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WHEN DOES THE FAMILY OF SINGULAR COMPACTIFICATIONS FORM A COMPLETE LATTICE?

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ABSTRACT. In this paper we provide a method of recognizing those spaces for which the supremum of all singular compactifications is βX. We also provide a method of recognizing those spaces for which the family of singular compactifications forms a complete lattice.

1. Introduction. All hypothesized topological spaces will be assumed to be locally compact and Hausdorff.

Two compactifications αX and γX of a space X are said to be *equivalent* if there is a homeomorphism $f : \alpha X \to \gamma X$ from αX onto γX which fixes the points of X. This defines an equivalence relation on the family of all compactifications of X. When we will speak of a compactification αX of X it will be understood that we are referring to the equivalence class of αX. The notation $\alpha X \cong \gamma X$ will mean that αX is equivalent to γX. We will say that the compactification αX is less than or equal to the compactification γX, denoted by $\alpha X \leq \gamma X$ if there is a continuous function $f : \gamma X \to \alpha X$ of γX onto αX which acts as the identity on X. This defines a partial order on the family $K(X)$ of all compactifications of X. It is well known that $K(X)$ is a complete lattice with respect to the partial order $\leq$ (see [**3**, 2.19]). If αX and γX are compactifications of X such that $\alpha X \leq \gamma X$, we will denote the projection map from γX onto αX which fixes the points of X by $\pi_{\gamma\alpha}$.

The family of compactifications studied here was first defined and discussed in [**6**]. We introduce the object of our study in the following definitions which appear in [**6**]. A *singular compactification induced by the function f* is constructed as follows: Let $f : X \to K$ be a continuous

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function from the space X into a compact set K. Let the *singular set*, $S(f)$, of f be defined as the set $\{x \in K : \text{for any neighborhood } U \text{ of } x, \text{cl}_X f^{\leftarrow}[U] \text{ is not compact}\}$. If $S(f) = K$, then f is said to be a *singular map*. It is easy to verify that $S(f)$ is closed in K and that if f is a singular map then $f[X]$ is dense in $S(f)$. If f is a singular map the *singular compactification of* X *induced by* f, denoted by $X \cup_f S(f)$, is the set $X \cup S(f)$ where the basic neighborhoods of the points in X are the same as in the original space X, and the points of $S(f)$ have neighborhoods of form $U \cup (f^{\leftarrow}[U] \backslash F)$ where U is open in $S(f)$ and F is a compact subset of X. This defines a compact Hausdorff topology on $X \cup_f S(f)$ in which X is a dense subspace. We will say that a compactification αX of X is a *singular compactification* if αX is equivalent to $X \cup_f S(f)$ for some singular map f.

An important characterization of a singular compactification is the following one: The singular compactifications of X are precisely those compactifications αX of X whose remainder $\alpha X \backslash X$ is a retract of αX. It is also known that if αX is a singular compactification and γX is any compactification of X less than αX, then γX is also a singular compactification. (The reader is referred to [**14, 5**] and [**1**] for more details). Hence the infimum of any family of singular compactifications is a singular compactification. However the supremum of a family of singular compactifications need not be a singular compactification. It is known, for example, that $\beta\mathbf{N}$ is the supremum of singular compactifications but $\beta\mathbf{N}$ is not a singular compactification (see [**14**, page 20] or [**1**]).

If G is contained in $C^*(X)$, the symbol $\omega_G X$ will denote the smallest compactification to which all functions in G extend (this notation was introduced in [**11**]). If f belongs to $C^*(X)$, $\omega_f X$ will denote the smallest compactification of X to which f extends. Let $G \subseteq C^*(X)$. The *evaluation map* e_G induced by G is the function $e_G : X \to \Pi\{I_g : g \in G\}$ (where, for each g, I_g is a closed interval containing $g[X]$) defined by $e_G(x) = \langle g(x) \rangle_{g \in G}$. The set S_γ will denote the set of all singular maps in $C_\gamma(X) = \{f|_x : f \in C(\gamma X)\}$. Thus S_β denotes the collection of all singular maps in $C^*(X)$. All other notation will be as described in [**1**] and [**15**].

It is known that every compactification αX of X can be expressed in the form $\omega_G X$, where $G \subseteq C^*(X)$, see [**11**, Theorem 1] or [**1**, 1.11]. In particular, αX is equivalent to $\omega_{C_\alpha(X)}(X)$. By Theorem 2.6 of [**1**], we know that if αX is singular, then αX is equivalent to $\omega_{S_\alpha}X$, the supremum of all singular compactifications less than or equal to αX. In fact, the following stronger statement is true: A compactification αX is the supremum of singular compactifications if and only if αX is equivalent to $\omega_G X$ for some $G \subseteq S_\beta$, see [**1**, 2.8]. But this does not imply that a compactification of form $\omega_G X$ for some $G \subseteq S_\beta$ is a singular compactification (as witnessed by the example of $\beta\mathbf{N}$). It is also known that not all compactifications are of the form $\omega_G X$ for some $G \subseteq S_\beta$. (The two-point compactification of $\mathbf{R}$ is an example; see [**1**, 2.7].)

2. The largest singular compactification. The main objective of this section is to develop a way of recognizing those locally compact noncompact Hausdorff spaces X which have a largest singular compactification. We begin by clearly defining the term *largest*.

Definitions 2.1. We will say that αX is the *largest singular compactification* of X if αX is a singular compactification and, whenever γX is a singular compactification of X, then $\gamma X \leq \alpha X$, i.e., X has a largest singular compactification if the supremum in $(K(X), \leq)$ of the set of all singular compactifications of X is a singular compactification. We say that the compactification γX is a *maximal* singular compactification if γX is singular and there does not exist a singular compactification ζX such that $\zeta X > \gamma X$.

Note. Recall that the family of all singular compactifications is a lower semi-lattice. Thus, to show that a locally compact Hausdorff space X has a largest singular compactification is equivalent to showing that the family of all singular compactifications of X is a complete lattice.

We begin by presenting the following previously proven results.

Proposition 2.2 [**11**]. *Let $G \subseteq C^*(X)$ and αX be a compactification of X. Then $\alpha X \cong \omega_G X$ if and only if each function g in G extends to g^α in $C(\alpha X)$ and G^α separates the points of $\alpha X \backslash X$.*

Theorem 2.3 [**5**]. *If αX is a compactification of X and $G \subseteq S_\alpha$, then $\alpha X = \sup\{X \cup_f S(f) : f \in G\}$ if and only if G^α separates the points of $\alpha X \backslash X$.*

Theorem 2.4 [**11**]. a) *Let $f \in C^*(X)$. Then $\omega_f X$ is equivalent to $X \cup^* S(f)$. In particular, if f is a singular map, then $\omega_f X$ is a singular compactification and $\omega_f X$ is equivalent to $X \cup_f S(f)$.*

b) *If $G \subseteq C^*(X)$ and $\omega_G X$ is a singular compactification, then $t = e_G^\omega \cdot {}_G \circ r|_X$ is a singular map, where $r : \omega_G X \to \omega_G X \backslash X$ is a retraction map, and $\omega_G X$ is equivalent to $X \cup_t S(t)$.*

Theorem 2.5 [**1**]. *If αX is a singular compactification, then αX is equivalent to $\omega_{S_\alpha} X$. Hence every singular compactification αX of X is the supremum of the family $\{X \cup_f S(f) : f \in S_\alpha\}$ of singular compactifications.*

The following proposition will help us formulate our problem in a more succinct way.

Proposition 2.6. *The compactification αX of X is the largest singular compactification of X if and only if $\alpha X \cong \omega_{S_\beta} X$ and $\omega_{S_\beta} X$ is singular.*

Proof. $\Rightarrow$. Suppose αX is the largest singular compactification of the space X. Then, by 2.5, $\alpha X \cong \omega_{S_\alpha} X$. Since $\omega_{S_\alpha} X$ is the smallest compactification to which all functions in $S_\beta \cap C_\alpha(X)$ extend, then $\omega_{S_\alpha} X \leq \omega_{S_\beta} X$. Now, if $f \in S_\beta$, then, by Theorem 2.4a), $\omega_f X \cong X \cup_f S(f) \leq \alpha X$ (since αX is the largest singular compactification). Let $\gamma X = \sup\{\omega_f X : f \in S_\beta\}$. Hence, $\gamma X \leq \alpha X$. By Theorem 2.3, S_β^γ separates points of $\gamma X \backslash X$, consequently γX must be the smallest compactification to which the set of all functions in S_β extend, or more succinctly, $\gamma X \cong \omega_{S_\beta} X$. It must then follow that $\alpha X \cong \omega_{S_\beta} X$.

$\Leftarrow$. Suppose $\alpha X \cong \omega_{S_\beta} X$ and that $\omega_{S_\beta} X$ is a singular compactification. By Proposition 2.2, S_β^α separates the points of $\alpha X \backslash X$ and, by Theorem 2.3, $\alpha X = \sup\{X \cup_f S(f) : f \in S_\beta\}$. Since every singular compactification is of the form $\omega_G X$ for some $G \subseteq S_\beta$, by Theorem 2.5, and as

$$\begin{aligned}\omega_G X &= \sup\{X \cup_f S(f) : f \in G\} \quad \text{(by 1.4)}\\ &\leq \sup\{X \cup_f S(f) : f \in S_\beta\} \quad (\text{since } G \subseteq S_\beta)\\ &= \alpha X,\end{aligned}$$

then $\omega_{S_\beta} X$ is the supremum of all singular compactifications; hence, αX is the largest singular compactification of X. □

We can now reformulate our question as follows:

When is the compactification $\omega_{S_\beta} X$ a singular compactification?

Definition 2.7. The compactification $\omega_{S_\beta} X$ will be denoted by μX (whether it is singular or not). When we will speak of the *μ-compactification* of X, we will mean μX.

Note that the μ-compactification of X exists for all completely regular spaces X. We know that in some cases the μ-compactification of X is equivalent to βX. (In 2.13 of [**1**] the author shows that, for a compactification αX of X, if $\alpha X \backslash X$ is not totally disconnected, then αX is equivalent to $\omega_{S_\alpha} X$. Also, in 2.14 of the same paper, we have the following result: If X is a strongly zero-dimensional not almost compact space, then βX is the supremum of the family of the two-point singular compactifications of X, hence $\beta X = \omega_{S_\beta} X$.) We will show that there is a multitude of spaces X whose μ-compactification μX is neither the Stone-Čech compactification nor the Freudenthal compactification. Note, however, that if $\mu X < \beta X$, then $\beta X \backslash X$ is totally disconnected, since, by 2.13 of [**1**] noted above, if $\beta X \backslash X$ is not totally disconnected, then $\beta X \cong \omega_{S_\beta} X = \mu X$. Hence, if $\mu X < \beta X$, then μX cannot be the Freudenthal compactification, since if $\beta X \backslash X$ is totally disconnected βX is the Freudenthal compactification.

