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HEREDITARY PROPERTIES FOR DUALS OF BOCHNER L_p-FUNCTION-SPACES

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ABSTRACT. For a finite and positive measure space (Ω, Σ, μ) hereditary results which hold for Bochner L_p-spaces are derived for their dual spaces. In addition, results of N. Kalton, G. Pisier and N. Randrianantoanina–E. Saab are given alternative proofs.

1. Introduction and preliminaries. There exists a long list on hereditary results for Bochner L_p-spaces, but virtually no equivalent statements are given for their dual spaces. The major problem is the lack of a satisfying representation of their dual spaces, respectively, meaning the second dual of the Bochner L_p-spaces. Let us mention some classical hereditary results concerning embedding classical sequence spaces. S. Kwapien showed that for $1 \leq p < \infty$, $c_0 \subset L_p(\mu, X)$, if and only if it embeds in X [**12**]. J. Mendoza proved that l_∞ can be found as a copy in $L_p(\mu, X)$, $1 \leq p < \infty$ if and only if it is isomorphically embedded in X, see [**14**]. Furthermore, the result that $l_1 \subset L_p(\mu, X)$ for $1 < p < \infty$ if and only if l_1 embeds in X, is due to G. Pisier [**17**]. We will show Pisier's result by different means, which allows the extension to the w^*-measurable case. In [**21**, p. 404] it was shown that some renorming properties of X are inherited by $L_p(\mu, X)$ for $1 \leq p < \infty$, the space of p-integrable vector-valued functions. M. Talagrand demonstrated in [**22**, p. 717] that weak sequential completeness of X passes to the function space $E(X)$, where E is an order continuous Banach lattice with a weak unit not containing c_0. N. Randrianantoanina and E. Saab obtained that $L_p(\mu, X)$, $1 < p < \infty$, enjoys the complete continuity property if and only if X does [**18**, p. 1111]. We will present an alternative proof and extend it to the dual of $L_p(\mu, X)$. The results are based on approximation results for operator-valued functions in Section 2, which may be of independent interest. In addition we extend results of J. Voigt [**23**], respectively the author [**20**], to the w^*-measurable case.

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First we will fix some notations and present some basic facts. Throughout the paper (Ω, Σ, μ) is a positive and finite measure space. X and Y are Banach spaces with duals X^* and Y^*. $B(X)$ is the unit ball while $S(X)$ denotes the unit sphere. $K(X,Y)$ and $W(X,Y)$, respectively $CW(X,Y)$, stands for the compact, weakly compact, respectively conditionally weakly compact operators.

For $A \in \Sigma$, $\mu(A) > 0$ and $f \in L_p(\mu, X)$, respectively $f \in L_p^{w^*}(\mu, X^*)$, $1 \leq p \leq \infty$, define

$$I_A(f) := \int_A f \, d\mu, \qquad N_A(f) := \frac{I_A(f)}{\mu(A)}.$$

$\mathcal{E}(\Omega, \Sigma, X) := \{f : \Omega \to X;\ f = \sum_{i=1}^n x_i \chi_{A_i}, A_1, \dots, A_n \in \Sigma$ pairwise disjoint, $x_i \in X\}$ is the set of the Σ-simple functions with values in X.

Let π be a finite Σ-partition. Then, for $1 \leq p \leq \infty$,

$$\mathbf{E}_\pi : L_p(\mu, X) \longmapsto L_p(\mu, X), \quad f \longrightarrow \sum_{A \in \pi} N_A(f) \chi_A$$

or

$$\mathbf{E}_\pi : L_p^{w^*}(\mu, X^*) \longmapsto L_p^{w^*}(\mu, X^*), \quad f \longrightarrow \sum_{A \in \pi} N_A(f) \chi_A$$

denote the *conditional expectation operator with respect to* π.

Let us collect some basic facts for w^*-measurable functions.

(1.1) Let $f : \Omega \to X^*$ be a function. Then f is called w^*-measurable, if $\Omega \ni \omega \mapsto \langle x, f(\omega) \rangle$ is measurable for all $x \in X$.

Let $1 \leq p < \infty$. Then we define

$$\mathcal{L}_p^{w^*}(\mu, X^*) := \Big\{ f : \Omega \to X^*; f \text{ is } w^*\text{-measurable and } \overline{\int_\Omega} \|f(\omega)\|^p \, d\mu(\omega) < \infty \Big\},$$

where

$$\overline{\int_\Omega} \|f(\omega)\|^p \, d\mu(\omega) := \inf \Big\{ \int_\Omega g(\omega) \, d\mu(\omega);\ g \in \mathcal{L}_1(\mu) \text{ and } \|f(\omega)\|^p \leq g(\omega) \text{ on } \Omega \Big\}.$$

[**10**].

For $p : 1 \le p \le \infty$ we always denote the conjugate of p by p'.

Let $1 \le p < \infty$, and let $f \in \mathcal{L}_p^{w^*}(\mu, X^*)$. We define

$$\|f\|_{p,*} := \sup\left\{ \int_\Omega |\langle f(\omega), g(\omega)\rangle| \, d\mu(\omega); g \in \mathcal{E}(\Omega, \Sigma, X), \|g\|_{p'} \le 1\right\},$$

which is a seminorm on $\mathcal{L}_p^{w^*}(\mu, X^*)$.

If $p = \infty$, then

$$\begin{aligned} \mathcal{L}_\infty^{w^*}(\mu, X^*) := \{f : \Omega \to X^*; f \text{ is } w^*\text{-measurable and} \\ \|f\|_{\infty,*} := \inf\{K > 0; \forall\, x \in B(X) : \\ \mu(\{\omega \in \Omega; |\langle x, f(\omega)\rangle| > K\}) = 0\} < \infty\}. \end{aligned}$$

According to the definition of the norm we have $L_p(\mu, X)^* = L_{p'}^{w^*}(\mu, X^*)$ for $1 \le p < \infty$.

Let $f : \Omega \to X^*$ be w^*-measurable. We define the norm function

$$\|f(\cdot)\| : \Omega \longrightarrow \mathbf{R}, \quad \omega \longmapsto \|f(\omega)\|.$$

If X is separable with a dense sequence $(x_n) \subset B(X)$, then every w^*-measurable function $f : \Omega \to X^*$ has measurable norm function, since $\|f(\omega)\| = \sup_{n\in\mathbf{N}} |\langle x_n, f(\omega)\rangle|$, $\omega \in \Omega$.

Since we are investigating operator-valued functions to deduce the hereditary properties, we have to make the following definitions.

