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OSCILLATORY PROPERTIES OF THE SOLUTIONS OF IMPULSIVE DIFFERENTIAL EQUATIONS WITH A DEVIATING ARGUMENT AND NONCONSTANT COEFFICIENTS

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ABSTRACT. Sufficient conditions are found for oscillation of all solutions of the impulsive differential equation with a deviating argument

$$x'(t) + p(t)x(t-\tau) = 0, \quad t \neq \tau_k,$$
$$\Delta x(\tau_k) = b_k x(\tau_k), \quad t = \tau_k,$$

where the function p is not of constant sign.

1. Introduction. In the last twenty years the number of investigations devoted to oscillatory and nonoscillatory behavior of solutions of functional differential equations has considerably increased. The greater part of the works on this subject published by 1977 are given in [**4**]. In the monographs [**2**] and [**3**] published respectively in 1987 and 1991, the oscillatory and asymptotic properties of the solutions of various classes of functional differential equations were systematically studied. The pioneer work devoted to the investigation of the oscillatory properties of the solutions of impulsive differential equations with a deviating argument was the work of Gopalsamy and Zhang [**1**]. In it the authors gave sufficient conditions for oscillation of the solutions of the impulsive differential equation with a deviating argument

$$\begin{aligned} x'(t) + p(t)x(t-\tau) &= 0, \qquad \tau = \text{const}, \quad t \neq \tau_k; \\ \Delta x(\tau_k) = x(\tau_k+0) - x(\tau_k - 0) &= b_k x(\tau_k - 0) \end{aligned} \tag{1}$$

where p is a nonnegative function. It is again there that conditions are given for the existence of nonoscillating solutions of the equation considered when p is a positive constant.

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Here by τ_k, $k = 1, 2, \dots$, the points of jump are denoted.

In the present paper sufficient conditions for oscillation of all solutions of equation (1) are found, where the function p is not of constant sign.

2. Preliminary notes. Consider the impulsive differential equation with a deviating argument (1) with initial condition

$$x(t) = \varphi(t), \qquad -\tau \le t \le 0. \tag{2}$$

Introduce the following conditions:

H1. τ is a positive constant.

H2. b_k are constants, $b_k > -1$, $k = 1, 2, \dots$.

H3. $0 < \tau_1 < \tau_2 < \cdots$, $\lim_{k\to+\infty} \tau_k = \infty$.

H4. The function $p \in C([0,\infty), \mathbf{R})$ and $p(t) \ge 0$ at least in the sequence of intervals $\{(\xi_n, \eta_n)\}_{n=1}^{\infty}$, where $\eta_n < \xi_{n+1}$, $\eta_n - \xi_n = 2\tau$, $n = 1, 2, \dots$.

H5. $\varphi \in C([-\tau, 0], \mathbf{R})$, $p(t+\tau)\varphi(t) \ne 0$ for $t \in [-\tau, 0]$.

H6. There exists a constant $M > 0$ such that for each $k = 1, 2, \dots$ the inequality $0 < b_k \le M$ is valid.

We construct the sequence $\{t_i\}_{i=1}^{\infty}$ in the following way: Let $\tau_{i\tau} = \tau_i + \tau$, $i = 1, 2, \dots$; $\{t_i\}_{i=1}^{\infty} = \{\tau_i\}_{i=1}^{\infty} \cup \{\tau_{i\tau}\}_{i=1}^{\infty}$, and we assume that $t_i < t_{i+1}$, $i = 1, 2, \dots$.

Definition 1. By a *solution* of equation (1) with initial condition (2) we mean a function x: $[-\tau, \infty) \to \mathbf{R}$ for which the following conditions are valid:

1. If $-\tau \le t \le 0$, then $x(t) = \varphi(t)$.

2. If $0 \le t \le t_1 = \tau_1$, then the solution x coincides with the solution of the problem

$$\begin{aligned} x'(t) + p(t)x(t-\tau) = 0, \quad & t \in [0, \tau_1] \\ x(t) = \varphi(t), \quad & t \in [-\tau, 0]. \end{aligned}$$

3. If $t_i < t \le t_{i+1}$ and $t_i \in \{\tau_i\}_{i=1}^{\infty} \backslash \{\tau_{i\tau}\}_{i=1}^{\infty}$, then the solution x coincides with the solution of the problem

$$x'(t) + p(t)x(t-\tau) = 0 \qquad x(t_i + 0) = (1 + b_{k_i})x(t_i),$$

where k_i is determined from the equality $\tau_{k_i} = t_i$.