Before we answer the question stated above, we will develop in 2.9–2.12 a characterization of those spaces X such that μX is equivalent to βX. First we give an example of a space X such that μX is strictly less than βX.

Example 2.8. Let x and y be distinct points in $\beta\mathbf{R}\backslash\mathbf{R}$, and let $X = \beta\mathbf{R}\backslash\{x, y\}$, where X is equipped with the subspace topology inherited from $\beta\mathbf{R}$. If $f \in C^*(X)$, f can be extended to $\mathrm{cl}_{\beta\mathbf{R}}X$, via $f|_{\mathbf{R}}$, hence $\beta\mathbf{R} \cong \beta X$. Clearly X must be connected as $\mathbf{R} \subseteq X \subseteq \mathrm{cl}_{\beta\mathbf{R}}\mathbf{R}$, and $\mathbf{R}$ is connected. It follows that $\beta X\backslash X$ is not the continuous image of X. Then the one-point compactification is the only singular compactification, since $\beta X\backslash X$ cannot be a retract of βX. Hence, by Proposition 2.6, the one-point compactification of X is μX. Hence, $\mu X < \beta X$.

Note that those spaces X such that $\mu X < \beta X$ must be amongst those spaces which are not strongly zero-dimensional and whose outgrowth $\beta X\backslash X$ is totally disconnected (see the paragraph following Definition 2.7).

Theorem 2.9. *Let X be a topological space. Then $\mu X \cong \beta X$ if and only if S_β^β separates the points of $D \cap (\beta X\backslash X)$ for each connected component D of βX.*

Proof. $\Rightarrow$. Suppose X is a space such that $\beta X \cong \mu X \cong \omega_{S_\beta} X$. Then S_β^β separates the points of $\beta X\backslash X$. Hence S_β^β separates the points of $D \cap (\beta X\backslash X)$ for each connected component D of βX.

$\Leftarrow$. Suppose S_β^β separates the points of $D \cap (\beta X\backslash X)$ for each connected component D of βX. It will suffice to show that S_β^β separates points of $\beta X\backslash X$, since Proposition 2.2 will imply that $\mu X \cong \beta X$. Let x and y be distinct points in $\beta X\backslash X$. If x and y belong to distinct components of βX, then there exists a clopen subset U of βX which contains x but not y. The restriction of the characteristic function χ_U to X is a singular map whose extension to βX separates x and y. This fact, and our hypothesis, implies that $\beta X \cong \omega_{S_\beta} X \cong \mu X$. $\square$

The example of a space X such that μX is not equivalent to βX given in Example 2.8 is rather trivial. We will now investigate such spaces in order to construct more complex examples of such spaces. First we develop some more theory (in Theorem 2.11 and Example 2.13).

The following is Corollary 1.7 of [**1**].

Corollary 2.10. *Let $f : X \to K$ be a continuous map into a compact Hausdorff space such that $f[X]$ is dense in K. Let $E_f(X)$ denote the set of all compactifications αX of X such that $f : X \to K$ extends to $f^\alpha : \alpha X \to K$. Then f is a singular map if and only if $f^\alpha[\alpha X \backslash X]$ contains $f[X]$ for some (equivalently for all) $\alpha X \in E_f(X)$.*

Theorem 2.11. *If X is a connected noncompact space which is not almost compact, then the following are equivalent*:

1) $\mu X \cong \beta X$.

2) *There is a continuous function from $\beta X \backslash X$ onto a closed interval with nonempty interior.*

3) *The space X has a compactification αX whose outgrowth $\alpha X \backslash X$ is homeomorphic to a closed interval of real numbers* (*with nonempty interior*).

4) *The space X has a singular compactification which is not the one-point compactification ωX of X.*

5) *S_β contains a nonconstant function.*

Proof. We will prove the equivalence of these statements in the following order: $4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 5 \Rightarrow 4$.

$4 \Rightarrow 3$. Suppose X has a singular compactification αX such that $\alpha X \backslash X$ contains more than one point. By Theorem 2.5, αX is equivalent to $\omega_{S_\alpha} X$. Let x and y be distinct points in $\alpha X \backslash X$. Since S_α^α separates the points of $\alpha X \backslash X$, there is a function f in S_α such that $f^\alpha(x)$ is not equal to $f^\alpha(y)$. Since f is a singular map, the compactification $\omega_f X$ is a singular compactification (2.4). Also, since X is connected, then by 2.15 of [**1**], $\omega_f X \backslash X$ is homeomorphic to a closed interval in $\mathbf{R}$. Now $\omega_f X \backslash X$ contains more than one point, hence this interval has nonempty interior.

$3 \Rightarrow 2$. If X has a compactification αX such that $\alpha X \backslash X$ is homeomorphic to a closed interval of $\mathbf{R}$ (with nonempty interior), then the projection map, $\pi_{\beta\alpha}$, maps $\beta X \backslash X$ onto $\alpha X \backslash X$. This means $\beta X \backslash X$ can be mapped continuously onto a closed interval of $\mathbf{R}$ (with nonempty interior).

$2 \Rightarrow 1$. Suppose there is a continuous function f from $\beta X \backslash X$ onto a closed interval $[a,b] = I$ with nonempty interior. We must show that $\mu X \cong \beta X$. The reader will note that the connectedness of X does not play a role in the proof of $2 \Rightarrow 1$. We will suppose that $\beta X \backslash X$ is 0-dimensional, since, if $\beta X \backslash X$ is not 0-dimensional then, by 2.13 of [**1**], $\mu X \cong \beta X$. Let x and y be distinct points in $\beta X \backslash X$ for which $f(x) = f(y)$ (since $\beta X \backslash X$ is 0-dimensional and $[a,b]$ is not this implies that f cannot be one-to-one; thus such a pair of points can be found). Let us consider the case where $f(x)$ is a point in (a,b). (The proof for the case where $f(x)$ is a or b will be similar.) Let $M = (c,d)$ be an open interval containing $f(x)$ such that c is not a, and d is not b. Let U and V be disjoint clopen (in $\beta X \backslash X$) neighborhoods of x and y, respectively, such that both U and V are contained in $f^{\leftarrow}(M)$. Let $f^* : \beta X \backslash X \to \mathbf{R}$ be a function which agrees with f on $(\beta X \backslash X) \backslash (U \cup V)$ and which sends U and V to distinct points in $[a,b] \backslash M$. The function f^* is continuous. Let the function $h : [a,b] \to \mathbf{R}$ be defined as follows: $h(x) = x$ if $x \in [a,c]$, $h(x) = c$ if $x \in [c,d]$ and $h(x) = x - (d-c)$ if $x \in [d,b]$. The function h is continuous and has a range which is a closed interval. Then the function $h \circ f^*$ separates the points x and y and maps $\beta X \backslash X$ onto the closed interval $[a, b-(d-c)]$. Let $k : \beta X \to \mathbf{R}$ be an extension of $h \circ f^*$ to all of βX, and let $g = (k \wedge \mathbf{a}) \vee \mathbf{b} - (\mathbf{d} - \mathbf{c})$. Note that g maps βX into $g[\beta X \backslash X] = [a, b-(d-c)]$. Hence $g|_X$ is a singular function which separates the arbitrarily chosen points x and y in $\beta X \backslash X$. We have shown that S_β^β separates the points of $\beta X \backslash X$; hence $\beta X \cong \omega_{S_\beta} X = \mu X$ (by Proposition 2.2 and the definition of μX).

$1 \Rightarrow 5$. Suppose every function in S_β is constant. Then every function in S_β extends to ωX, hence $\mu X = \omega_{S_\beta} X \cong \omega X$. As X is not almost compact and as $|\omega X \backslash X| = 1$, we have μX is not equivalent to βX.

$5 \Rightarrow 4$. Suppose S_β contains a function f which is not a constant function. Since f is a singular function $\omega_f X$ is a singular compactification and $\omega_f X$ is equivalent to $X \cup_f S(f)$ (by Theorem 2.4). Since f maps X into $S(f)$ and $f[X]$ contains at least two points, then $\omega_f X$ is a singular compactification which is not the one-point compactification of X. $\square$

We now give a general characterization of spaces X such that $\mu X \cong \beta X$.

Theorem 2.12. *Let X be a locally compact space. Then the following are equivalent*:

1) $\mu X \cong \beta X$.

2) *At least one of the two following conditions is satisfied*:

a) *Any two points of $\beta X \backslash X$ are contained in distinct connected components of βX.*

b) *There is a continuous function from $\beta X \backslash X$ onto a closed interval with nonempty interior.*

Proof. $1 \Rightarrow 2$. Suppose the space X is such that $\mu X \cong \beta X$ and that $\beta X \backslash X$ contains a pair of points, say x and y, which both belong to the same connected component C of βX. Since $\mu X \cong \beta X$, then βX is equivalent to $\omega_{S_\beta} X$. Hence there is a function f in S_β such that f^β separates x and y (2.2). Since f is a singular map, $f^\beta[\beta X]$ is contained in $f^\beta[\beta X \backslash X]$ (2.10). Also, since C is connected and f^β separates x and y, $f^\beta[C]$ is a closed interval, say $[a, b]$, with nonempty interior, that is, a is not equal to b. Then $f^\beta[C] = [a, b]$ is contained in $f^\beta[\beta X \backslash X]$. Let $h = (f^\beta \wedge \mathbf{a}) \vee \mathbf{b}$. Since h maps $\beta X \backslash X$ continuously onto $[a, b]$, we are done.

$2 \Rightarrow 1$. Suppose any two points in $\beta X \backslash X$ are contained in distinct connected components of βX. Let x and y be any two points in $\beta X \backslash X$, and let M and L be distinct connected components of βX such that x is in M and y is in L. Then there exists a clopen (in βX) subset U of βX which contains M but not L. If f is a characteristic map which sends U to zero and $\beta X \backslash U$ to one, then $f|_X$ is a singular function whose extension to βX separates x and y. Since x and y were arbitrarily chosen in $\beta X \backslash X$, S_β^β separates the points of $\beta X \backslash X$, hence βX is equivalent to $\mu X = \omega_{S_\beta} X$.

We now consider the other hypothesis of 2). Suppose $\beta X \backslash X$ can be mapped by a continuous function f onto some closed interval $[a, b]$ of $\mathbf{R}$. In $2 \Rightarrow 1$ of Theorem 2.11 we have proven that this hypothesis implies that $\mu X \cong \beta X$ (without using the hypothesis that X is connected). The theorem follows. $\square$

We now provide a method for constructing spaces X such that μX is not equivalent to βX. Recall that a function $f : X \to Y$ is called irreducible if f does not map any proper closed subset of X onto Y. Also recall that a topological space is a scattered space if it contains no nonempty dense-in-itself subset (see 30E of [**25**]).

Example 2.13. Let $\mathbf{R}^+ = \{x \in \mathbf{R} : x \geq 0\}$ and ∞ be a point in $\beta\mathbf{R}^+\backslash\mathbf{R}^+$. Let S be an infinite compact scattered space and $Y = \beta\mathbf{R}^+\backslash\{\infty\}$. Let u and v be distinct points in S, $X = S \times Y$ and X^* be the quotient space of X obtained by collapsing to a single point the doubleton $\{(u,0),(v,0)\}$ and fixing all other points of X. Then $\mu(X^*)$ is not equivalent to $\beta(X^*)$.