(1.2) Let (Ω, Σ, μ) be a positive measure space. A function $U : \Omega \to L(X, Y)$ is called *strongly measurable* if:

$$\forall\, x \in X : U(\cdot)(x) : \Omega \longrightarrow Y \text{ is } \mu\text{-measurable.}$$

(1.3) Let (Ω, Σ, μ) be a positive measure space. A function $U : \Omega \to L(X, Y^*)$ is w^*-measurable, if

$$\forall\, x \in X : U(\cdot)(x) \text{ is } w^*\text{-measurable.}$$

(1.4) Let $1 \le p \le \infty$, and let $T : X \to L_p(\mu, Y)$, respectively $T : X \to L_p^{w^*}(\mu, Y^*)$, be linear. Then

$U : \Omega \to L(X, Y)$ is a *strongly measurable operator-valued density* of T, respectively $U : \Omega \to L(X, Y^*)$ is a *w^*-measurable operator-valued density* of T, if

$$U \text{ is strongly measurable, respectively } w^*\text{-measurable, and}$$
$$\forall\, x \in X : U(\cdot)(x) = T(x) \quad \text{in } L_p(\mu, Y),$$
$$\text{respectively in } L_p^{w^*}(\mu, Y^*).$$

In this case define $\overline{U} : X \to L_p(\mu, Y)$, respectively $\overline{U} : X \to L_p^{w^*}(\mu, Y^*)$, by $\overline{U}(x) = U(\cdot)(x)$, $x \in X$.

(1.5) Let ρ be a lifting of $\mathcal{L}_\infty := \mathcal{L}_\infty(\mu)$.

Let $U : \Omega \to L(X, Y^*)$ be a function.

i) If $\sup_{\omega\in\Omega} \|U(\omega)\| < \infty$, then we write $\rho(U) = U$, if $\rho(\langle y, U(\omega)(x)\rangle) = \langle y, U(\omega)x\rangle$ for all $\omega \in \Omega$, $x \in X$, $y \in Y$.

ii) If $\sup_{\omega\in\Omega} \|U(\omega)\| \leq \infty$, then we write $\rho[U] = U$, if there exists a partition of Ω, $(A_n) \subset \Sigma$, such that for all $n \in N$, $\omega \in \Omega$, $x \in X$, $y \in Y : \rho(\langle y, U(\omega)(x)\rangle \chi_{A_n}) = \langle y, U(\omega)(x)\rangle \rho(\chi_{A_n})$.

An application of a result of N. Dinculeanu [**8**, p. 269] gives the following fundamental theorem for the lifting.

Theorem 1.6. *Let ρ be a lifting of $\mathcal{L}_\infty$. Then there exists a map $\tau : L_1^{w^*}(\mu, X^*) \to \mathcal{L}_1^{w^*}(\mu, X^*)$, which has the following properties.*

a) $\tau(f) \in f$, $\rho[\tau(f)] = \tau(f)$, $f \in L_1^{w^*}(\mu, X^*)$.

b) $\|\tau(f)(\cdot)\| \in \mathcal{L}_1(\mu)$ *and* $\|f\|_{1,*} = \int_\Omega \|\tau(f)(\omega)\|\, d\mu(\omega)$ *for* $f \in L_1^{w^*}(\mu, X^*)$.

c) *For all $\alpha, \beta \in \mathbf{R}$, for all $f_1, f_2 \in L_1^{w^*}(\mu, X^*)$:*

$$\tau(\alpha f_1 + \beta f_2)(\omega) = \alpha\tau(f_1)(\omega) + \beta\tau(f_2)(\omega)$$
$$\textit{for almost all } \omega \in \Omega.$$

In the sequel we will write f^ρ instead of $\tau(f)$, where ρ is a lifting of $\mathcal{L}_\infty$.

Let us state some well-known facts for Bochner-integrable functions in the w^*-measurable setting. The proofs are straightforward applications of the classical results.

(1.7) Extended Lebesgue domination theorem. Let $(f_n) \subset L_1^{w^*}(\mu, X^*)$. Let $\hat{f} : \Omega \to X^*$ be a function and $g \in \mathcal{L}_1(\mu)$ such that

i) $f_n^\rho(\omega) \to \hat{f}(\omega)$ for almost all $\omega \in \Omega$,

ii) for all $n \in \mathbf{N}$, $\|f_n^\rho(\omega)\| \le g(\omega)$ for almost all $\omega \in \Omega$. Then

$$\hat{f} \in \mathcal{L}_1^{w^*}(\mu, X^*) \quad \text{and} \quad \|f_n - f\|_{1,*} \longrightarrow 0,$$

where f is the equivalence class of $\hat{f}$.

(1.8) Extended Egorov theorem. Let $(f_n) \subset L_1^{w^*}(\mu, X^*)$. Let $f : \Omega \to X^*$ be a function such that $f_n^\rho \to \hat{f}$ pointwise almost everywhere. Then

a) $\hat{f}$ is w^*-measurable and $\|\hat{f}(\cdot)\|$ is μ-measurable,

b) for all $\varepsilon > 0$, there exists $\Omega_\varepsilon \in \Sigma$, $\mu(\Omega \backslash \Omega_\varepsilon) < \varepsilon$: $\sup_{w \in \Omega_\varepsilon} \|f_n^\rho(\omega) - \hat{f}(\omega)\| < \varepsilon$.

(1.9) Let $(f_n) \subset L_1^{w^*}(\mu, X^*)$, $f \in L_1^{w^*}(\mu, X^*)$, such that $\|f - f_n\|_{1,*} \to 0$. Then there is a subsequence (f_{n_k}) such that $f_{n_k}^\rho \to f^\rho$ pointwise almost everywhere.

As an analogon to J. Diestel, W. Ruess and W. Schachermayer [**6**], we present a result for the weak compactness of w^*-measurable functions. We will sketch the proof, since it uses mainly ideas of [**6**, p. 448]. But we also have to refer to results of M. Talagrand [**22**, p. 720], since a representation of the duals of $L_p^{w^*}(\mu, X^*)$ as function spaces is not at hand and, thus, the proof of [**6**, p. 448] cannot be transferred literally.

Theorem 1.10. *Let $1 \le p < \infty$, and let $K \subset L_p^{w^*}(\mu, X^*)$ be bounded. Let ρ be a lifting of $\mathcal{L}_\infty$. Then the following conditions are equivalent*:

a) *K is relatively weakly compact.*

b) i) *K is uniformly integrable.*

ii) *For all $(f_n) \subset K$ there exists (g_n), $g_n \in \mathrm{co}\,(f_n, f_{n+1}, \dots)$, such that $(g_n^\rho(\omega))$ is converging in norm for almost all $\omega \in \Omega$.*

c) i) *K is uniformly integrable.*

ii) *For all $(f_n) \subset K$ there exists (g_n), $g_n \in \mathrm{co}\,(f_n, f_{n+1}, \dots)$, such that $(g_n^\rho(\omega))$ is converging weakly for almost all $\omega \in \Omega$.*

If $p > 1$*, then* i) *is superfluous in* b) *and* c).

Proof. a) $\Rightarrow$ b). Copy the proof in [**6**] and use (1.9).

b) $\Rightarrow$ c). Trivial.

c) $\Rightarrow$ b). First start with the arguments as presented in [**6**, p. 448] to reduce the problem to a set K uniformly bounded in $L_\infty^{w^*}(\mu, X^*)$. Then the assertion follows by results of Talagrand's [**22**, p. 720]. □

2. Approximation of w^*-measurable operator-valued functions.

The following basic result is taken from [**19**, p. 380].