4. If $t_i < t \le t_{i+1}$ and $t_i \in \{\tau_{i\tau}\}_{i=1}^{\infty} \backslash \{\tau_i\}_{i=1}^{\infty}$, then the solution x coincides with the solution of the problem

$$x'(t) + p(t)x(t-\tau+0) = 0, \qquad x(t_i+0) = x(t_i).$$

5. If $t_i < t \le t_{i+1}$ and $t_i \in \{\tau_i\}_{i=1}^{\infty} \cap \{\tau_{i\tau}\}_{i=1}^{\infty}$, then the solution x coincides with the solution of the problem

$$x'(t) + p(t)x(t-\tau+0) = 0, \qquad x(t_i+0) = (1+b_{k_i})x(t_i),$$

where k_i is determined from the equality $\tau_{k_i} = t_i$.

Definition 2. A nonzero solution x of the problem (1), (2) is said to be *nonoscillating* if there exists $t_0 \ge 0$ such that $x(t)$ is of constant sign for $t \ge t_0$. Otherwise, the solution is said to *oscillate*.

3. Main results.

Theorem 1. *Let the following conditions hold*:

1. *Conditions* H1—H5 *are met.*
2. $\tau_{n+1} - \tau_n \ge 2\tau$, $\eta_n - \tau_n < \tau$ *for* $n = 1, 2, \ldots$, $\tau_n \in (\xi_n, \eta_n)$.
3. $\limsup_{n\to\infty}(1/(1+b_n)) \int_{\tau_n}^{\eta_n} p(s)\,ds > 1$.

Then all solutions of the problem (1), (2) *oscillate.*

Proof. Let a nonoscillating solution x of the problem (1), (2) exist. Without loss of generality we may assume that $x(t) > 0$ for $t \ge t_0$ for some $t_0 \ge 0$. Then $x(t-\tau) > 0$ too for $t \ge t_0 + \tau$. (The case when $x(t) < 0$ for $t \ge t_0$ is considered analogously.)

From (1) and condition H4 it follows that x is a nonincreasing function in $\cup_{\xi_n \ge t_0+\tau}(\xi_n, \eta_n)$.

Let τ_n be a point of jump in the interval (ξ_n, η_n).

Integrate equation (1) from τ_n to η_n and obtain that

$$x(\eta_n) - x(\tau_n+0) + \int_{\tau_n}^{\eta_n} p(s)x(s-\tau)\,ds = 0. \tag{3}$$

From (3) and the fact that x is a nonincreasing function in (τ_n, η_n) it follows that

$$x(\eta_n) - x(\tau_n + 0) + x(\eta_n - \tau) \int_{\tau_n}^{\eta_n} p(s)\, ds \leq 0. \tag{4}$$

From (4) and the inequality $x(\eta_n - \tau) \geq x(\tau_n)$ there follows the inequality

$$x(\eta_n) - x(\tau_n + 0) + x(\tau_n) \int_{\tau_n}^{\eta_n} p(s)\, ds \leq 0. \tag{5}$$

Replace in (5) $x(\tau_n)$ by $x(\tau_n + 0)/(1 + b_n)$ and obtain that

$$x(\eta_n) + x(\tau_n + 0) \left[\frac{1}{1 + b_n} \int_{\tau_n}^{\eta_n} p(s)\, ds - 1 \right] \leq 0,$$

whence it follows that

$$\limsup_{n \to \infty} \frac{1}{1 + b_n} \int_{\tau_n}^{\eta_n} p(s)\, ds \leq 1.$$

The last inequality contradicts condition 3 of Theorem 1. □

Theorem 2. *Let the following conditions hold*:

1. *Conditions* H1, H3—H6 *and condition* 2 *of Theorem* 1 *are met.*
2. $\liminf_{n \to \infty} \int_{\eta_n - \tau}^{\eta_n} p(s)\, ds > 1 + M$.

Then all solutions of equation (1) *oscillate.*

Proof. Let a nonoscillating solution x of equation (1) exist. Without loss of generality we may assume that $x(t) > 0$ for $t \geq t_0$ for some $t_0 \geq 0$. Then $x(t - \tau) > 0$ too for $t \geq t_0 + \tau$. (The case when $x(t) < 0$ for $t \geq t_0$ is considered analogously.)