Proof. Let S, Y, X and X^* be as described in the statement of the theorem. Then βY is $\beta\mathbf{R}^+$, the one-point compactification of Y. Since Y is pseudocompact, $\beta X = S \times \beta Y$ (see 8.12 and 8.20 of [**24**]). It is easily verified that $\beta X^* = X^* \cup \{(x,\infty) : x \in S\}$. Then $\beta X^*\backslash X^*$ is the scattered space $S^* = \{(x,\infty) : x \in S\}$ which is homeomorphic to S itself. Since the perfect image of a scattered space is scattered, there is no continuous surjection from S^* onto a closed interval I with nonempty interior. We have just produced a completely regular nonconnected Hausdorff space X^* whose outgrowth $\beta X^*\backslash X^*$ cannot be mapped continuously onto a closed interval. Note that the points (u,∞) and (v,∞) are not contained in distinct connected components of βX^*. Then, by Theorem 2.12 $1 \Rightarrow 2$, μX^* is not equivalent to βX^*. $\square$

The rest of this section (2.14 to 2.34) is devoted to solving the question (stated earlier): When is the supremum, μX ($\cong \omega_{S_\beta} X$), of all singular compactifications a singular compactification?

Recall that a subset B of X is called a P-set if any G_δ containing B is a neighborhood of B.

Lemma 2.14. *If D is a closed C-embedded copy of* $\mathbf{N}$ *in a locally compact space X, then* $(\mathrm{cl}_{\beta X} D)\backslash D$ *is a P-set of* $\beta X\backslash X$.

Proof. Let D be a closed C-embedded copy of $\mathbf{N}$ in a space X. It suffices to show that, if $(\mathrm{cl}_{\beta X} D)\backslash D$ is contained in a zero-set Z

in $\beta X\backslash X$, then it must be contained in its $\beta X\backslash X$-interior. Let $f \in C^*(\beta X\backslash X)$ be such that $(\mathrm{cl}_{\beta X} D)\backslash D \subseteq Z(f)$. Let g be a function in $C(\beta X)$ such that $g|_{\beta X\backslash X} = f$. Since $(\mathrm{cl}_{\beta X} D)\backslash D \subseteq Z(g)$, then if $D = \{d_i : i \in \mathbf{N}\}$, $\{g(d_i) : i \in \mathbf{N}\}$ converges to zero. For each $i \in \mathbf{N}$, choose a neighborhood V_i of d_i such that the closures in X of the V_i neighborhoods form a pairwise disjoint family of compact sets and $|g(x)-g(d_i)|, 1/i$ for all x in V_i. Let $h : X \to \mathbf{R}$ be a continuous function such that $h[d_i] = 1$ for each $i \in \mathbf{N}$ and $h[X\backslash \cup\{V_i : i \in \mathbf{N}\}] = \{0\}$. (By 9M1 of [**15**] such a function exists). Let h^β denote the extension of h to βX. Since $h^\beta[\mathrm{cl}_{\beta X} D] = \mathrm{cl}_{\mathbf{R}} h[D] = \{1\}$, then $h^\beta[\mathrm{cl}_{\beta X} D\backslash D] = \{1\}$, hence $\mathrm{cl}_{\beta X} D\backslash D \subseteq Cz(h^\beta)$. Since $X\backslash(\cup\{V_i : i \in \mathbf{N}\}) \subseteq Z(h^\beta)$, $\mathrm{cl}_{\beta X}(X\backslash(\cup\{V_i : i \in \mathbf{N}\}) \subseteq Z(h^\beta)$. Let p be an arbitrary point in $(\beta X\backslash X) \cap Cz(h^\beta)$. Then p contains a βX-neighborhood which misses $X\backslash \cup \{V_i : i \in \mathbf{N}\}$. Furthermore, any βX-neighborhood S of p must meet infinitely many V_i's since $\mathrm{cl}_X V_i$ is compact for all i. Suppose $g(p) \neq 0$. Observe that $\lim_{i\to\infty}[\sup\{|g(x)| : x \in V_i\}] = 0$ (since $|g(x) - g(d_i)|, 1/i$ for all x in V_i and $\{g(d_i) : i \in \mathbf{N}\}$ converges to zero). If $g(p) \neq 0$, then there exists an open interval T (in $\mathbf{R}$) containing $g(p)$ such that $\mathrm{cl}_{\mathbf{R}} T$ does not contain the point 0. But $g^{\leftarrow}[T]$ meets infinitely many V_is. Since $\lim_{i\to\infty}[\sup\{|g(x)| : x \in V_i\}] = 0$, the point 0 must belong to $\mathrm{cl}_{\mathbf{R}} T$. Since this is a contradiction, $g(p) = 0 = f(p)$ (since $\gamma|_{\beta X\backslash X} = f$). Hence $p \in Z(f)$. Since p was arbitrarily chosen in $\beta X\backslash X \cap Cz(h^\beta)$, $\beta X\backslash X \cap Cz(h^\beta) \subseteq Z(f)$. Hence $Z(f)$ is a $\beta X\backslash X$-neighborhood of $\mathrm{cl}_{\beta X} D\backslash D$. Thus $\mathrm{cl}_{\beta X} D\backslash D$ is a P-set of $\beta X\backslash X$. $\square$

In 6.6 of [**24**], W.W. Comfort shows (by assuming the continuum hypothesis) that, if βX is a singular compactification, then X must be pseudocompact. In 2.16 we have a generalization of Comfort's result. We prove it in ZFC. (In [**12**] the author also presents a proof of Lemma 2.15 and Theorem 2.16). We begin by proving the following lemma.

Lemma 2.15. *If X contains a C-embedded copy of $\mathbf{N}$, i.e., if X is not pseudocompact, then $\mu X \cong \beta X$.*

Proof. Suppose X contains a C-embedded copy of $\mathbf{N}$. Let x and y be distinct points in $\beta X\backslash X$. We will show that there exists a singular function $t : X \to [0, 1]$ whose extension to βX separates x and y. Let u, p and z be distinct points in $\beta D \cap (\beta X\backslash X)$. If x

belongs to $\beta D\backslash D$, let $u = x$ and if y belongs to $\beta D\backslash D$, let $z = y$. Let U, V and M be pairwise disjoint open subsets of βX such that $u \in U$, $z \in V$ and $p \in M$. Let $f : \beta D \to [0,1]$ be a continuous function such that $f(u) = 0$, $f(z) = 1$ and f is a bijection from $M \cap D$ onto $\mathbf{Q} \cap (0,1)$. Since the subsets U, M and V are pairwise disjoint, the subset $M \cap D$ is infinite and C-embedded in X, and the subset $\{u\} \cup \{z\} \cup \mathrm{cl}_{\beta D}(M \cap D)$ is compact, then such a function exists. Note that $f[\mathrm{cl}_{\beta D}(M \cap D)] = \mathrm{cl}_{\mathbf{R}} f[M \cap D] = [0,1]$. Let $h : \beta D \cup \{x\} \cup \{y\} \to [0,1]$ be defined as follows: $h = f$ on βD; if x does not belong to βD, let $h(x) = 0$, and if y does not belong to βD, let $h(y) = 1$. Observe that $\beta D \cup \{x\} \cup \{y\}$ is C-embedded in βX (since it is compact). Thus, h extends to a function k on βX such that $k|_{\beta D} = f$. Let $t = \mathbf{0} \vee (k|_X \wedge \mathbf{1})$; thus, $t^\beta = \mathbf{0} \vee (k \wedge \mathbf{1})$. Consequently, t^β maps βX onto $[0,1]$. If S is an open subset of $[0,1]$, $\mathrm{cl}_{\beta X} t^{\leftarrow}[S]$ will meet $(\mathrm{cl}_{\beta X} D)\backslash D = \beta D\backslash D$ since $t^\beta|_{M\cap D}$ is a bijection from $M \cap D$ onto $(0,1) \cap \mathbf{Q}$. Hence t is a singular map. Observe that the extension of the singular function t to t^β on βX separates x from y. Thus S_β^β separates the points of $\beta X\backslash X$. By Proposition 2.2, μX is equivalent to βX. This proves the lemma. □

Theorem 2.16. *If X has a largest singular compactification μX, then X does not contain a C-embedded copy of* $\mathbf{N}$*, i.e., X is pseudo-compact.*

Proof. Suppose μX is singular. We will suppose that X contains a C-embedded copy D of $\mathbf{N}$ and show that this leads to a contradiction. If D is a C-embedded copy of $\mathbf{N}$ in X, then, by 6.9 of [**15**], $\mathrm{cl}_{\beta X} D \cap \beta D$. Since D is closed in X, $\beta D\backslash D$ is contained in $\beta X\backslash X$. By Lemma 2.15, μX is equivalent to βX.

Let $r : \beta X \to \beta X\backslash X$ be a retraction from βX onto $\beta X\backslash X$ (the retraction r will exist by Lemma 2.15). The following construction will reveal a contradiction. First note that, since $r[\beta D]$ must be separable and $r[\beta D\backslash D] = \beta D\backslash D$, then $r[D]\backslash r[\beta D\backslash D]$ must contain infinitely many points. By Lemma 2.14, the set $V = (\beta X\backslash X)\backslash(r[D]\backslash(\beta D\backslash D))$ is a neighborhood of $\beta D\backslash D$ in $\beta X\backslash X$. But V contains an open neighborhood W of $\beta D\backslash D$, so $r[D]\backslash W$ has finitely many elements, thus providing a contradiction. □

The converse of Theorem 2.16 fails. In 8.23 of [**24**] it is shown that the product space $[0, \omega_1) \times [0, \omega_1)$ does not have a largest singular compactification even though it clearly does not contain a closed C-embedded copy D of $\mathbf{N}$. Moreover, this illustrates that countably compact spaces need not have a largest singular compactification. On the other hand, the Tychonoff plank $\mathbf{T}$ is not countably compact and yet possesses a largest singular compactification $\beta\mathbf{T} = (\omega T)$ induced by any constant map on $\mathbf{T}$. (Note that $\mathbf{T}$ is almost compact noncompact, hence $\beta\mathbf{T}$ is singular as clearly there is a retraction $r : \beta\mathbf{T} \to \beta\mathbf{T}\backslash\mathbf{T}$.)

The following definition leads us to a useful characterization of pseudocompact spaces.

Definition 2.17. The subset $C^{\#}(X)$ of $C(X)$ is the set of all real-valued functions f such that for every maximal ideal M in $C(X)$ there exists a real number r such that $f - \mathbf{r} \in M$.

The following theorem is an easy consequence of Theorem 5.8 (b) in [**15**].