Theorem 2.1. *Every bounded operator* $T : X \to L_\infty^{w^*}(\mu, Y^*)$ *admits of a* w^**-measurable density* $U : \Omega \to L(X, Y^*)$*.*

In particular, every bounded operator $T : X \to L_\infty(\mu, Y)$ *has a strongly measurable density* $U : \Omega \to L(X, Y)$*.*

Remark 2.2. Let ω_1 be the first uncountable ordinal. Let $(\phi_\alpha)_{0 \le \alpha < \omega_1}, \phi_\alpha \colon \Omega \to \mathbf{R}$ be a family of measurable functions such that

$$\forall\, \alpha, \beta, \quad \alpha < \beta, \quad \forall\, \omega \in \Omega : \phi_\alpha(\omega) \le \phi_\beta(\omega).$$

Then there exists a countable ordinal β_0, such that

$$\forall\, \beta \ge \beta_0 : \phi_\beta = \phi_{\beta_0} \quad \text{a.e. on } \Omega.$$

A straightforward consequence of the foregoing remark is the following proposition.

Proposition 2.3. *Let* $1 \le p \le \infty$*, and let* ρ *be a lifting of* $\mathcal{L}_\infty$*.*

a) *Let* $T : X \to L_p(\mu, Y)$ *be linear and bounded.* T *has a strongly measurable density if and only if* $T|_{X_0}$ *has a strongly measurable density* $U : \Omega \to L(X_0, Y)$ *for all separable subspaces* $X_0 \subset X$*.*

b) *Let* $T : X \to L_p^{w^*}(\mu, Y^*)$ *be bounded and linear.* T *has a* w^**-measurable density if and only if* $T|_{X_0}$ *has a* w^**-measurable density* $U : \Omega \to L(X_0, Y^*)$ *for all separable subspaces* $X_0 \subset X$*.*

Theorem 2.4. *Let ρ be a lifting of $\mathcal{L}_\infty$. Let (Θ, Ψ, ν) be a finite and positive measure space, and let $T : L_1(\nu) \to L_1^{w^*}(\mu, X^*)$ be bounded. Then there exists a w^*-measurable density $U : \Omega \to L(L_\infty, X^*)$ of $T|_{L_\infty}$ such that $\rho[U] = U$. If, in addition, $l_\infty \not\subset X^*$, then $U : \Omega \to W(L_\infty, X^*)$.*

Proof. We assume that (Θ, Ψ, ν) is the Lebesgue measure space. Thus, let $(\pi_n)_{n\in\mathbf{N}}$ be the sequence of dyadic partitions of the interval $[0,1]$. Then we defined for $n \in \mathbf{N}$ and $\tau = (\tau_i) \in \{-1,1\}^{2^n} : f_{n,\tau} := \sum_{i=1}^{2^n} \tau_i \chi_{A_i}$, where $A_i \in \pi_n$, $i = 1, \dots, 2^n$. We define for $n \in \mathbf{N}$, $K > 0$,

$$A_{K,n} := \{\omega \in \Omega; \exists \tau \in \{-1,1\}^{2^n} \|(T(f_{n,\tau}))^\rho(\omega)\| > K\}.$$

According to (1.2) $(T(f_{n,\tau}))^\rho$ has a measurable norm function. Thus, the sets $A_{K,n}$ are measurable.

$$\sup_{n\in\mathbf{N}} \mu(A_{K,n}) \overset{K\to\infty}{\longrightarrow} 0. \tag{1}$$

For the proof first we note that $A_{K,n} \subset A_{K,n+1}$ for all $K > 0$. Let $n \in \mathbf{N}$ and $K > 0$. We define $\sigma : A_{K,n} \to \{-1,1\}^{2^n}$ as follows. Let $\{\tau_1, \dots, \tau_m\}$ be a counting of the set $\{-1,1\}^{2^n}$ with $m = 2^{2^n}$. For $1 \le j \le m$ we define $A_{K,n,j} := \{\omega \in A_{K,n} : \|T(f_{n,\tau_j})^\rho(\omega)\| > K\}$ and have $A_{K,n} = \cup_{j=1}^m A_{K,n,j}$ by definition. Let $C_1 := A_{K,n,1}$ and $C_j := A_{K,n,j} \setminus \cup_{i=1}^{j-1} C_i$ for $j = 2, \dots, m$. Evidently $(C_j)_{j=1}^m$ is a Σ-partition of $A_{K,n}$. Now we define

$$\sigma : A_{K,n} \longrightarrow \{-1,1\}^{2^n}, \quad \omega \longmapsto \tau_j,$$

where $\omega \in C_j$. The map $\omega \mapsto T(f_{n\sigma(\omega)})^\rho$ is measurable, since $(C_j) \subset \Sigma$.

Then we estimate

$$\begin{aligned}\mu(A_{K,n})K &\leq \int_{A_{K,n}} \|(T(f_{n,\sigma(\omega)}))^\rho(\omega)\|\, d\mu(\omega) \\ &= \int_{A_{K,n}} \left\| T\left(\sum_{i=1}^{2^n} \sigma(\omega)_i \chi_{A_i}\right)^\rho(\omega)\right\| d\mu(\omega) \quad (1.6c) \\ &= \int_{A_{K,n}} \left\| \sum_{i=1}^{2^n} \sigma(\omega)_i T(\chi_{A_i})^\rho(\omega)\right\| d\mu(\omega) \\ &\leq \int_{A_{K,n}} \sum_{i=1}^{2^n} \|\sigma(\omega)_i T(\chi_{A_i})^\rho(\omega)\|\, d\mu(\omega) \\ &= \int_{A_{K,n}} \sum_{i=1}^{2^n} \|T(\chi_{A_i})^\rho(\omega)\|\, d\mu(\omega) \\ &= \sum_{i=1}^{2^n} \int_{A_{K,n}} (\|T(\chi_{A_i})^\rho(\omega)\|)\, d\mu(\omega) \\ &\leq \sum_{i=1}^{2^n} (\|T\|(\lambda(A_i))) = \|T\|\lambda([0,1]) = \|T\|.\end{aligned}$$

This finishes the proof for (1).

Since $B(L_\infty)$ is weakly compact in L_1, $T|_{L_\infty}$ is weakly compact, thus uniformly integrable (see Theorem 1.10). Hence, for a given $\varepsilon > 0$, there is a $\delta > 0$ such that, for all $A \in \Sigma$, $\mu(A) < \delta$,

$$\sup_{g \in B(L_\infty)} \int_A \|(T(g))^\rho\|\, d\mu < \varepsilon. \tag{2}$$