Let τ_n be a point of jump in the interval $(\eta_n - \tau, \eta_n)$ where $\xi_n > t_0 + \tau$.

Integrate equation (1) from $\eta_n - \tau$ to η_n and obtain that

$$x(\tau_n) - x(\eta_n - \tau) + x(\eta_n) - x(\tau_n + 0) + \int_{\eta_n - \tau}^{\eta_n} p(s) x(s - \tau)\, ds = 0,$$

i.e.,

$$\int_{\eta_n-\tau}^{\eta_n} p(s)x(s-\tau)\,ds = x(\eta_n-\tau) - x(\eta_n) + b_n x(\tau_n).$$

From the last inequality we find that

$$\inf_{s\in[\eta_n-\tau,\eta_n]} x(s-\tau)\int_{\eta_n-\tau}^{\eta_n} p(s)\,ds \le x(\eta_n-\tau) + b_n x(\tau_n). \tag{6}$$

From (1) it follows that $x(t)$ is a nonincreasing function in the intervals (ξ_n,τ_n) and (τ_n,η_n). Then

$$\begin{gathered} x(\tau_n) < x(\eta_n-\tau) \\ \inf_{\eta_n-\tau\le s\le\eta_n} x(s-\tau) = \inf_{\xi_n\le s\le \eta_n-\tau} x(s) = x(\eta_n-\tau). \end{gathered} \tag{7}$$

From (6) and (7) we obtain that

$$x(\eta_n-\tau)\int_{\eta_n-\tau}^{\eta_n} p(s)\,ds \le x(\eta_n-\tau) + b_n x(\eta_n-\tau),$$

whence it follows that

$$\int_{\eta_n-\tau}^{\eta_n} p(s)\,ds \le 1 + b_n \le 1 + M.$$

The last inequality contradicts condition 2 of Theorem 2. $\square$

Theorem 3. *Let the following conditions hold*:

1. *Conditions* H1, H3–H6 *are met.*
2. *The number of points of jump of the solutions of the problem* (1), (2) *in the interval* (ξ_n,η_n) *is* k_n, $n=1,2,\dots$.
3. *There exists a constant* k *such that* $k_n < k$, $n=1,2,\dots$.
4. $\liminf_{n\to\infty}\int_{\eta_n-\tau}^{\eta_n} p(s)\,ds > (1+M)^k$.

Then all solutions of the equation (1) *oscillate.*

Proof. Denote the points of jump in the interval (ξ_n,η_n) by $\tau_n^{(1)},\tau_n^{(2)},\dots,\tau_n^{(k_n)}$, $(\tau_n^{(1)} < \tau_n^{(2)} < \cdots < \tau_n^{(k_n)})$ and the corresponding constants by $b_n^{(i)}$, $i=1,2,\dots,k_n$.

Let a nonoscillating solution x of the equation (1) exist. Without loss of generality we may assume that $x(t) > 0$ for $t \geq t_0$ for some $t_0 \geq 0$. Then $x(t-\tau) > 0$ too for $t \geq t_0+\tau$. (The case when $x(t) < 0$ for $t \geq t_0$ is considered analogously.)

In the interval $(\xi_n, \eta_n - \tau)$ let r points of jump exist, and in the interval $(\eta_n - \tau, \eta_n)$ let l points of jump exist ($k_n = r + l$, $\xi_n \geq t_0 + \tau$).

Integrate equation (1) from $\eta_n - \tau$ to η_n and obtain that

$$\int_{\eta_n-\tau}^{\eta_n} p(s)x(s-\tau)\,ds = x(\eta_n-\tau) - x(\eta_n) + \sum_{i=1}^{l} b_n^{(r+i)} x(\tau_n^{(r+i)}). \tag{8}$$