Theorem 2.18. *The space X is pseudocompact if and only if $C(X) = C^{\#}(X)$.*

Theorem 2.19 [**2**, 1.6]. *The following are equivalent for f in $C^*(X)$*

1) *f belongs to $C^{\#}(X)$.*
2) *For every open subset U of $\beta X f[U \cap X] = f^{\beta}[U]$.*
3) $\mathrm{Cl}_{\beta X} Z(f - \mathbf{r}) = Z(f^{\beta} - \mathbf{r})$ *for any $r \in \mathbf{R}$.*
4) *f maps zero sets to closed sets.*

Lemma 2.20. *If X is a noncompact pseudocompact space and αX is a compactification of X then, for each $f \in S_{\alpha}$, $Z(f)$ is not compact whenever $Z(f^{\alpha})$ is nonempty. Furthermore,* $\mathrm{cl}_{\alpha X} Z(f) = Z(f^{\alpha})$ *for all $f \in C_{\alpha}(X) = \{f|_X : f \in C(\alpha X)\}$.*

Proof. Since X is pseudocompact, then $\mathrm{cl}_{\beta X} Z(f) = Z(f^{\beta})$ for all f in $C^*(X)$ (by Theorems 2.18 and 2.19 and also by 8.8 (b) together with

8A (4) of [**15**]). Let $f \in S_\alpha$. Then $f[X] \subseteq S(f) = f^\alpha[\alpha X \backslash X]$ (2.10). Hence $Z(f^\alpha) \cap (\alpha X \backslash X)$ is nonempty if $Z(f)$ is nonempty. Note that $Z(f^\alpha) = \pi_{\beta\alpha}[Z(f^\beta)] = \pi_{\beta\alpha}[\mathrm{cl}_{\beta X} Z(f)] = \mathrm{cl}_{\alpha X} \pi_{\beta\alpha}[Z(f)] = \mathrm{cl}_{\alpha X} Z(f)$ for all $f \in C_\alpha(X)$. It follows that $Z(f^\alpha) = \mathrm{cl}_{\alpha X} Z(f)$ for all f in $C_\alpha(X)$. Hence $Z(f)$ is not compact if $Z(f^\alpha)$ is nonempty. $\square$

Proposition 2.21. *If X is pseudocompact and $\alpha X = X \cup_f S(f)$ is a singular compactification of X such that $S(f)$ is homeomorphic to a subset of* $\mathbf{R}$*, then $f^{\leftarrow}(x)$ is noncompact for any $x \in S(f)$ and $f[X] = S(f)$.*

Proof. Suppose X is pseudocompact and $\alpha X = X \cup_f S(f)$ is a singular compactification of X such that $S(f)$ is a subset of $\mathbf{R}$. By the lemma above, $Z(f^\alpha) = \mathrm{cl}_{\alpha X} Z(f)$ for all f in $C_\alpha(X)$. Also $Z(f)$ is not compact if $Z(f^\alpha)$ is nonempty. Hence $f^{\leftarrow}(x) = Z(f - \mathbf{x})$ is not compact for any $x \in S(f)$ (since f is a singular real-valued function). By applying Theorem 2.18 and the equivalence of Theorem 2.19 (1) and (4), we also conclude that $f[X] = S(f)$ (since $f[X]$ is dense in $S(f)$.

Suppose $S(f)$ is homeomorphic to a subset K of $\mathbf{R}$. Let $h : S(f) \to K$ be a function which maps $S(f)$ homeomorphically onto K. By the above, $(h \circ f)^{\leftarrow}(x)$ is noncompact for all x in K. Hence $f^{\leftarrow}(y)$ is noncompact for all y in $S(f)$. $\square$

If X is not pseudocompact, then the above proposition may fail as the following example illustrates.

Example 2.22. Let $X^* = [0,1] \times [0,1] \cup \{(-2,0)\}$ viewed as a subspace of the product space $\mathbf{R}^2$. Then X^* is a compactification of the space $X = X^* \backslash ([0,1] \times \{1\})$ and $X^* \backslash X$ is homeomorphic to the closed interval $[0,1]$. Clearly X is not pseudocompact. Let us define the function $r : X^* \to [0,1] \times \{1\}$ as follows: $r((-2,0)) = (0,1)$ and, for $a \in [0,1]$, $r((a,b)) = ((a-1)b+1, 1)$, i.e., r linearly maps the closed interval $\{a\} \times [0,1]$ onto $[a,1] \times \{1\}$ carrying $(a,1)$ to $(a,1)$ and $(a,0)$ to $(1,1)$. Observe that r is a well-defined continuous real-valued function and that r maps any point of $X^* \backslash X$ to itself; hence, $X^* \backslash X$ is a retract of X^* and $r|_X$ is singular. Also note that $(0,1)$

and $(-2,0)$ are the only two points in X^* which are carried to $(0,1)$. Hence, $\mathrm{cl}_{X^*}(X \cap r^{\leftarrow}((0,1)) = \mathrm{cl}_{X^*}\{(-2,0)\} = (-2,0) = r|_X^{\leftarrow}((0,1))$. Thus $r|_X^{\leftarrow}((0,1))$ is compact.

Lemma 2.23. *If $\{f_n : n \in \mathbf{N}\}$ is a sequence of real-valued singular functions which converges uniformly to a function f in $C^*(X)$, then f is also a singular function.*

Proof. Let $x \in X$, $f(x) = r$ and U be an open interval in $\mathbf{R}$ which contains r. Let $\varepsilon > 0$ such that $(r-\varepsilon, r+\varepsilon) \subseteq U$. Since $\{f_n : n \in \mathbf{N}\}$ converges uniformly to f, there exists a number N such that, for all $n \geq N$, $\|f_n - f\| < \varepsilon/3$. It follows that $|f_N(x) - f(x)| < \varepsilon/3$. Let $z = f_n(x)$; then $z \in (r - \varepsilon/3, r + \varepsilon/3)$. Let V be an open neighborhood of z such that $V \subseteq (r-\varepsilon/3, r+\varepsilon/3)$. We claim that $f_n^{\leftarrow}[V] \subseteq f^{\leftarrow}[U]$ for all $n \geq N$. Let $t \in f_m^{\leftarrow}[V]$ for some $m \geq N$. Then $|f_m(t) - f(t)| < \varepsilon/3$; hence, $f(t) \in (r-\varepsilon, r+\varepsilon) \subseteq U$. Thus, $f|f_m^{\leftarrow}[V]] \subseteq U$. Since $f^{\leftarrow}[U] = \{x \in X : f(x) \in U\}$, $f_m^{\leftarrow}[V] \subseteq f^{\leftarrow}[U]$. This establishes the claim. Since $f_N \in S_\beta$, $\mathrm{cl}_X f_N^{\leftarrow}[V]$ is not compact. Hence $\mathrm{cl}_X f^{\leftarrow}[U]$ cannot be compact since $\mathrm{cl}_X f_N^{\leftarrow}[V] \subseteq \mathrm{cl}_X f^{\leftarrow}[U]$ (by the above claim). This implies that f is a singular map. □

In Remark 2.9 of [**1**] the author shows that, for $G \subseteq S_\beta$, "$\omega_G X$ being singular does not imply that e_G is singular." In Theorem 2.26 we show that if G is a subalgebra of $C^*(X)$ which is contained in S_α, then $\omega_G X$ is singular and so is e_G.

First we require the following results from [**1**].

Theorem 2.24 [**1**, 2.11]. *Let αX be a singular compactification of X. Let $r : \alpha X \to \alpha X \backslash X$ be a retraction map, and define F to be $\{f \circ r|_X : f \in C(\alpha X)\}$. Then $F \subseteq S_\alpha$, F is a subalgebra of $C_\alpha(X)$, e_F i a singular map, e_F^α separates points of $\alpha X \backslash X$, and $\alpha X \cong X \cup e_F S(e_F) \cong \omega_F X$.*

In what follows, we will require the following concepts. If B is a collection of functions in $C^*(X)$, a maximal stationary set of B is a subset of X maximal with respect to the property that every f in B is constant on it.

The maximal stationary sets of a subalgebra are briefly discussed in 16.31 of [**15**].

Let $G \subseteq C^*(X)$, x a point in X and $G^+ = \{f - \mathbf{r} : r \in G, r \in \mathbf{R}\}$. The symbol ${}_xK_G$ will denote the set $\cap\{Z(f) : f \in G^+, x \in Z(f)\}$. Thus, $y \in {}_xK_G$ if and only if $f(y) = f(x)$ for each $f \in G$. Suppose αX is a compactification of X such that G (hence G^+) is a subset of $C_\alpha(X)$. For $x \in \alpha X$, let ${}_xK_G^\alpha = \cap\{Z(f^\alpha) : f \in G^+, x \in Z(f^\alpha)\}$. It is clear that the subset ${}_xK_G$ (${}_xK_G^\alpha$) is a maximal stationary set of G (G^α) which contains the point x. It is easily observed that, given $G \subseteq C^*(X)$, the collection $\{{}_xK_G : x \in X\}$ forms a partition of X.

Theorem 2.25 [**1**, 2.12]. *Let αX be a compactification of X. Let G be a subset of S_α such that the evaluation map $e_G^\alpha : \alpha X \to \Pi_{f \in G} S(f)$ separates the points of $\alpha X \backslash X$. Then αX is equivalent to $\omega_G X$. Furthermore, the following are equivalent*:

1) *e_G is a singular map and $\omega_G X (\cong \alpha X)$ is equivalent to the singular compactification $X \cup_{e_G} S(e_G)$.*

2) $e_G[X] \subseteq e_G^{\omega_G}[\omega_G X \backslash X]$.

3) *e_G is a singular map.*

4) *e_F is a singular map for every finite subset F of G.*

5) *${}_xK_G^{\omega_G} \cap (\omega_G X \backslash X)$ is a singleton set for every $x \in X$.*

Theorem 2.26. *A compactification αX of X is singular if and only if S_α contains a subalgebra G of $C^*(X)$ such that G^α separates the points of $\alpha X \backslash X$. Furthermore, if G is a subalgebra of $C^*(X)$ which is contained in S_α such that G^α separates the points of $\alpha X \backslash X$, then e_G is a singular map and $\alpha X \cong \omega_G X \cong X \cup_{e_G} S(e_G)$ (a singular compactification).*

Proof. $\Rightarrow$. Suppose αX is a singular compactification. Then, by Theorem 2.5, αX is equivalent to $\omega_{S_\alpha} X$. By Theorem 2.24, S_α contains a subalgebra G of $C^*(X)$ such that e_G is singular and $\alpha X \cong X \cup_{e_G} S(e_G)$. Since e_{G^α} separates the points of $\alpha X \backslash X$, then so does G^α (by 1.10 of [**1**]).

$\Leftarrow$. Suppose αX is a compactification of X and G is a subalgebra of $C^*(X)$ which is contained in S_α such that G^α separates the points of $\alpha X\backslash X$. To obtain our result we will show that ${}_xK_{G^\alpha} \cap (\alpha X\backslash X)$ is a singleton for each $x \in X$ and then apply the equivalence of Theorem 2.25 (1) and (5).