Hence, by (1), there exists a $K > 0$ such that $\mu(\cup_{n\in\mathbf{N}} A_{K,n}) \leq \delta$. Again, let $\Omega_\varepsilon := \Omega \backslash (\cup_{n\in\mathbf{N}} A_{K,n})$. We have $\|(T(f_{n,\tau}))^\rho \chi_{\Omega_\varepsilon}\|_{\infty,*} \leq K$, $n \in \mathbf{N}$, $\tau \in \{-1,1\}^{2^n}$. Let $g \in B(L_\infty)$. There exists a sequence (g_k), $g_k \in \operatorname{co}\{f_{n,\tau} : n \in \mathbf{N}, \tau \in \{-1,1\}^{2^n}\}$, such that $g_k \to g$ in L_1. By the $\|\cdot\|_1 - \|\cdot\|_{1,*}$-continuity of T, $\|T(g_k) - T(g)\|_{1,*} \to 0$. By (1.9), there is a subsequence such that $(T(g_{k_j}))^\rho \to (T(g))^\rho$ pointwise almost everywhere. Hence,

$$\|(T(g))^\rho \chi_{\Omega_\varepsilon}\|_{\infty,*} \leq K. \tag{3}$$

So we are able to define

$$\tilde{T} : L_\infty \longrightarrow L_\infty^{w^*}(\mu, X^*), \quad \tilde{T}(g) := T(g)^\rho \chi_{\Omega_\varepsilon}.$$

From (2) and (3) we obtain $\|\tilde{T}\| \leq K$ and $\|T - \tilde{T}\|_{L(L_\infty, L_1^{w^*}(\mu, X^*))} < \varepsilon$. The existence of an operator-valued density of $\tilde{T}$ follows from Theorem 2.1. Since $\varepsilon > 0$ is arbitrary, we find a strongly measurable density on $\cup_{n\in\mathbf{N}}\Omega_{1/n} = \Omega$ up to a set of measure 0. If (Θ, Ψ, ν) is an arbitrary, finite, separable and positive measure space, then we can split Ω in a purely atomic and a purely nonatomic part. According to the Carathéodory representation, the purely nonatomic part is measure isomorphic to $([0,1], \mathcal{L}, \lambda)$, which is treated before, while its purely atomic part is done by a similar estimation as in (1).

The general case is done according to Proposition 2.3b). Let $L \subset L_\infty(\nu)$ be separable. Then there is a separable sub-σ-algebra $\Psi_0 \subset \Psi$ so that $L \subset L_\infty(\nu|_{\Psi_0})$. Thus, according to the separable case above, there is a strongly measurable density $U : \Omega \to L(L_1(\nu|_{\Psi_0}), X)$. Proposition 2.3b) finishes the proof to the general case.

[**8**, p. 215] tells us that we may assume $\rho[U] = U$, see 1.14 for a definition. If $l_\infty \not\subset X^*$, by a result of H. Rosenthal, see [**7**, p. 156], we get the w^*-measurable operator-valued density $U : \Omega \to W(L_\infty, X^*)$. $\square$

Theorem 2.5. *Let $l_\infty \not\subset X^*$, and let $T : L_1 \to L_1^{w^*}(\mu, X^*)$ be a bounded operator. Let ρ be a lifting of $\mathcal{L}_\infty$. Then $T(B(L_1))$ is uniformly integrable if and only if for all $\varepsilon > 0$ there is a w^*-measurable $U_\varepsilon : \Omega \to L(L_1, X^*)$ such that $\|T - U_\varepsilon\|_{L(L_1, L_1^{w^*}(\mu, X^*))} < \varepsilon$, $U_\varepsilon : L_1 \to L_\infty^{w^*}(\mu, X^*)$ and $\rho(U_\varepsilon) = U_\varepsilon$.*

Proof. Let $T(B(L_1))$ be uniformly integrable. Let ρ be a lifting of $\mathcal{L}_\infty$. According to Theorem 2.4 there exists a w^*-measurable function $U : \Omega \to W(L_\infty, X^*)$, such that $T|_{L_\infty} = \overline{U}$ and $\rho[U] = U$.

Let $\varepsilon > 0$ be given. Since T is bounded and $T(B(L_1))$ is uniformly integrable, there exists an $M > 0$ such that

$$\sup_{f \in B(L_1)} \|T(f)^\rho - T(f)^\rho \chi_{\{\|T(f)^\rho\| \leq M\}}\|_{1,*} < \varepsilon.$$

For all $A \in \mathcal{L}$, let $B_A := \{\omega \in \Omega; \|T(\chi_A/\lambda(A))^\rho(\omega)\| \leq M\}$.

Let $(\pi_n)_{n\in\mathbf{N}}$ be the sequence of the dyadic partitions of the interval $[0,1]$. We define the operator

$$\begin{aligned} T_n : L_1 &\longrightarrow L_\infty^{w^*}(\mu, X^*) \\ f &\longmapsto \sum_{A\in\pi_n} \Big(\int_A f\,d\lambda\Big)\overline{U}\Big(\frac{\chi_A}{\lambda(A)}\Big)\chi_{B_A} \\ &= \sum_{A\in\pi_n} \Big(\int_A f\,d\lambda\Big)T\Big(\frac{\chi_A}{\lambda(A)}\Big)\chi_{B_A} \end{aligned}$$

and obtain

$$\|T_n(f)\|_{\infty,*} \le \sum_{A\in\pi_n} \Big|\int_A f\,d\lambda\Big| \Big\|T\Big(\frac{\chi_A}{\lambda(A)}\Big)\chi_{B_A}\Big\|_{\infty,*} \le \|f\|_1 M.$$

Next we consider for $n, m \in \mathbf{N}$, $n \le m$ a set $D \in \sigma(\pi_n)$. We have

$$\begin{aligned} &\Big\|T\Big(\frac{\chi_D}{\lambda(D)}\Big)^\rho - T_m\Big(\frac{\chi_D}{\lambda(D)}\Big)^\rho\Big\|_{1,*} \\ &\qquad = \Big\|\sum_{\substack{A\in\pi_n,\\ A\subset D}} \Big(\frac{\lambda(A)}{\lambda(D)}T\Big(\frac{\chi_A}{\lambda(A)}\Big)^\rho - \frac{\lambda(A)}{\lambda(D)}T\Big(\frac{\chi_A}{\lambda(A)}\Big)^\rho\chi_{B_A}\Big)\Big\|_{1,*} \\ &\qquad \le \sum_{\substack{A\in\pi_n,\\ A\subset D}} \frac{\lambda(A)}{\lambda(D)}\Big\|T\Big(\frac{\chi_A}{\lambda(A)}\Big)^\rho - T\Big(\frac{\chi_A}{\lambda(A)}\Big)^\rho\chi_{B_A}\Big\|_{1,*} < \varepsilon. \end{aligned}$$

Hence, for all $f \in \operatorname{aco}(\chi_A/\lambda(A), A \in \pi_n)$ and $m \ge n$ we have $\|(T(f))^\rho - T_m(f)^\rho\|_{1,*} < \varepsilon$. Thus we can conclude that for all $m \ge n$ and for each $f \in B(L_1)$, $f = \mathbf{E}_{\pi_n} f$:

$$\|T(f) - T_m(f)\|_{1,*} < \varepsilon.$$

Let $f \in B(L_\infty)$. Then for almost every $\omega \in \Omega$,

$$((T_n(f))^\rho(\omega))_{n\in\mathbf{N}} \subset X^* \quad \text{is relatively weakly compact.} \tag{4}$$