The lefthand side of (8) can be represented in the form

$$\begin{aligned}
\int_{\eta_n-\tau}^{\eta_n} p(s)x(s-\tau)\,ds &= \int_{\eta_n-2\tau}^{\eta_n-\tau} p(s+\tau)x(s)\,ds \\
&= \int_{\xi_n}^{\tau_n^{(1)}} p(s+\tau)x(s)\,ds \\
&\quad + \sum_{i=1}^{r-1} \int_{\tau_n^{(i)}}^{\tau_n^{(i+1)}} p(s+\tau)x(s)\,ds \\
&\quad + \int_{\tau_n^{(r)}}^{\eta_n-\tau} p(s+\tau)x(s)\,ds \\
&\geq x(\tau_n^{(1)}) \int_{\xi_n}^{\tau_n^{(1)}} p(s+\tau)\,ds \\
&\quad + \int_{i=1}^{r-1} x(\tau_n^{(i+1)}) \int_{\tau_n^{(i)}}^{\tau_n^{(i+1)}} p(s+\tau)\,ds \\
&\quad + x(\eta_n-\tau) \int_{\tau_n^{(r)}}^{\eta_n-\tau} p(s+\tau)\,ds.
\end{aligned} \tag{9}$$

Since $x(\tau_n^{(i)}) = x(\tau_n^{(i)}+0)/(1+b_n^{(i)})$ and x is a nonincreasing function for

$$t \in (\xi_n, \tau_n^{(1)}) \cup \left[\cup_{i=1}^{k_n-1} (\tau_n^{(i)}, \tau_n^{(i+1)}) \right] \cup (\tau_n^{(k_n)}, \eta_n),$$

then

$$
\begin{aligned}
x(\tau_n^{(1)}) &= \frac{x(\tau_n^{(1)}+0)}{1+b_n^{(1)}} \geq \frac{x(\tau_n^{(2)})}{1+b_n^{(1)}} \\
&= \frac{x(\tau_n^{(2)}+0)}{(1+b_n^{(1)})(1+b_n^{(2)})} \qquad (10)\\
&\geq \cdots \geq \frac{x(\eta_n-\tau)}{\prod_{i=1}^{r}(1+b_n^{(i)})}.
\end{aligned}
$$

Substitute (10) into (9) and obtain that

$$
\begin{aligned}
\int_{\eta_n-\tau}^{\eta_n} p(s)x(s-\tau)\,ds \geq x(\eta_n-\tau)
&\cdot\Bigg[\frac{1}{\prod_{i=1}^{r}(1+b_n^{(i)})}\int_{\xi_n}^{\tau_n} p(s+\tau)\,ds \\
&+ \frac{1}{\prod_{i=1}^{r-1}(1+b_n^{(i)})}\int_{\tau_n^{(1)}}^{\tau_n^{(2)}} p(s+\tau)\,ds \\
&+\cdots+\int_{\tau_n^{(r)}}^{\eta_n-\tau} p(s+\tau)\,ds\Bigg].
\end{aligned}
$$

From condition H6 it follows that

$$
\begin{aligned}
\int_{\eta_n-\tau}^{\eta_n} p(s)x(s-\tau)\,ds &\geq x(\eta_n-\tau) \\
&\cdot\Bigg[\frac{1}{(1+M)^r}\int_{\xi_n}^{\tau_n^{(1)}} p(s+\tau)\,ds \\
&+ \frac{1}{(1+M)^{r-1}}\int_{\tau_n^{(1)}}^{\tau_n^{(2)}} p(s+\tau)\,ds \qquad (11)\\
&+\cdots+\int_{\tau_n^{(r)}}^{\eta_n-\tau} p(s+\tau)\,ds\Bigg] \\
&\geq \frac{x(\eta_n-\tau)}{(1+M)^r}\int_{\xi_n}^{\eta_n-\tau} p(s+\tau)\,ds \\
&= \frac{x(\eta_n-\tau)}{(1+M)^r}\int_{\eta_n-\tau}^{\eta_n} p(s)\,ds.
\end{aligned}
$$

From (9) and (11) we obtain that

$$\begin{aligned}\frac{x(\eta_n-\tau)}{(1+M)^r}\int_{\eta_n-\tau}^{\eta_n}p(s)\,ds &\le x(\eta_n-\tau)-x(\eta_n)\\ &\qquad+\sum_{i=1}^{l}b_n^{(r+i)}x(\tau_n^{(r+i)})\\ &\le x(\eta_n-\tau)+\sum_{i=1}^{l}b_n^{(r+i)}x(\tau_n^{(r+i)})\\ &\le x(\eta_n-\tau)+M\sum_{i=1}^{l}x(\tau_n^{(r+i)}).\end{aligned}\tag{12}$$