Let k be a point in X, and let $H = \{Z(f^\alpha) \cap \alpha X\backslash X : f \in G^+, k \in Z(f)\}$. It is easily seen that $\cap H = {}_kK_{G^\alpha} \cap \alpha X\backslash X$. We wish to show that $\cap H$ is nonempty by verifying that H possesses the finite intersection property. Let $M = \{Z(f_i^\alpha) \cap \alpha X\backslash X : i \in F\}$ be a finite subcollection of H. Note that $\cap M = Z(\sum_{i\in F}(f_i^\alpha)^2) \cap (\alpha X\backslash X)$. Since G is a subalgebra of $C^*(X)$ and each f_i belongs to G^+, the function $\sum_{i\in F}(f_i)^2$ belongs to G^+; hence, by 1.16 of [**1**], it belongs to S_α. Thus, by Corollary 2.10, $(\sum_{i\in F} f_i^2)[X] \subseteq (\sum_{i\in F} f_i^2)^\alpha[\alpha X\backslash X] = (\sum_{i\in F}(f_i^2)^\alpha)[\alpha X]$. As $k \in Z(\sum_{i\in F}(f_i^2)^\alpha)$, it follows that $\cap M$ is nonempty. Hence, H has the finite intersection property. Since $\alpha X\backslash X$ is compact, $\cap H = {}_kK_{G^\alpha} \cap (\alpha X\backslash X)$ is nonempty. Since G^α separates the points of $\alpha X\backslash X$, ${}_kK_{G^\alpha} \cap (\alpha X\backslash X)$ is a singleton set in $\alpha X\backslash X$. By Theorem 2.25 (5) implies Theorem 2.25 (1), e_G is a singular map and $\alpha X \cong \omega_G X \cong X \cup_{e_G} S(e_G)$, a singular compactification. □

Suppose αX is a singular compactification and $r : \alpha X \to \alpha X\backslash X$ is a retraction map. It is worth noting that the subalgebra $G = \{f \circ r|_X : f \in C(\alpha X)\}$ (see Theorem 2.24) contains the constant functions, hence $G = G^+$. This follows from the following fact.

Fact. *If $g \in S_\alpha$ is so that g^α is constant on $\alpha X\backslash X$, then g is constant.*

Proof. Since g is singular $g[X] \subseteq g^\alpha[\alpha X\backslash X]$, by Corollary 2.10. Since $g^\alpha[\alpha X\backslash X]$ is a singleton, $g[X]$ is as well. □

Proposition 2.27 [**1**, 1.6]. *If αX is a compactification of X, K is a compact Hausdorff space and $f : X \to K$ is a continuous function which extends to $f^\alpha : \alpha X \to K$ then $f^\alpha[\alpha X\backslash X] = S(f)$.*

Theorem 2.28. *Let αX be a compactification of the space X. There is a one-to-one correspondence between the retraction maps from αX onto $\alpha X\backslash X$, and the subalgebras G of $C_\alpha(X)$ such that $G \subseteq S_\alpha$ and*

$G^\alpha|_{\alpha X\backslash X} = C(\alpha X\backslash X)$. If αX is not a singular compactification, then no such retraction map r or such a subalgebra G exist.

Proof. If αX is not a singular compactification, then there does not exist a retraction map $r : \alpha X \to \alpha X\backslash X$. Also, by Theorem 2.26, S_α does not contain a subalgebra G of $C_\alpha(X)$ such that G^α separates the points of $\alpha X\backslash X$. Hence, $C_\alpha(X)$ does not contain a subalgebra G satisfying the properties described in the statement of the theorem.

Suppose αX is a singular compactification. Then there exists a retraction map $r : \alpha X \to \alpha X\backslash X$ from αX onto $\alpha X\backslash X$. By Theorem 2.24, the family $G = \{f \circ r|_X : f \in C(\alpha X)\}$ i a subalgebra of $C_\alpha(X)$, e_G is a singular map, e_{G^α} separates points of $\alpha X\backslash X$, and $\alpha X \cong X \cup e_G S(e_G)$. Observe that $G^\alpha = \{f \circ r : f \in C(\alpha X\backslash X)\}$ and that $G^\alpha|_{\alpha X\backslash X} = C(\alpha X\backslash X)$ (since $r|_{\alpha X\backslash X}$ is the identity function on $\alpha X\backslash X$). We have shown that we can associate to each retraction map $r : \alpha X \to \alpha X\backslash X$ a subalgebra $G = \{f \circ r|_X : f \in C(\alpha X)\}$ of $C_\alpha(X)$ which is contained in S_α such that $G^\alpha|_{\alpha X\backslash X} = C(\alpha X\backslash X)$.

Let $F_r = \{f \circ r|_X : f \in C(\alpha X)\}$ and $F_s = \{f \circ s|_X : f \in C(\alpha X)\}$, where $r : \alpha X \to \alpha X\backslash X$ and $s : \alpha X \to \alpha X\backslash X$ are retractions. We want to show that if $r \neq$, then $F_r \neq F_s$, i.e., that the map $r| \to F_r$ is one-to-one. If $r \neq s$, there exists $x_0 \in X$ such that $r(x_0) \neq s(x_0)$. As $C(\alpha X)$ separates the points of $\alpha X\backslash X$, there exists $f \in C_\alpha(X)$ such that $f^\alpha(r(x_0)) \neq f^\alpha(s(x_0))$, i.e., $(f^\alpha|_{\alpha X\backslash X} \circ r)(x_0) \neq (f^\alpha|_{\alpha X\backslash X} \circ s)(x_0)$. Now $f^\alpha|_{\alpha X\backslash X} \circ r \in F_r$; we will show that $f^\alpha|_{\alpha X\backslash X} \circ r \notin F_s$, thereby showing that $F_r \neq F_s$. Consequently, if $t \in \alpha X\backslash X$, then $s(t) = r(t) = t$ (as r and s are retractions) and $g^\alpha(t) = g^\alpha(s(t)) = f^\alpha(r(t)) = f^\alpha(t)$. Hence, in particular, $g^\alpha(s(x_0)) = f^\alpha(s(x_0))$. But, by the above, $f^\alpha(s(x_0)) \neq f^\alpha(r(x_0))$. Thus, $(g^\alpha|_{\alpha X\backslash X} \circ s)(x_0) \neq (f^\alpha|_{\alpha X\backslash X} \circ r)(x_0)$, in contradiction to the definition of g. Hence $f^\alpha|_{aX\backslash X} \circ r \notin F_s$, $F_r \neq F_s$, and $r| \to F_r$ is a one-to-one map.

We will now show that, for every subalgebra G of $C_\alpha(X)$ such that $G \subseteq S_\alpha$ and $G^\alpha|_{\alpha X\backslash X} = C(\alpha X\backslash X)$ there exists a retraction map $r : \alpha X \to \alpha X\backslash X$ from αX onto $\alpha X\backslash X$ such that $G = \{f \circ r|_X : f \in C(\alpha X)\}$. Let G be a subalgebra of $C_\alpha(X)$ such that $G \subseteq S_\alpha$ and $G^\alpha|_{\alpha X\backslash X} = C(\alpha X\backslash X)$. We have shown (in Theorem 2.26), that if G is a subalgebra of $C_\alpha(X)$ such that $G \subseteq S_\alpha$ and G^α separates the points of $\alpha X\backslash X$, then e_G is a singular map and $\alpha X \cong X \cup_{e_G} S(e_G)$.

Since G^α separate the points of $\alpha X\backslash X$, then e_{G^α} is one-to-one on $\alpha X\backslash X$; hence, the function $(e_{G^\alpha}|_{\alpha X\backslash X})^{\leftarrow}\circ e_{G^\alpha}:\alpha X\to\alpha X\backslash X$ is a retraction map (since e_G is singular and, by Proposition 2.27, $e_G^\alpha[\alpha X\backslash X]=S(e_G)=e_G^\alpha[\alpha X]$; thus, $(e_G^\alpha|_{\alpha X\backslash X})^{\leftarrow}$ is a well-defined map whose domain is $(e_G^\alpha|_{\alpha X\backslash X})[\alpha X\backslash X]$). We claim that $G^\alpha=\{f^\alpha|_{\alpha X\backslash X}\circ[(e_G^\alpha|_{\alpha X\backslash X})^{\leftarrow}\circ e_G^\alpha]:f\in C_\alpha(X)\}$. We begin by proving that $G^\alpha\subseteq\{f^\alpha|_{\alpha X\backslash X}\circ[(e_G^\alpha|_{\alpha X\backslash X})^{\leftarrow}\circ e_G^\alpha]:f\in C_\alpha(X)\}$. Let $g\in G$ and $x\in X$. Since $g\in G\subseteq S_\alpha$, g extends to a function g^α on αX. Then $e_G^{\alpha\leftarrow}\circ e_G^\alpha(x)$ is a subset of αX which meets $\alpha X\backslash X$ in a singleton set, say $\{y\}$, (since, by Proposition 2.27 $e_G[X]\subseteq e_G^\alpha[\alpha X\backslash X]$ and G^α separates the points of $\alpha X\backslash X$, hence e_{G^α} is one-to-one on $\alpha X\backslash X$). Hence, $e_G^\alpha|_{\alpha X\backslash X}^{\leftarrow}\circ e_G^\alpha(x)=\{y\}$. Observe that $e_G^{\alpha\leftarrow}(e_G^\alpha(x))\subseteq{}^{\alpha\leftarrow}g(g^\alpha(x))$ (since $g^\alpha\in G^\alpha$ and $e_G^{\alpha\leftarrow}(e_G^\alpha(x)=\{y\}$. Observe that $e_{G^{\alpha\leftarrow}}(e_{G^\alpha}(x))\subseteq g^{\alpha\leftarrow}(g^\alpha(x))$ (since $g^\alpha\in G^\alpha$ and $e_G^{\alpha\leftarrow}(e_G^\alpha(x))=\cap\{f^{\alpha\leftarrow}(f(x)):f\in G\}$). Thus, $y\in g^{\alpha\leftarrow}(g(x))$. Therefore $g^\alpha(y)=g^\alpha(x)=g(x)$. We have just shown that $g^\alpha|_{\alpha X\backslash X}([e_G^\alpha|_{\alpha X\backslash X^{\leftarrow}}\circ e_G](x))=g^\alpha(y)=g(x)$ for an arbitrary point x (hence for all x) in X. Thus $g^\alpha=g^\alpha|_{\alpha X\backslash X}\circ[e_G^\alpha|_{\alpha X\backslash X^{\leftarrow}}\circ e_G^\alpha]\in\{f^\alpha|_{\alpha X\backslash X}\circ[(e_G^\alpha|_{\alpha X\backslash X})^{\leftarrow}\circ e_G^\alpha]:f\in C_\alpha(X)\}$. This proves that $G^\alpha\subseteq\{f^\alpha|_{\alpha X\backslash X}\circ[(e_G^\alpha|_{\alpha X\backslash X})^{\leftarrow}\circ e_G^\alpha]:f\in C_\alpha(X)\}$. We now prove $G^\alpha\supseteq\{f^\alpha|_{\alpha X\backslash X}\circ[(e_G^\alpha|_{\alpha X\backslash X})^{\leftarrow}\circ e_G^\alpha]:f\in C_\alpha(X)\}$. Let $k\in\{f^\alpha|_{\alpha X\backslash X}\circ[(e_G^\alpha|_{\alpha X\backslash X})^{\leftarrow}\circ e_G^\alpha]:f\in C_\alpha(X)\}$. Observe that if $t\in C_\alpha(X)$ such that $k=t^\alpha|_{\alpha X\backslash X}\circ[(e_G^\alpha|_{\alpha X\backslash X})^{\leftarrow}\circ e_G^\alpha]$, then $k|_{\alpha X\backslash X}=t^\alpha|_{\alpha X\backslash X}$ on $\alpha X\backslash X$; hence, $k=k|_{\alpha X\backslash X}\circ[(e_G^\alpha|_{\alpha X\backslash X})^{\leftarrow}\circ e_G^\alpha]$. Note that $k|_{\alpha X\backslash X}$ extends to a function $g\in G^\alpha$ (since $k|_{\alpha X\backslash X}\in C(\alpha X\backslash X)$ and, by hypothesis, $G^\alpha|_{\alpha X\backslash X}=C(\alpha X\backslash X)$). Obviously, $k|_{\alpha X\backslash X}=g|_{\alpha X\backslash X}$ on $\alpha X\backslash X$. Let $x\in X$. The argument in the proof above shows that (since $g\in G^\alpha$) $g|_{\alpha X\backslash X}\circ[e_G^\alpha|_{\alpha X\backslash X}^{\leftarrow}\circ e_G](x)=g(x)$ (for all x in X). Hence, $k(x)=k^\alpha|_{\alpha X\backslash X}\circ[e_G^\alpha|_{\alpha X\backslash X}^{\leftarrow}\circ e_G](x)=g|_{\alpha X\backslash X}\circ[e_G^\alpha|_{\alpha X\backslash X}^{\leftarrow}\circ e_G](x)=g(x)$ (for all x in X). Hence, $k=g^\alpha\in G^\alpha$. We have shown that $G^\alpha\supseteq\{f^\alpha|_{\alpha X\backslash X}\circ[(e_G^\alpha|_{\alpha X\backslash X})^{\leftarrow}\circ e_G^\alpha]:f\in C_\alpha(X)\}$. We conclude that $G^\alpha=\{f\circ[(e_G^\alpha|_{\alpha X\backslash X})^{\leftarrow}\circ e_G^\alpha]:f\in C_\alpha(X)\}$. Hence, for every subalgebra G of $C_\alpha(X)$ such that $G\subseteq S_\alpha$ and $G^\alpha|_{\alpha X\backslash X}=C(\alpha X\backslash X)$ there exists a retraction map $r:\alpha X\to\alpha X\backslash X$ from αX onto $\alpha X\backslash X$ (in this case $r=[e_G^\alpha]_{\alpha X\backslash X}^{\leftarrow}\circ e_G^\alpha]$) such that $G=\{f^\alpha|_{\alpha X\backslash X}\circ r|_X:f\in C_\alpha(X)\}$.

