. □

Corollary 2.7. *If $|X| = p$, p is prime, and if (X,τ) is a disconnected homogeneous topological space then τ is the discrete topology.*

Proof. Since (X,τ) is homogeneous, therefore, there are $\tau(p)$ topologies on X (up to homeomorphism). Since p is prime, then $\tau(p) = 2$. Hence τ is either τ_{dis} or τ_{ind}, but (X,τ) is disconnected, thus $\tau = \tau_{\text{dis}}$.
□

To give our last result we need the following definition.

Definition 2.8. A topological space (X,τ) is said to have the *fixed point property* if and only if every continuous map $f : X \to X$ has a fixed point.

Corollary 2.9. *If X is a nonempty finite set with $|X| > 1$ and (X,τ) is a homogeneous space, then (X,τ) does not have the fixed point property.*

Proof. Since (X, τ) is a finite homogeneous space, therefore, (X, τ) is the sum of mutually homeomorphic indiscrete spaces according to Corollary 2.3. Thus, $\tau = \tau_{\text{ind}}$ (which is not T_0 since $|X| > 1$) or (X, τ) is disconnected. In either case, (X, τ) does not have the fixed point property. □

In general the condition that X is finite of the last result cannot be removed as we shall see in the following example.

Example 2.10. Take X to be $\mathbf{R}P(2n)$, the real projective space in even dimensions. Then such spaces are compact connected manifolds with $H_i(\mathbf{R}P(2n), \mathbf{R})$ the torsion for all $i > 0$. Now $\mathbf{R}P(2n)$ are homogeneous spaces for all n because every manifold is homogeneous. Next, we show that every continuous map from $\mathbf{R}P(2n)$ has a fixed point. If f did not have such a fixed point, then $\operatorname{tr}(f_*) = 0$ where $f_* : H_*(\mathbf{R}P(2n), \mathbf{R}) \to H_*(\mathbf{R}P(2n), \mathbf{R})$. Since $\mathbf{R}P(2n)$ has torsion reduced homology, the only nontrivial homology with coefficients in the reals is $H_0(\mathbf{R}P(2n), \mathbf{R})$, which is $\mathbf{R}$, because $\mathbf{R}P(2n)$ is connected. Thus $f_* : \mathbf{R} \to \mathbf{R}$ in the case, with $f_*(x) = x$. Hence $\operatorname{tr}(f_*) = 1 \neq 0$, and the proof is complete.

Acknowledgment. Authors are thankful to the referee for his constructive suggestions.

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