For the proof of (4) we realize first that, for $n \in \mathbf{N}$ and $A \in \pi_n$: $|\int_A f\,d\lambda| \le \lambda(A)$,

$$\begin{aligned}(T_n(f))^\rho(\omega) &\overset{\rho[U]=U}{=} \sum_{A\in\pi_n}\left(\int_A f\,d\lambda\right)U(\omega)\left(\frac{\chi_A}{\lambda(A)}\right)\chi_{B_A}(\omega)\\ &= \sum_{A\in\pi_n} N_A(f)U(\omega)(\chi_A)\chi_{B_A}(\omega)\\ &= \sum_{\substack{A\in\pi_n,\\ \omega\in B_A}} N_A(f)U(\omega)(\chi_A)\\ &= U(\omega)\left(\sum_{\substack{A\in\pi_n,\\ \omega\in B_A}} N_A(f)\chi_A\right)\\ &\in U(\omega)(B(L_\infty)) \quad \text{since } |N_A(f)|\le 1.\end{aligned}$$

Since $U(\omega)(B(L_\infty))$ is weakly compact in X^*, by Theorem 1.10 we have for all $f \in L_\infty$ that $(T_n(f))_{n\in\mathbf{N}}$ is relatively weakly compact in $L_1^{w^*}(\mu, X^*)$.

Since L_1 is separable, there is a subsequence T_{n_k} such that $T_{n_k}(f_j) \overset{k\to\infty}{\to} \hat{T}(f_j)$ weakly in $L_1^{w^*}(\mu, X^*)$ for all $j \in \mathbf{N}$ where $\{f_1, f_2, \dots\} \subset L_\infty$ is an L_1-dense subset. Thus $\hat{T}$ is linear on L_∞. The uniform estimate for the sequence (T_n) shows that $\hat{T}$ is a $\|\cdot\|_1 - \|\cdot\|_{1,*}$ continuous operator on an L_1-dense, linear subspace of L_∞, since this holds for all T_{n_k}, $k \in \mathbf{N}$.

Hence there is an extension to a linear operator $\hat{T} : L_1 \to L_1^{w^*}(\mu, X^*)$ such that $T_{n_k}(f) \overset{k\to\infty}{\to} \hat{T}(f)$ weakly for all $f \in L_1$. Because $\|T_{n_k}(f)\|_{\infty,*} \le M$ for all $f \in B(L_1)$ and $k \in \mathbf{N}$, it follows that $\|\hat{T}(f)\|_{\infty,*} \le M$. Furthermore, $\|T_{n_k}(f) - T(f)\|_{1,*} < \varepsilon$ for all $f \in B(L_1)$, $f = \mathbf{E}_{\pi_n} f$, $n \in \mathbf{N}$ and $k \in \mathbf{N}$, such that $n_k \ge n$. Thus we have

$$\hat{T} : L_1 \longrightarrow L_\infty^{w^*}(\mu, X^*),$$
$$\text{with } \|\hat{T} - T\|_{L(L_1, L_1^{w^*}(\mu, X^*))} < \varepsilon.$$

By Theorem 2.1 there exists a map $U_\varepsilon : \Omega \to L(L_1, X^*)$ with $\hat{T} = \overline{U}_\varepsilon$. With [**8**, p. 215], we may assume $\rho(U_\varepsilon) = U_\varepsilon$.

The uniform integrability is provided by the fact that for all $\varepsilon > 0$ the image $\overline{U}_\varepsilon(B(L_1)) \subset L_1^{w^*}(\mu, X^*)$ is L_∞-bounded, hence uniformly integrable. The approximation gives the assertion and finishes the proof of the theorem. □

We state the same results for the Bochner function spaces, where the proofs are omitted, since they run parallel and turn out to be easier.

An operator $T : X \to L_p(\mu, Y)$ is called *σ_p^*-compact* if $T(B(X))$ is relatively $\sigma(L_p(\mu, Y), L_{p'}(\mu, Y^*))$-compact.

We note that the $\sigma(L_p(\mu, Y), L_{p'}(\mu, Y^*))$-topology coincides with the weak topology if and only if Y^* has the RNP with respect to (Ω, Σ, μ), see [**7**, p. 98].

Theorem 2.6. *Let K be a compact totally disconnected Hausdorff space. Let $1 \leq p < \infty$ and $T : C(K) \to L_p(\mu, Y)$ be bounded. We consider the following properties*:

a) *There exists a subspace $Y_0 \subset Y$, $l_\infty \not\subset Y_0$, such that $T(C(K)) \subset L_p(\mu, Y_0)$.*

b) *T is σ^*-compact.*

c) *T is weakly compact.*

Then a) $\Rightarrow$ b) $\Leftrightarrow$ c). *If (Ω, Σ, μ) is separable, then* a) *and* b) *are equivalent. In particular, l_∞ embeds in $L_p(\mu, Y)$ if and only if it embeds in Y.*

Proof. a) $\Rightarrow$ b). Since $l_\infty \not\subset Y_0$ and $T(C(K)) \subset L_p(\mu, Y_0)$, we can use a result of J. Batt and W. Hiermeyer [**1**, p. 411] and a result of H. Rosenthal [**7**, p. 156] to conclude that T is σ_p^*-compact.

b) $\Rightarrow$ c). Suppose T is not weakly compact; then, by a result of H. Rosenthal, see [**7**, p. 156], there exists a copy of $l_\infty \subset T(C(K))$. According to b), $T(C(K)) \subset L$ where $L := \overline{\operatorname{span}}\, M$ with M σ_p^*-compact. But this contradicts a result in [**21**, p. 409] which says that L admits an equivalent locally uniformly convex norm. But this fails for l_∞ [**5**, p. 120].

c) $\Rightarrow$ b). Trivial.

b) $\Rightarrow$ a). If (Ω, Σ, μ) is separable, there exists a countable set $\mathcal{C} \subset \Sigma$ such that the generated σ-algebra of $\mathcal{C}$ is dense in Σ and for all $\varepsilon > 0$ the set $\{A \in \mathcal{C}; \mu(A) > \varepsilon\}$ is finite. Let $K := T(B(C(K))$. Then the uniform integrability of K [**1**, p. 411] implies that $\{A \in \mathcal{C}; \sup_{f \in T(B(C(K))} \|I_A(f)\| \geq \varepsilon\}$ is finite for all $\varepsilon > 0$. For all $A \in \mathcal{C}$ the set $I_A(K)$ is relatively weakly compact in Y [**1**, p. 411]. We define $D := \cup_{A \in \mathcal{C}} I_A(K)$. By the above D is relatively weakly compact in Y. Let $Y_0 := \overline{\operatorname{span}}\, D$. Then Y_0 is weakly compactly generated and hence does not contain a copy of l_∞. Since the algebra generated by $\mathcal{C}$ is dense in Σ, we have $T(C(K)) \subset L_1(\mu, Y_0)$. □

The technique presented in Theorem 2.4, respectively Theorem 2.5, can be transmitted to the strongly measurable case almost literally. Hence we get, using Theorem 2.6 for Theorem 2.7,

Theorem 2.7. *Let (Θ, Ψ, ν) be a finite and positive measure space and $T : L_1(\nu) \to L_1(\mu, X)$ be bounded. Then $T|_{L_\infty}$ has a strongly measurable density $U : \Omega \to L(L_\infty, X)$. If (Θ, Ψ, ν) is separable, then $U : \Omega \to W(L_\infty, X)$.*

The previous theorem should be compared with [**11**, p. 314].