We have thus shown that there is a one-to-one correspondence between the retraction maps r from αX onto $\alpha X\backslash X$ and the subalgebras G of $C_\alpha(X)$ such that $G\subseteq S_\alpha$ and $G^\alpha|_{\alpha X\backslash X}=C(\alpha X\backslash X)$. $\square$

Theorem 2.29. *Let αX be a compactification of X. Then the following are equivalent*:

1) *αX is a singular compactification.*

2) *S_α contains a subalgebra G of $C^*(X)$ such that G^α separates the points of $\alpha X \backslash X$.*

3) *S_α contains a closed subalgebra G of $C_\alpha(X)$ such that the mapping $\phi : G \to C(\alpha X \backslash X)$ from G onto $C(\alpha X \backslash X)$ defined by $\phi(f) = f^\alpha|_{\alpha X \backslash X}$ is an isomorphism.*

Furthermore, the subalgebra described in statement 3) *is the closure* (*in $C^*(X)$*) *of the subalgebra described in statement* 2).

Proof. $1 \Leftrightarrow 2$. This is Theorem 2.26.

$2 \Rightarrow 3$. Suppose there exists a subalgebra G of $C^*(X)$ contained in S_α such that G^α separates the points of $\alpha X \backslash X$. As $S_\alpha \subseteq C_\alpha(X)$ clearly $\{f^\alpha|_{\alpha X \backslash X} : f \in S_\alpha\}$ is contained in $C(\alpha X \backslash X)$. As $\alpha X \backslash X$ is compact and G^α separates points of $\alpha X \backslash X$, the collection $G^\alpha|_{\alpha X \backslash X} = \{f^\alpha|_{\alpha X \backslash X} : f \in G\}$ is a subalgebra of $C(\alpha X \backslash X)$ which separates the points and closed sets of $\alpha X \backslash X$. Without loss of generality, we may suppose that G contains the constant functions since, if k is an number and $f \in S_\alpha$, $f + \mathbf{k}$ and $\mathbf{k}f$ are both singular maps. Thus, $G^\alpha|_{\alpha X \backslash X}$ contains the constant functions and separates points and closed sets of $\alpha X \backslash X$.

We claim that $C(\alpha X \backslash X) = (\mathrm{cl}_{C_\alpha(X)} G)^\alpha|_{\alpha X \backslash X}$. By the Stone-Weirstrass theorem, $(\mathrm{cl}_{C_\alpha(X)} G)^\alpha|_{\alpha X \backslash X} \subseteq C(\alpha X \backslash X)$.

Observe that $\mathrm{cl}_{C(\alpha X \backslash X)}(G^\alpha|_{\alpha X \backslash X}) = C(\alpha X \backslash X)$ (again by the Stone-Weirstrass theorem). Hence, it will suffice to show that

$$\mathrm{cl}_{C(\alpha X \backslash X)}(G^\alpha|_{\alpha X \backslash X}) \subseteq (\mathrm{cl}_{C_\alpha(X)} G)^\alpha|_{\alpha X \backslash X}.$$

Let $f \in \mathrm{cl}_{C(\alpha X \backslash X)}(G^\alpha|_{\alpha X \backslash X})$. Then we can construct a sequence $C = \{f_i : i \in \mathbf{N}\}$ in $G^\alpha|_{\alpha X \backslash X}(C(C(\alpha X \backslash X), \|\ \|))$ whose only cluster point is f.

We wish to show that $f \in (\mathrm{cl}_{C_\alpha(X)} G)^\alpha|_{\alpha X \backslash X}$. Now every function f_i in C extends to a function f_i^* in G^α. Let $C^* = \{f_i^* : i \in \mathbf{N}\} \subseteq C(\alpha X)$. Let g be a cluster point of C^*. Then $g \in \mathrm{cl}_{C(\alpha X)} C^* \subseteq \mathrm{cl}_{C(\alpha X)}(G^\alpha)$. We will first show that $g|_{\alpha X \backslash X} = f$. We can construct a sequence

$D = \{f_{ij} : j \in \mathbf{N}\} \subseteq C^*$ whose only cluster point is g. Then, for every $\varepsilon > 0$ there exists a number $N(\varepsilon)$ such that $\|f^*_{ij} - g\| < \varepsilon$ for every $j > N(\varepsilon)$. Thus, $\|f_{ij} - g|_{\alpha X\setminus X}\| < \varepsilon$ for every $j > N(\varepsilon)$. It follows that $g|_{\alpha X\setminus X}$ is a cluster point of C. Since C has only one cluster point, namely f, $g|_{\alpha X\setminus X} = f$.

We will now show that $g|_{\alpha X\setminus X} \in (\mathrm{cl}_{C_\alpha(X)}G)^\alpha|_{\alpha X\setminus X}$. It is easily seen that $g|_X \in \mathrm{cl}_{C_\alpha(X)}\{f^*_i|_x : i \in \mathbf{N}\} \subseteq \mathrm{cl}_{C_\alpha(X)}G$. Hence, $g \in (\mathrm{cl}_{C_\alpha(x)}\{f^*_i|_x : i \in \mathbf{N}\})^\alpha \subseteq (\mathrm{cl}_{C_\alpha(X)}G)^\alpha$. Thus, $g|_{\alpha X\setminus X} \in (\mathrm{cl}_{C_\alpha(X)}G)^\alpha|_{\alpha X\setminus X}$. Since $g|_{\alpha X\setminus X} = f$, $f \in (\mathrm{cl}_{C_\alpha(X)}G)^a|_{\alpha X\setminus X}$. The claim is established, i.e., $C(\alpha X\setminus X) = (\mathrm{cl}_{C_\alpha(X)}G)^\alpha|_{\alpha X\setminus X}$. By Lemma 2.23, $\mathrm{cl}_{C_\alpha(X)}G$ is contained in S_α.

We now define the function $\phi : \mathrm{cl}_{C_\alpha(X)}G \to C(\alpha X\setminus X)$ from $\mathrm{cl}_{C_\alpha(X)}G$ into $C(\alpha X\setminus X)$ as $\phi(f) = f^\alpha|_{\alpha X\setminus X}$. Clearly ϕ is a homomorphism. By the above claim ϕ is onto $C(\alpha X\setminus X)$. We now show that ϕ is one-to-one. Let f and g be two functions in $\mathrm{cl}_{C_\alpha(X)}G$ such that $f^\alpha|_{\alpha X\setminus X} = g^\alpha|_{\alpha X\setminus X}$. Since f and g are both singular maps and $\mathrm{cl}_{C_\alpha(X)}G$ is a subalgebra which is contained in S_α, then $f - g$ is singular. Then $(f^\alpha - g^\alpha)[X] = (f^\alpha - g^\alpha)[\alpha X\setminus X] = (f^\alpha|_{\alpha X\setminus X} - g^\alpha|_{\alpha X\setminus X})[\alpha X\setminus X] = \{0\}$, (2.10). Hence, $f = g$. It follows that the map ϕ is an isomorphism.

$3 \Rightarrow 2$. Suppose S_α contains a closed subalgebra G of $C_\alpha(X)$ such that the mapping $\phi : G \to C(\alpha X\setminus X)$ from G onto $C(\alpha X\setminus X)$ defined by $\phi(f) = f^\alpha|_{\alpha X\setminus X}$ is an isomorphism. Then clearly G^α separate the points of $\alpha X\setminus X$. $\square$

Let $C_\infty(X)$ denote the family of all functions f in $C^*(X)$ for which the set $\{x \in X : |f(x)| \geq 1/n\}$ is compact for all n in $\mathbf{N}$. These functions are said to "vanish at infinity," (see 7FG of [**15**]). It is easily verified that $C_\infty(X)$ is an ideal in the ring $C^*(X)$.