From the fact that x is a nonincreasing function for

$$t\in(\eta_n-\tau,\tau_n^{(1)})\cup\left[\bigcup_{i=1}^{l-1}(\tau_n^{(r+i)},\tau_n^{(r+i+1)})\right]\cup(\tau_n^{(r+l)},\eta_n),$$

there follow the inequalities

$$\begin{aligned}x(\tau_n^{(r+1)}) &\le x(\eta_n-\tau)\\ x(\tau_n^{(r+2)}) &\le x(\tau_n^{(r+1)}+0)\\ &=(1+b_n^{(r+1)})x(\tau_n^{(r+1)})\\ &\le(1+M)x(\eta_n-\tau)\\ &\vdots\\ x(\tau_n^{(r+l)}) &\le(1+M)^{l-1}x(\eta_n-\tau)\\ x(\eta_n) &\le x(\tau_n^{(r+l)}+0)\\ &\le(1+M)^l x(\eta_n-\tau)\end{aligned}$$

and the estimate

$$\begin{aligned}\sum_{i=1}^{l}x(\tau_n^{(r+i)}) &\le x(\eta_n-\tau)[1+(1+M)+\cdots+(1+M)^{l-1}]\\ &=x(\eta_n-\tau)\frac{1-(1+M)^l}{1-1-M}\\ &=\frac{x(\eta_n-\tau)}{-M}[1-(1+M)^l].\end{aligned}\tag{13}$$

Substitute (13) into (12) and obtain that

$$\frac{x(\eta_n-\tau)}{(1+M)^r}\int_{\eta_n-\tau}^{\eta_n} p(s)\,ds \le x(\eta_n-\tau) - x(\eta_n-\tau)[1-(1+M)^l]$$
$$\int_{\eta_n-\tau}^{\eta_n} p(s)\,ds \le (1+M)^{l+r} = (1+M)^{k_n} \le (1+M)^k.$$

The last inequality contradicts condition 4 of Theorem 3. $\square$

Corollary 1. *Let the conditions of Theorem* 2 *hold. Then*:

1. *The inequality*

$$\begin{aligned} x'(t)+p(t)x(t-\tau) &\le 0, \quad t\ne\tau_k;\\ \Delta x(\tau_k) &= b_k x(\tau_k), \quad t=\tau_k \end{aligned} \tag{14}$$

has no positive solutions.

2. *The inequality*

$$\begin{aligned} x'(t)+p(t)x(t-\tau) &\ge 0, \quad t\ne\tau_k;\\ \Delta x(\tau_k) &= b_k x(\tau_k), \quad t=\tau_k \end{aligned} \tag{15}$$

has no negative solutions.

Corollary 2. *Let the conditions of Theorem* 3 *hold. Then*:

1. *The inequality* (14) *has no positive solutions.*
2. *The inequality* (15) *has no negative solutions.*

The proofs of Corollary 1 and Corollary 2 are carried out analogous to the proofs of Theorem 2 and Theorem 3, respectively.

Theorem 4. *Let the following conditions hold*:

1. *Conditions* H1, H3, H5 *and* H6 *are met.*
2. *The function* $p \in C([0,\infty),\mathbf{R})$ *and there exists a sequence of intervals* $\{(\xi_n,\eta_n)\}_{n=1}^{\infty}$ *such that* $p(t)\ge 0$ *for* $t\in(\xi_n,\eta_n)$, $\eta_n<\xi_{n+1}$, $2\tau<\eta_n-\xi_n\le 5\tau/2$, $n=1,2,\dots$.

3. $\tau_{n+1} - \tau_n \geq \tau$, $n = 1, 2, \dots$.

4. $\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds > (1+M)/e$, $e = \exp$, $t \in \cup^{\infty}(\xi_n + 2\tau, \eta_n)$.

Then all solutions of the equation (1) *oscillate.*

Proof. Let a nonoscillating solution x of the equation (1) exist. Without loss of generality we may assume that $x(t) > 0$ for $t \geq t_0$ for some $t_0 \geq 0$. Then $x(t-\tau) > 0$ too for $t \geq t_0 + \tau$. (The case when $x(t) < 0$ for $t \geq t_0$ is considered analogously.)

Define the function $w(t) = x(t-\tau)/x(t)$ for $t \in (\xi_n + 2\tau, \eta_n)$, $\xi_n \geq t_0 + \tau$. Let $\tau_n \in (\eta_n - \tau, \xi_n + 2\tau)$, i.e., for $t \in (\xi_n + 2\tau, \eta_n)$, τ_n is a point of jump in the interval $(t-\tau, t)$.