Theorem 2.8. *Let $T : L_1 \to L_1(\mu, X)$ be a bounded operator. Then $T(B(L_1))$ is uniformly integrable if and only if for all $\varepsilon > 0$ there is a strongly measurable map $U_\varepsilon : \Omega \to L(L_1, X)$ such that $\|T - U_\varepsilon\|_{L(L_1, L_1(\mu, X))} < \varepsilon$ and $\overline{U}_\varepsilon : L_1 \to L_\infty(\mu, X)$.*

3. Hereditary properties for w^*-measurable function spaces. If X has the RNP, then by [**7**, p. 66], every bounded, linear operator $T : L_1([0,1]) \to X$ is Dunford-Pettis, i.e., it maps weakly convergent sequences to norm convergent sequences. Thus the following condition is weaker than the RNP.

(3.1) A convex, bounded and closed set $K \subset X$ has the *complete continuity property* (CCP) if every bounded, linear operator $T : L_1([0,1]) \to X$, such that $\{T(\chi_A/\lambda(A)); A \in \mathcal{L}, \lambda(A) > 0\} \subset K$ is a Dunford Pettis operator, see [**2, 9**] for more details.

A Banach space X has the CCP, if $B(X)$ enjoys the CCP, see [**9**, p. 59; p. 78].

The following results are an extension of the result of J. Voigt that the strong integral of compact operators is again compact [**23**].

Theorem 3.2. *Let* $l_1 \not\subset X$ *and let* $U : \Omega \to K(X, Y^*)$ *be* w^**-measurable and bounded. Then for* $1 \leq p < \infty$*,*

$$\overline{U} : X \longrightarrow L_p^{w^*}(\mu, Y^*)$$

is compact.

In particular,

$$\int_\Omega U : X \longrightarrow Y^*, \qquad x \longmapsto \int_\Omega U(\omega)(x)\, d\mu(\omega)$$

is compact.

Proof. Let $(x_n) \subset B(X)$ be a given sequence. Since $l_1 \not\subset X$, we assume without loss of generality that (x_n) is weakly Cauchy according to the Rosenthal dichotomy. Let us first assume that Y is separable. Then, according to the introduction, every $f \in \mathcal{L}_p^{w^*}(\mu, Y^*)$ has a measurable norm-function. Hence $\Omega \ni \omega \longmapsto \|U(\omega)(x)\|$ is measurable for all $x \in X$. Since $U(\omega)$ is compact, $(U(\omega)(x_n))$ converges in norm to some $\hat{f}(\omega) \in Y^*$ for all $\omega \in \Omega$. The convergence implies that $\hat{f} : \Omega \to Y^*$ is w^*-measurable. Since U is bounded and $\|U(\omega)(x_n) - \hat{f}(\omega)\|^p \to 0$ for all $\omega \in \Omega$, the classical Lebesgue domination theorem gives the assertion.

We consider now the nonseparable case of Y and assume that $(\overline{U}(x_n))$ is not norm-Cauchy. Then there is an $\varepsilon > 0$ and a subsequence $(\overline{U}(x_{n_k}))$ such that

$$\forall\, k \in \mathbf{N} : \|\overline{U}(x_{n_{k+1}}) - \overline{U}(x_{n_k})\|_{p,*} > \varepsilon.$$

For all $k \in \mathbf{N}$ there is a $g_k \in \mathcal{E}(\Omega, \Sigma, Y)$, $\|g_k\|_{p'} \leq 1$, such that

$$\forall\, k \in \mathbf{N} : |\langle g_k, \overline{U}(x_{n_{k+1}}) - \overline{U}(x_{n_k})\rangle| > \varepsilon. \tag{5}$$

Let $Y_0 \subset Y$ be a separable subspace such that $g_k \in \mathcal{E}(\Omega, \Sigma, Y_0)$ for all $k \in \mathbf{N}$. Then let us consider the embedding $\imath : Y_0 \hookrightarrow Y$ and its adjoint

$\imath^* : Y^* \to Y_0^*$. Evidently $\imath^* \circ U : \Omega \to K(X, Y_0^*)$ is w^*-measurable and the separable case of Y is applicable. But we have, according to (5),

$$\forall\, k \in \mathbf{N} : \|\overline{\imath^* \circ U}(x_{n_{k+1}}) - \overline{\imath^* \circ U}(x_{n_k})\|_{p,*} \\ \geq |\langle g_k, \overline{U}(x_{n_{k+1}}) - \overline{U}(x_{n_k})\rangle| > \varepsilon,$$

which is a contradiction. □

Corollary 3.3. *Let $C \subset X$ be conditionally weakly compact and $U : \Omega \to L(X, Y^*)$ be bounded and w^*-measurable, such that $U(\omega)(C)$ is relatively compact in Y^* for $\omega \in \Omega$. Then $\overline{U}(C)$ is relatively compact in $L_p^{w^*}(\mu, Y^*)$.*

Proof. Let $K := \overline{\text{aco}}\, C$. Then K is conditionally weakly compact. Let p_K denote the Minkowski-functional. Let $X_0 := \{x \in X; p_k(x) < \infty\}$. Then (X_0, p_k) is a Banach space not containing a copy of l_1 and $C \subset B(X_0)$ [**4**, p. 237]. Let $\text{id} : X_0 \hookrightarrow X$ be the continuous inclusion. We consider $V : \Omega \to L(X_0, Y^*)$, $V(\omega) = U(\omega) \circ \text{id}$, $\omega \in \Omega$. Evidently V is w^*-measurable and $V(\omega) \in K(X_0, Y^*)$ for all $\omega \in \Omega$. Hence Theorem 3.2 implies that $\overline{V} : X_0 \to L_p^{w^*}(\mu, Y^*)$ is compact. Since $\overline{U}(C) \subset \overline{U}(\text{id}(B(X_0))) = \overline{V}(B(X_0))$, the proof is done. □

Proposition 3.4. *Let X^* have the* CCP. *Then every bounded and uniformly integrable operator $T : L_1 \to L_1^{w^*}(\mu, X^*)$ is Dunford-Pettis.*

Proof. X^* enjoys the CCP, thus $l_\infty \not\subset X^*$ since $c_0 \not\subset X^*$. Because $T(B(L_1))$ is uniformly integrable and X^* does not contain l_∞, according to Theorem 2.5 for a given $\varepsilon > 0$ we find a w^*-measurable function $U : \Omega \to L(L_1, X^*)$ so that $\overline{U} : L_1 \to L_\infty^{w^*}(\mu, X^*)$ and $\|T - \overline{U}\| < \varepsilon$. Since X^* has the CCP, we conclude that $U(\omega)(B(L_\infty))$ is compact for all $\omega \in \Omega$. Since $B(L_\infty) \subset L_1$ is weakly compact, thus conditionally weakly compact, Corollary 3.3 is applicable. This reveals that $\overline{U}(B(L_\infty))$ is compact, too. Thus $\overline{U}|_{L_\infty}$ is compact, which demonstrates the compactness of T since $\varepsilon > 0$ is arbitrary. □

The strong measurable case reads as follows and the proof is the same as before, just exchanging Corollary 3.3 and Theorem 2.5 by [**23**, p. 260] and Theorem 2.8, respectively.