We now know that if αX is a singular compactification of X then S_α contains a closed subalgebra G of $C^*(X)$ such that G^α separates the points of $\alpha X\setminus X$. The following theorem tells us that such a subalgebra G of $C_\alpha(X)$ is isomorphic to the quotient ring $C_\alpha(X)/C_\infty(X)$ under the canonical homomorphism $\sigma : G \to C_\alpha(X)/C_\infty(X)$ defined by $\sigma(f) = C_\infty(X) + f$.

Theorem 2.30. *Let αX be a compactification of X. then αX is a singular compactification of X if and only if $C_\alpha(X)/C_\infty(X)$ is the* *isomorphic image of a closed subring F (of $C_\alpha(X)$) $\subseteq S_\alpha$ under the homomorphism $\sigma : F \to C_\alpha(X)/C_\infty(X)$ defined by $\sigma(f) = C_\infty(X) + f$.*

Proof. $\Rightarrow$. Suppose αX is a singular compactification. Then, by Theorem 2.26, there exists a subalgebra F of $C^*(X)$ which is contained in S_α such that F^α separates the points of $\alpha X \backslash X$ and such that αX is equivalent to $X \cup_{e_F} S(e_F)$. By Theorem 2.29, the homomorphism $\phi : \mathrm{cl}_{C_\alpha(X)} F \to C(\alpha X \backslash X)$ defined by $\phi(f) = f^\alpha|_{\alpha X \backslash X}$ is a ring isomorphism. Let $\tau : C_\alpha(X) \to C(\alpha X \backslash X)$ be the homomorphism from $C_\alpha(X)$ onto $C(\alpha X \backslash X)$ defined by $\tau(f) = f^\alpha|_{\alpha X \backslash X}$. We now define the mapping $\psi : C_\alpha(X) \to \mathrm{cl}_{C_\alpha(X)} F$ as $\psi = \phi^\leftarrow \circ \tau$. (Note that $\psi(f)$ is the unique $g \in \mathrm{cl}_{C_\alpha(X)} F$ for which $g^\alpha|_{\alpha X \backslash X} = f^\alpha|_{\alpha X \backslash X}$). Observe that the kernel of ψ is $\psi^\leftarrow(\mathbf{0}) = (\phi^\leftarrow \circ \tau)^\leftarrow(\mathbf{0}) = \tau^\leftarrow \circ \phi(\mathbf{0}) = \tau^\leftarrow(\mathbf{0}^\alpha|_{\alpha X \backslash X}) = C_\infty(X)$. Hence, by the fundamental theorem of homomorphisms, the function $\zeta : C_\alpha(X)/C_\infty(X) \to \mathrm{cl}_{C_\alpha(X)} F$ defined by $\zeta(C_\infty(X) + f) = \psi(f)$ maps $C_\alpha(X)/C_\infty(X)$ isomorphically onto the image $\mathrm{cl}_{C_\alpha(X)} F$ of $C_\alpha(X)$ under ψ. Observe that, if $g \in \mathrm{cl}_{C_\alpha(X)} F$, then $\psi(g) = \phi^\leftarrow \circ \tau(g) = \phi^\leftarrow(g^\alpha|_{\alpha X \backslash X}) = g$ (since ϕ is one-to-one and onto $C(\alpha X \backslash X)$). Hence, for $g \in \mathrm{cl}_{C_\alpha(X)} F$, $\zeta(C_\infty(X) + g) = \psi(g) = g$. It then follows that the canonical homomorphism $\sigma : \mathrm{cl}_{C_\alpha(X)} F \to C_\alpha(X)/C_\infty(X)$ defined by $\sigma(f) = C_\infty(X) + f$ is onto $C_\alpha(X)/C_\infty(X)$ (since, if $g \in C_\alpha(X)$, then $C_\infty(X) + g = \zeta^\leftarrow(\psi(g)) = \zeta^\leftarrow(\psi(\psi(g)) = C_\infty(X) + \psi(g)$; hence, $\sigma(\psi(g)) = C_\infty(X) + \psi(g) = C_\infty(X) + g$). Hence, the canonical homomorphism σ maps $\mathrm{cl}_{C_\alpha(X)} F$ isomorphically onto $C_\alpha(X)/C_\infty(X)$.

$\Leftarrow$. Suppose now that $C_\alpha(X)/C_\infty(X)$ is the isomorphic image of a closed subring F (of $C_\alpha(X)$) $\subseteq S_\alpha$ under the homomorphism $\sigma : F \to C_\alpha(X)/C_\infty(X)$ defined by $\sigma(f) = C_\infty(X) + f$. We claim that F^α separates the points of $\alpha X \backslash X$. For any $g \in C_\alpha(X)$ there is a function $f \in F$ such that $C_\infty(X) + f = C_\infty(X) + g$ (since σ maps F onto $C_\alpha(X)/C_\infty(X)$). It follows that, for every function g in $C_\alpha(X)$, there is a function f_g in F and a function h_g in $C_\infty(X)$ such that $g = f_g + h_g$. Observe that the function h_g^α is zero on $\alpha X \backslash X$ for each g in $C_\alpha(X)$. Since the collection $\{g^\alpha : g \in C_\alpha(X)\}$ separate the points of $\alpha X \backslash X$, then the subset $\{f_g^\alpha : g \in C_\alpha(X)\}$ of F^α must separate the points of $\alpha X \backslash X$. Then, by Theorem 2.26, αX is a singular compactification. $\square$

Theorem 2.31. *Let αX be a compactification of X. then αX is a singular compactification if and only if $C_\alpha(X) = C_\infty(X) \oplus G$ (the vector space direct sum) for some closed subalgebra G of $C^*(X)$ contained in S_α.*

Proof. $\Rightarrow$. Suppose αX is a singular compactification. We proceed as in the first half of the proof of Theorem 2.30. By Theorem 2.29, S_α contains a closed subalgebra F of $C_\alpha(X)$ such that the mapping $\phi : F \to C(\alpha X \backslash X)$ from F onto $C(\alpha X \backslash X)$ defined by $\phi(f) = f^\alpha|_{\alpha X \backslash X}$ is an isomorphism. Let $\tau : C_\alpha X) \to C(\alpha X \backslash X)$ be the homomorphism from $C_\alpha(X)$ onto $C(\alpha X \backslash X)$ defined by $\tau(f) = f^\alpha|_{\alpha X \backslash X}$. We now define the mapping $\psi : C_\alpha(X) \to F$ as $\psi = \phi^\leftarrow \circ \tau$. The kernel of ψ is $\psi^\leftarrow(\mathbf{0}) = (\phi^\leftarrow \circ \tau)^\leftarrow(\mathbf{0}) = \tau^\leftarrow \circ \phi(\mathbf{0}) = \tau^\leftarrow(\mathbf{0}^\alpha|_{\alpha X \backslash X}) = C_\infty(X)$. Observe that, for every f in $C_\alpha(X)$, $f - \psi(f) = f_\infty$ for some f_∞ in $C_\infty(X)$. Also if $h \in F \cap C_\infty(X)$, then $\tau(h) = h^\alpha|_{\alpha X \backslash X} = \mathbf{0}$ (as $h \in C_\infty(X)$). But $\tau(h) = \phi(h) = h^\alpha|_{\alpha X \backslash X}$. Consequently $\phi(h) = \mathbf{0}$. As ϕ is one-to-one, $\mathbf{h} = \mathbf{0}$. Hence, $F \cap C_\infty(X) = \{0\}$. Thus, $C_\alpha(X) = C_\infty(X) \oplus F$.

$\Leftarrow$. Suppose αX is a compactification of X such that $C_\alpha(X) = C_\infty(X) \oplus G$, where G is a closed subalgebra of $C^*(X)$ which is contained in S_α. Since $f[\alpha X \backslash X] = \{0\}$ for every function f in $C_\infty(X)$, then G^α must separate the points of $\alpha X \backslash X$. It follows that αX is equivalent to $\omega_G X$, (by 2.2), and that $e_{G^\alpha} : \alpha X \to \prod_{f \in G} S(f)$ separates the points of $\alpha X \backslash X$.

Let x be a point in X. Recall that the set ${}_xK_G = \cap\{Z(f) : f \in G^+, x \in Z(f)\}$ is the maximal stationary set of G which contains the point x (see the paragraph preceding Theorem 2.25). Let ${}_xK_{G^\alpha} = \cap\{Z(f^\alpha) : f \in G^+; x \in Z(f)\}$ be the maximal stationary set of G^α which contains the point x. Let $H_x = \{Z(f^\alpha) \cap (\alpha X \backslash X) : f \in G^+, x \in Z(f)\}$. Then $\cap H_x = {}_xK_{G^\alpha} \cap \alpha X \backslash X$.

We wish to show that $\cap H_x$ is a singleton set and then apply $5 \Rightarrow 1$ of Theorem 2.25 to obtain our result. Since g^α separates the points of $\alpha X \backslash X$, it will suffice to show that $\cap H_x$ is nonempty. In fact, since every element of H_x is compact, it will suffice to show that H_x possesses the finite intersection property. Let $M = \{Z(f_i^\alpha) \cap \alpha X \backslash X : i \in F\}$ be a finite subcollection of H_x. Note that $\cap M = Z(\sum_{i \in F}(f_i^\alpha)^2) \cap \alpha X \backslash X$. Since G^+ is a subalgebra, $\sum_{i \in F}(f_i^\alpha)^2$ is an element of $G^+ \subseteq S_\beta$.

Since $(\sum_{i\in F}(f_i^{\alpha})^2)[X] \subseteq [\sum_{i\in F}(f_i^{\alpha})^2][\alpha X\backslash X]$ (by Corollary 2.10), then $Z(\sum_{i\in F}(f_i^{\alpha})^2)\cap \alpha X\backslash X$ is nonempty. Thus, H_x possesses the finite intersection property. It follows that $\cap H_x = {}_xK_{G^\alpha}\cap \alpha X\backslash X$ is a singleton set. By $5 \Rightarrow 1$, e_G is a singular map and $\alpha X (\cong \omega_G X$, by Proposition 2.2) is the singular compactification $X\cup_{e_G} S(e_G)$ induced by e_G. $\square$

Recall that an upward directed partially ordered set $(X, \leq)$ must satisfy the following condition: If a and b are elements of X, then there exists an element c of X such that c is greater than or equal to both a and b. We now present an example of an upward directed family A of singular compactifications whose supremum is not a singular compactification.

Example 2.32. Let ω_1 denote the first uncountable ordinal and $[0,\omega_1)$ be the space of all ordinals less than ω_1. Let $X = [0,\omega_1)\times[0,\omega_1)$ (equipped with the product topology). The space X is pseudocompact (see 8.21 of [**24**]). In 8.23 of [**24**], it is shown that $\beta X = [0,\omega_1]\times[0,\omega_1]$ and that βX is not a singular compactification. We will show that the lattice of all compactifications of X contains a subfamily A of singular compactifications which is totally ordered and whose supremum is βX. Since a totally ordered family is clearly upward directed, we will have shown that an upward directed family of singular compactifications does not necessarily have a supremum which is a singular compactification.