From (1) and condition 2 of Theorem 4 it follows that x is a nonincreasing function in (ξ_n, τ_{n-1}), (τ_{n-1}, τ_n) and (τ_n, η_n), where τ_{n-1} and τ_n are the points of jump in (ξ_n, η_n). Then

$$x(t-\tau) \geq x(\tau_n) = \frac{x(\tau_n + 0)}{1 + b_n} \geq \frac{x(t)}{1 + b_n} \geq \frac{x(t)}{1 + M},$$

or $w(t) \geq (1+M)^{-1}$.

We shall prove that the function w is bounded from above.

Case 1. Let $\tau_n \in (t - \tau/2, t)$ for $t \in (\xi_n + 2\tau, \eta_n)$, i.e., $\tau_n \in (\eta_n - \tau/2, \xi_n + 2\tau)$. Integrate (1) from $t - \tau/2$ to t and obtain that

$$x(t) - x\left(t - \frac{\tau}{2}\right) - b_n x(\tau_n) + \int_{t-\tau/2}^{t} p(s) x(s-\tau)\,ds = 0$$

whence it follows that

1.1. If $\tau_{n-1} \in (t - 3\tau/2, t-\tau)$, i.e., $\tau_{n-1} \in (\eta_n - 3\tau/2, \xi_n + \tau)$, then

$$\begin{aligned} x\left(t - \frac{\tau}{2}\right) &\geq \int_{t-\tau/2}^{t} p(s) x(s-\tau)\,ds - b_n x(\tau_n) \\ &= \int_{t-3\tau/2}^{\tau_{n-1}} p(s+\tau) x(s)\,ds + \int_{\tau_{n-1}}^{t-\tau} p(s+\tau) x(s)\,ds - b_n x(\tau_n) \\ &\geq x(\tau_{n-1}) \int_{t-3\tau/2}^{\tau_{n-1}} p(s+\tau)\,ds \end{aligned}$$

$$
\begin{aligned}
&+ x(t-\tau)\int_{\tau_{n-1}}^{t-\tau} p(s+\tau)\,ds - b_n x(\tau_n)\\
&= \frac{x(\tau_{n-1}+0)}{1+b_{n-1}}\int_{t-3\tau/2}^{\tau_{n-1}} p(s+\tau)\,ds\\
&\quad + x(t-\tau)\int_{\tau_{n-1}}^{t-\tau} p(s+\tau)\,ds - b_n x(\tau_n)\\
&\geq \frac{x(t-\tau)}{1+M}\bigg[\int_{t-3\tau/2}^{\tau_{n-1}} p(s+\tau)\,ds + \int_{\tau_{n-1}}^{t-\tau} p(s+\tau)\,ds\\
&\qquad\qquad + M\int_{\tau_{n-1}}^{t-\tau} p(s+\tau)\,ds - M - M^2\bigg]\\
&\geq \frac{x(t-\tau)}{1+M}\bigg[\int_{t-\tau/2}^{t} p(s)\,ds - M - M^2\bigg],
\end{aligned}
$$

i.e.,

$$
x\left(t-\frac{\tau}{2}\right) \geq \frac{x(t-\tau)}{1+M}\bigg[\int_{t-\tau/2}^{t} p(s)\,ds - M - M^2\bigg]. \tag{16}
$$

1.2. If $\tau_{n-1} \notin (t-3\tau/2, t-\tau)$, i.e., $\tau_{n-1} < \eta_n - 3\tau/2$, then

$$
\begin{aligned}
x\left(t-\frac{\tau}{2}\right) &\geq \int_{t-3\tau/2}^{t-\tau} p(s+\tau)x(s)\,ds - b_n x(\tau_n)\\
&\geq x(t-\tau)\int_{t-\tau/2}^{t} p(s)\,ds - b_n x(\tau_n)\\
&\geq x(t-\tau)\bigg[\int_{t-\tau/2}^{t} p(s)ds - M\bigg],
\end{aligned}
$$

i.e.,

$$
x\left(t-\frac{\tau}{2}\right) \geq x(t-\tau)\bigg[\int_{t-\tau/2}^{t} p(s)\,ds - M\bigg]. \tag{17}
$$