Proposition 3.5. *Let X have the* CCP*. Then every bounded and uniformly integrable operator $T : L_1 \to L_1(\mu, X)$ is Dunford-Pettis.*

Lemma 3.6. *Let $1 < p < \infty$, let X^* enjoy the* CCP *and let $T : L_1 \to L_p^{w^*}(\mu, X^*)$ be bounded. Let ρ be a lifting of $\mathcal{L}_\infty$. Then*

$$\forall\, \varepsilon > 0, \quad \exists \delta > 0, \quad \forall\, A \in \Sigma, \quad \mu(A) < \delta, \quad \forall\, g \in B(L_\infty) : \\ \| \, \|T(g)^\rho(\cdot)\|_{X^*} - \|T(g)^\rho(\cdot)\|_{X^*} \chi_{\Omega \setminus A} \|_{p,*} < \varepsilon.$$

Proof. Straightforward. □

Theorem 3.7. *Let $1 < p < \infty$, and let X^* enjoy the* CCP*. Then $L_p^{w^*}(\mu, X^*)$ also has the* CCP*.*

Proof. Let $T : L_1 \to L_p^{w^*}(\mu, X^*)$ be a bounded operator. The embedding $\imath : L_p^{w^*}(\mu, X^*) \hookrightarrow L_1^{w^*}(\mu, X^*)$ is continuous and uniformly integrable. Thus, for the moment, we consider $T : L_1 \to L_1^{w^*}(\mu, X^*)$ to be uniformly integrable. Corollary 3.3 reveals that

$$T(B(L_\infty)) \quad \text{is compact in} \quad L_1^{w^*}(\mu, X^*). \tag{6}$$

We wish to prove that $(T(g_n))$ is relatively compact in $L_p^{w^*}(\mu, X^*)$ for an arbitrary sequence $(g_n)_{n \in \mathbf{N}} \subset B(L_\infty)$. Since $B(L_\infty)$ is weakly compact in L_1, we may assume that (g_n) converges weakly to some $g \in B(L_\infty)$ in L_1. Hence $(T(g_n))_{n \in \mathbf{N}}$ converges in norm to $T(g)$ in $L_1^{w^*}(\mu, X^*)$, which gives a subsequence converging pointwise almost everywhere on Ω in X^* by (1.9). For the sake of simplicity, we assume that $T(g_n) \overset{n \to \infty}{\to} T(g)$ almost everywhere on Ω. According to the extended Egorov theorem, we find for all $\varepsilon > 0$ a set $\Omega_\varepsilon \subset \Omega$, $\mu(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ such that $T(g_n)(\cdot) \overset{n \to \infty}{\to} T(g)(\cdot)$ uniformly on Ω_ε in X. With Lemma 3.6 this implies that $T(g_n) \overset{n \to \infty}{\to} T(g)$ in $L_p^{w^*}(\mu, X^*)$. Thus the image of $B(L_\infty)$ in $L_p^{w^*}(\mu, X^*)$ is compact. □

Since all ingredients for the proof to Theorem 3.7 have strongly measurable equivalence, see Theorems 2.8 and Proposition 3.5, we have an alternative proof of the result obtained in [**18**].

Corollary 3.8. *For $1 \le p \le \infty$, $L_p(\mu, X)$ has the* CCP *if and only if X has it and $1 < p < \infty$.*

We turn now to the embedding result of G. Pisier's and will show his result [**17**] by different means, which allows the extension to the w^*-measurable case. For this purpose we show that, in principle, on bounded sets in $L_p^{w^*}(\mu, X^*)$, $1 < p < \infty$, the weak topologies of $L_p^{w^*}(\mu, X^*)$ and $L_1^{w^*}(\mu, X^*)$ coincide.

Lemma 3.9. *Let $1 < p < \infty$.*

a) *Let $(f_n) \subset L_p(\mu, X)$ be bounded. Then*

$$(f_n) \text{ is weakly Cauchy in } L_p(\mu, X) \iff (f_n) \text{ is weakly Cauchy in } L_1(\mu, X).$$

b) *Let $(f_n) \subset L_p^{w^*}(\mu, X^*)$ be bounded. Then*

$$(f_n) \text{ is weakly Cauchy in } L_p^{w^*}(\mu, X^*) \iff (f_n) \text{ is weakly Cauchy in } L_1^{w^*}(\mu, X^*).$$

Proof. We give the proof only for b), since the proof for a) is even easier. First we note for $1 < p < \infty$ and a sequence of pairwise disjoint elements $(A_n) \subset \Sigma$, such that $\cup_{n \in \mathbf{N}} A_n = \Omega$:

$$(7) \qquad T : L_{p'}(\mu, X) \longrightarrow (\oplus L_{p'}(\mu | A_n, X))_{l_{p'}}, \quad f \longmapsto (f\chi_{A_n})$$

is an isometry.

$\Rightarrow$. This implication follows directly from the continuity of the inclusion map $\imath : L_p^{w^*}(\mu, X^*) \hookrightarrow L_1^{w^*}(\mu, X^*)$.

$\Leftarrow$. For $A \in \Sigma$ and $G \in L_p^{w^*}(\mu, X^*)^*$, let

$$G_{\chi_A} : L_p^{w^*}(\mu, X^*) \longrightarrow \mathbf{R}, \qquad f \longmapsto \langle f\chi_A, G\rangle.$$

Then $G\chi_A \in L_p^{w^*}(\mu, X^*)^*$. If we use (7) and Theorem 1.6c), we get for a disjoint sequence $(A_n) \subset \Sigma$, $\cup_{n \in \mathbf{N}} A_n = \Omega$:

$$(8) \quad T^{**} : L_p^{w^*}(\mu, X^*)^* \longrightarrow (\oplus L_p^{w^*}(\mu|_{A_n}, X^*)^*)_{l_{p'}}, \quad G \longmapsto (G\chi_{A_n})$$

is an isometry. Let $G \in B(L_p^{w^*}(\mu, X^*)^*)$. Our main aim is to prove

$$\text{(9)}\qquad \begin{gathered} \forall\, \varepsilon > 0, \quad \exists \Omega_\varepsilon \in \Sigma, \qquad \mu(\Omega \backslash \Omega_\varepsilon) < \varepsilon \\ G\chi_{\Omega_\varepsilon} \in L_1^{w^*}(\mu, X^*)^* \quad \text{and} \quad \|G\chi_{\Omega\backslash\Omega_\varepsilon}\| < \varepsilon \quad \text{in } L_p^{w^*}(\mu, X^*)^*. \end{gathered}$$

We will divide the proof to (9) into three steps.