Let λ be a nonlimit ordinal such that λ is less than ω_1. Let $\alpha_\lambda X$ be the decomposition space obtained by collapsing to a point the subset $([\lambda,\omega_1]\times\{\omega_1\})\cup(\{\omega_1\}\times[\lambda,\omega_1])$ of βX and fixing all other points of βX. Clearly $\alpha_\lambda X$ is a compactification of X. Note that, if κ is a nonlimit ordinal such that $\lambda < \kappa < \omega_1$, then $\alpha_\lambda X < \alpha_\kappa X < \beta X$. Hence the family $A = \{\alpha_\kappa X : 0 \leq \kappa < \omega_1, \kappa$ a nonlimit ordinal$\}$ is a totally ordered collection of compactifications of X whose supremum is βX. We now claim that every member of A is a singular compactification. Let $\alpha_\lambda X$ be a member of A. Let us denote by $[\omega_1]$ the point of $\alpha_\lambda X$ which is formed by collapsing to a single point the subset $[\lambda,\omega_1]\times\{\omega_1\}\cup\{\omega_1\}\times[\lambda,\omega_1]$ of βX. If $\kappa < \lambda$, let $F_\kappa = \{\kappa\}\times[0,\omega_1)$, and $H_\kappa = [\lambda,\omega_1)\times\{\kappa\}$ (both subsets of X). Let $K = [\lambda,\omega_1)\times[\lambda,\omega_1)$. Observe that the elements of the collection

$D = \{F_\kappa : \kappa < \lambda\} \cup \{H_\kappa : \kappa < \lambda\} \cup \{K\}$ of subsets are pairwise disjoint. Consider the function $r : \alpha_\lambda X \to \alpha_\lambda X \backslash X$ defined as follows: $r[F_\kappa] = (\kappa, \omega_1)$ if $\kappa < \lambda$, $r[H_\kappa] = (\omega_1, \kappa)$ if $\kappa < \lambda$, and $r[K] = [\omega_1]$. It is easily verified that r is continuous and is a retraction map. Hence, $\alpha_\lambda X$ is a singular compactification. Then A is an upward directed family of singular compactifications whose supremum is βX, a compactification of X which is not singular.

We summarize the main result of this section in the following theorem.

Theorem 2.33. *If X is a locally compact and Hausdorff space, then the following are equivalent*:

1) *The space X has a largest singular compactification, i.e., μX is a singular compactification.*

2) *The set S_μ contains a subalgebra G of $C_\mu(X)$ such that G^μ separates the points of $\mu X \backslash X$.*

3) *The set S_μ contains a closed subalgebra G of $C_\mu(X)$ such that the mapping $\phi : G \to C(\mu X \backslash X)$ from G onto $C(\mu X \backslash X)$ defined by $\phi(f) = f^\mu|_{\mu X \backslash X}$ is an isomorphism.*

4) *The quotient ring $C_\mu(X)/C_\infty(X)$ is the isomorphic image of a closed subring F (of $C_\mu(X)$) $\subseteq S_\mu$ under the homomorphism $\sigma : F \to C_\alpha(X)/C_\infty(X)$ defined by $\sigma(f) = C_\infty(X) + f$.*

5) *The set $C_\mu(X) = C_\infty(X) \oplus G$ (the vector space direct sum) for some closed subalgebra G of $C^*(X)$ contained in S_μ.*

Proof. $1 \Leftrightarrow 2$. This is Theorem 2.26.

$1 \Leftrightarrow 3$. This is Theorem 2.29.

$1 \Leftrightarrow 4$. This is Theorem 2.30.

$1 \Leftrightarrow 5$. This is Theorem 2.31. $\square$

By Theorem 2.16, any one of the above five conditions on μX implies that X is pseudocompact.

Recall that a space X is said to be retractive if $\beta X \backslash X$ is a retract of βX, i.e., βX is a singular compactification. W.W. Comfort has shown using CH that retractive spaces are locally compact and pseudocompact (see 6.6 of [**24**]). A precise characterization of retractive spaces can now be given.

Corollary 2.34. *For a locally compact Hausdorff space X the following are equivalent*:

1) *The space X is retractive, i.e., βX is a singular compactification.*

2) *The set S_β contains a subalgebra G of $C^*(X)$ such that G^β separates the points of $\beta X \backslash X$.*

3) *The set S_β contains a closed subalgebra G of $C^*(X)$ such that the mapping $\phi : G \to C(\beta X \backslash X)$ from G onto $C(\beta X \backslash X)$ defined by $\phi(f) = f^\beta|_{\beta X \backslash X}$ is an isomorphism.*

4) *The quotient ring $C^*(X)/C_\infty(X)$ is the isomorphic image of a closed subring F (of $C_\mu(X)$) $\subseteq S_\mu$ under the homomorphism $\sigma : F \to C^*(X)/C_\infty(X)$ defined by $\sigma(f) = C_\infty(X) + f$.*

5) *The set $C^*(X) = C_\infty(X) \oplus G$ (the vector space direct sum) for some closed subalgebra G of $C^*(X)$ contained in S_β.*

Proof. The equivalence of the statements 1 to 5 follow directly from Theorem 2.33. ◻

In the introductory paragraph of [**7**] the authors make the following conjecture.

Conjecture. *The singular compactifications of a space X forms a lattice if and only if βX is singular.*

We will show that this conjecture fails by constructing a space X whose family of singular compactifications forms a (complete) lattice even though βX is not singular.

Example 2.35. Let Y be a locally compact connected space such that $\beta Y \backslash Y$ is finite and has more than one point. (The space $Y = (\beta \mathbf{R}) \backslash F$ where F is a finite subset of $\beta \mathbf{R} \backslash \mathbf{R}$ is an example of such a space). Let $\mathbf{N}$ denote the natural numbers and $\omega \mathbf{N}$ denote its one-point compactification. Let $X = \omega \mathbf{N} \times Y$ (with the product topology). By 9D 3) of [**15**], Y is pseudocompact. By 8.12 and 8.20 of [**24**], $\beta X = \omega\mathbf{N} \times \beta Y$. We claim that $\mu X = \omega\mathbf{N} \times \omega Y$ (where ωY denotes the one-point compactification of Y). Let u and v be distinct points in $\beta Y\backslash Y$. Let $f \in S_\beta$ and x_0 be a point in $\omega\mathbf{N}$. Then f extends to the function $f^\beta : \beta X \to \mathbf{R}$. Let $x_0 \in \omega\mathbf{N}$ and suppose f^β separates the points (x_0, u) and (x_0, v). Since f is singular, $f^\beta[\beta X] = f^\beta[\beta X\backslash X]$ (by Corollary 2.10) and $f^\beta[\{x_0\} \times \beta Y] \subseteq f^\beta[\beta X\backslash X]$, which is a totally disconnected set (since it is countable). Since f^β separates (x_0, u) and (x_0, v), then $f^\beta[\{x_0\} \times \beta Y]$ is not a singleton, hence is not connected. This contradicts the fact that $f^\beta[\{x_0\} \times \beta Y]$ is connected (being the continuous image of the connected set $\{x_0\} \times \beta Y$). Hence, for any $x \in \omega\mathbf{N}$, every singular function f in S_β has an extension f^β which is constant on $(\mathrm{cl}_{\beta X}(\{x\} \times Y))\backslash(\{x\} \times Y)$. Thus, for each x in $\omega\mathbf{N}$, $(\mathrm{cl}_{\mu X}(\{x\} \times Y))\backslash(\{x\} \times Y)$ is a singleton set, (this follows from the facts that $(\mathrm{cl}_{\mu X}(\{x\} \times Y))\backslash(\{x\} \times Y)$ is either a singleton or contains finitely many elements, and the collection S_{β^μ} separates the points of $\mu X\backslash X$). Let x_0 and y_0 be distinct points in $\omega\mathbf{N}$. Since $\{x_0\} \times \beta Y$ and $\{y_0\} \times \beta Y$ are distinct connected components of βX, then there exists a clopen subset U of βX such that $\{x_0\} \times \beta Y \subseteq U$ and $\{y_0\} \times \beta Y \subseteq \beta X\backslash U$. Let $g : \beta X \to \{0, 1\}$ denote the characteristic function with respect to U. Then the function $g|_X$ is a singular function whose extension to βX separates $\{x_0\} \times \beta Y$ and $\{y_0\} \times \beta Y$. Hence S_β^β separates the connected components $\{\{x\} \times \beta Y : x \in \omega\mathbf{N}\}$ of βX. This implies that μX is the union of the *disjoint* collection $\{\mathrm{cl}_{\mu X}(\{x\} \times \beta Y) : x \in \omega\mathbf{N}\}$. The map r defined by $r[\mathrm{cl}_{\mu X}(\{x\} \times \beta Y)] = \mathrm{cl}_{\mu X}(\{x\} \times \beta Y)\backslash(\{x\} \times \beta Y)$ (where $x \in \omega\mathbf{N}$) is easily seen to be a retraction map from μX onto $\mu X\backslash X$. Thus we conclude that μX is a singular compactification. Since μX is the supremum of all singular compactifications, the collection of all singular compactifications forms a (complete) lattice (see the note following Definition 2.1). Since $(\mathrm{cl}_{\mu X}(\{x\} \times Y))\backslash(\{x\} \times Y)$ is a singleton for each $x \in \omega\mathbf{N}$, then μX is strictly less than βX. Hence βX is not a singular compactification.

In [**5**] the authors wonder whether the following statement is true: "If the set of singular compactifications of a space X forms a lattice, then it forms a complete lattice." The truth or falsity of this statement remains an open question.

We consider a simple problem. In the following example, we show that a subfamily F of the family of all singular compactifications of a space X may form a lattice which is not complete.

Example 2.36. A lattice of singular compactifications of a space X is not necessarily a complete lattice.

Proof. In Example 2.32, we have shown that the family of all singular compactifications of the space $X = [0, \omega_1) \times \{0, \omega_1)$ contains a totally ordered lattice $A = \{\alpha_\kappa X : 0 \leq \kappa < \omega_1, \kappa$ a nonlimit ordinal$\}$ of singular compactifications whose supremum is βX, a compactification which is not singular. $\square$

Observe that the family of all singular compactifications of the space $X = [0, \omega_1) \times \{0, \omega_1)$ does not form a lattice. To see this, let αX be the decomposition space obtained by collapsing to a point the subset $\{\omega_1\} \times [0, \omega_1]$ of $\mu X = [0, \omega_1] \times [0, \omega_1]$ (and fixing all other points). Clearly αX is a compactification of X. Let γX be the decomposition space obtained by collapsing to a point the subset $[0, \omega_1] \times \{\omega_1\}$ of $\beta X = \mu X = [0, \omega_1] \times [0, \omega_1]$ (and fixing all other points). It is easy to verify that both αX and γX are singular compactifications. Note that the supremum of αX and γX is μX, a nonsingular compactification (since $[0, \omega_1] \times [0, \omega_1]$ is not singular).

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