Integrate (1) from $t-\tau$ to $t-\tau/2$ and obtain that

$$
x\left(t-\frac{\tau}{2}\right) + x(t-\tau) + \int_{t-2\tau}^{t-3\tau/2} p(s+\tau)x(s)\,ds = 0
$$

whence it follows that

$$x(t-\tau) \geq x\left(t-\frac{3\tau}{2}\right) \int_{t-\tau}^{t-\tau/2} p(s)\, ds. \tag{18}$$

From (18) and (16) we obtain that

$$\frac{x(t-3\tau/2)}{x(t-\tau/2)} \leq \frac{1+M}{\int_{t-\tau}^{t-\tau/2} p(s)\, ds \left[\int_{t-\tau/2}^{t} p(s)\, ds - M - M^2\right]} \leq N, \tag{19}$$

for $t \in (\xi_n + 2\tau, \eta_n)$.

From (17) and (18) we obtain that

$$\frac{x(t-3\tau/2)}{x(t-\tau/2)} \leq \frac{1}{\int_{t-\tau}^{t-\tau/2} p(s)\, ds \left[\int_{t-\tau/2}^{t} p(s)\, ds - M\right]} \leq N_1, \tag{20}$$

for $t \in (\xi_n + 2\tau, \eta_n)$.

From (19) and (20) we obtain that $w(t) \leq \text{const}$, for $t \in (\xi_n + 2\tau, \eta_n)$, i.e., we have proved that the function w is bounded from above.

Case 2. Let $\tau_n \in (t-\tau, t-\tau/2)$ for $t \in (\xi_n + 2\tau, \eta_n)$, i.e., $\tau_n \in (\eta_n - \tau, \xi_n + 3\tau/2)$. Analogously to the Case 1 we obtain that the function w is bounded from above.

We divide both sides of the equation (1) by $x(t) > 0$ for $t \in (\xi_n + 2\tau, \eta_n)$, integrate from $t-\tau$ to t and obtain that

$$\ln\left[\frac{x(t-\tau)}{x(t)}(1+b_n)\right] = \int_{t-\tau}^{t} p(s) \frac{x(s-\tau)}{x(s)}\, ds. \tag{21}$$

Introduce the notation

$$w_0 = \liminf_{t\to\infty} w(t).$$

It is clear that $0 < w_0 < \infty$. Then from (21) we obtain that

$$\ln[(1+M)w(t)] \geq w_0 \int_{t-\tau}^{t} p(s)\, ds,$$

i.e.,

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds \le \frac{\ln[(1+M)w_0]}{w_0} \le \frac{1+M}{e}$$

for $t \in \cup^{\infty}(\xi_n + 2\tau, \eta_n)$, $\xi_n \ge t_0 + \tau$, which contradicts condition 4 of Theorem 4. □

Corollary 3. *Let the conditions of Theorem* 4 *hold. Then*:

1. *The inequality* (14) *has no positive solutions.*
2. *The inequality* (15) *has no negative solutions.*

The proof of Corollary 3 is carried out analogous to the proof of Theorem 4.

Theorem 5. *Let the following conditions hold*:

1. *Conditions* 1 *and* 2 *of Theorem* 4 *are met.*

2. *The number of the points of jump of the solutions of the problem* (1), (2) *in the interval* (ξ_n, η_n) *is* k_n, $n = 1, 2, \dots$.

3. *There exists a constant* k *such that* $k_n < k$, $n = 1, 2, \dots$.

4. $\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds > (1+M)^k/e$, $e = \exp$, $t \in \cup^{\infty}(\xi_n + 2\tau, \eta_n)$.

Then all solutions of the equation (1) *oscillate.*

Proof. Analogous to the proof of Theorem 3 and Theorem 4 we obtain that

$$\ln\left[\frac{x(t-\tau)}{x(t)} \prod_{i=1}^{k}(1+b_n^{(i)})\right] \ge \int_{t-\tau}^{t} p(s)\frac{x(s-\tau)}{x(s)}\,ds.$$

Then

$$\ln[(1+M)^k w(t)] \ge w_0 \int_{t-\tau}^{t} p(s)\,ds.$$

From the last inequality it follows that

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds \le \frac{(1+M)^k}{e}, \quad t \in \cup^{\infty}(\xi_n + 2\tau, \eta_n)$$

which contradicts condition 4 of Theorem 5. □

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