Step 1.

$$\begin{gathered} \exists A \in \Sigma, \quad \mu(A) > 0, \quad \exists n \in \mathbf{N}, \quad \forall\, B \subset A, \quad \mu(B) > 0 : \\ \|G\chi_B\| < n\mu(B)^{1/p}. \end{gathered}$$

Proof to Step 1. We suppose the contrary is true. Then

$$\begin{gathered} \forall\, A \in \Sigma, \quad \mu(A) > 0, \quad \forall\, n \in \mathbf{N}, \quad \exists B \subset A, \\ \mu(B) > 0 : \|G\chi_B\| \geq n\mu(B)^{1/p'}. \end{gathered}$$

Then by applying the exhaustion argument [**7**, p. 70], we can find a sequence of pairwise disjoint elements $(B_j) \subset \Sigma$, $\cup_{j\in\mathbf{N}} B_j = \Omega$, such that for all $j \in \mathbf{N} : \|G\chi_{B_j}\| \geq n\mu(B_j)^{1/p'}$. Using (8), we can conclude for $n \in \mathbf{N}$:

$$\|G\| \geq n\mu(\Omega)^{1/p'}.$$

Since $n \in \mathbf{N}$ is arbitrary, we get $G \notin L_p^{w^*}(\mu, X^*)^*$, which is a contradiction, and Step 1 is proved.

Step 2. With the result of Step 1, we can find a sequence $(C_n) \subset \Sigma$, $\mu(C_n) > 0$ and an increasing sequence $(M_n) \subset \mathbf{R}_+$ such that

$$\begin{gathered} \forall\, n \in \mathbf{N}, \quad \forall\, B \subset C_n, \quad \mu(B) > 0 : \\ \|G\chi_B\| < M_n\mu(B)^{1/p'} \quad \text{and} \quad \bigcup_{n\in\mathbf{N}} C_n = \Omega. \end{gathered}$$

The sequence can be assumed to be increasing. Thus we can define a pairwise disjoint sequence by induction:

$$A_1 := C_1, \qquad A_n := C_n \Big\backslash \bigcup_{j=1}^{n-1} A_j.$$

Then, according to (8), we deduce:

$$\lim_{m\to\infty}\left\|G\chi_{\cup_{j=m}^{\infty}A_j}\right\| = \lim_{m\to\infty}\left\|\sum_{j=m}^{\infty}G\chi_{A_j}\right\| = 0.$$

Hence, $\lim_{n\to\infty}\|G\chi_{\Omega\setminus C_n}\| = 0$. Hence, for a given $\varepsilon > 0$ there is an $n \in \mathbf{N}$ such that

$$\forall\, B \subset C_n, \quad \mu(B) > 0 : \|G\chi_B\| < M_n\mu(B)^{1/p'} \tag{10}$$

and

$$\|G\chi_{\Omega\setminus C_n}\| < \varepsilon.$$

Step 3. Let $\Omega_\varepsilon := C_n$ according to (10). All we have to show is $G\chi_{\Omega_\varepsilon} \in L_1^{w^*}(\mu, X^*)^*$.

Proof. Let ρ be a lifting of $\mathcal{L}_\infty$. Let $f \in B(L_1^{w^*}(\mu, X^*))$ be given. For $m \in \mathbf{N}_0$, $k \in \mathbf{N}$ and $1 \le i \le k$, we define

$$A_{m,i} := \left\{\omega \in \Omega_\varepsilon; m + \frac{i-1}{k} \le \|f^\rho(\omega)\| < m + \frac{i}{k}\right\}.$$

Then we can compute with (10) and the definition of $A_{m,i}$:

$$\begin{aligned}|G\chi_{A_{m,i}}(f)| &\le M_n\left(m + \frac{i}{k}\right)\mu(A_{m,i})^{1/p}\mu(A_{m,i})^{1/p'}\\ &= M_n\left(m + \frac{i}{k}\right)\mu(A_{m,i})\\ &\le M_n\left(\|f\chi_{A_{m,i}}\|_{1,*} + \frac{1}{k}\mu(A_{m,i})\right).\end{aligned}$$

Since $\Omega_\varepsilon = \cup_{m\in\mathbf{N}_0}\cup_{i=1}^{k} A_{m,i}$ and the $A_{m,i}$ are pairwise disjoint, we obtain

$$\begin{aligned}|G\chi_{\Omega_\varepsilon}(f)| &\le \sum_{m=0}^{\infty}\sum_{i=1}^{k} M_n\left(\|f\chi_{A_{m,i}}\|_{1,*} + \frac{1}{k}\mu(A_{m,i})\right)\\ &\le M_n\left(\|f\|_{1,*} + \frac{1}{k}\mu(\Omega_\varepsilon)\right).\end{aligned}$$

So we have $|G\chi_{\Omega_\varepsilon}(f)| \leq M_n$ if $k \to \infty$ and hence $G\chi_{\Omega_\varepsilon} \in L_1^{w^*}(\mu, X^*)^*$.

The final arguments are devoted to the proof of the lemma. For this purpose we take a bounded sequence $(f_n) \subset L_p^{w^*}(\mu, X^*)$, which is weakly Cauchy in $L_1^{w^*}(\mu, X^*)$. Let $G \in B(L_p^{w^*}(\mu, X^*)^*)$ be given and let $\varepsilon > 0$. We may assume that $(f_n) \subset B(L_p^{w^*}(\mu, X^*))$. According to (9) there is a set $\Omega_\varepsilon \in \Sigma$ such that

$$G\chi_{\Omega_\varepsilon} \in L_1^{w^*}(\mu, X^*)^* \quad \text{and} \quad \|G\chi_{\Omega\setminus\Omega_\varepsilon}\| < \varepsilon/4 \quad \text{in } L_p^{w^*}(\mu, X^*)^*.$$

Thus $(G\chi_{\Omega_\varepsilon}(f_n))$ is a Cauchy sequence in $\mathbf{R}$ and we find an $n_0 \in \mathbf{N}$ such that, for all $n, m \geq n_0$: $|G\chi_{\Omega_\varepsilon}(f_n - f_m)| < \varepsilon/2$. Finally, for $n, m \geq n_0$, we have

$$\begin{aligned} |G(f_n - f_m)| &\leq |G\chi_{\Omega_\varepsilon}(f_n - f_m)| + |G\chi_{\Omega\setminus\Omega_\varepsilon}(f_n - f_m)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

The proof to the lemma is done. □

For concluding the final and main result on the embedding (3.11) we need the following w^*-analogon to [**20**, Theorem 2.3].

Theorem 3.10. *Let $U : \Omega \to CW(X, Y^*)$ be w^*-measurable and bounded, where $CW(X, Y^*)$ is the ideal of conditionally weakly compact operators. Let ρ be a lifting of $\mathcal{L}_\infty$ such that $\rho(U) = U$. Then, for $1 \leq p < \infty$,*

$$\overline{U} : X \longrightarrow L_p^{w^*}(\mu, Y^*) \quad \textit{is conditionally weakly compact.}$$

In particular,

$$\begin{aligned} \int_\Omega U : X &\longrightarrow Y^* \\ x &\longmapsto \int_\Omega U(\omega)(x)\, d\mu(\omega) \end{aligned}$$

is conditionally weakly compact.

Proof. We follow the idea of the proof in [**20**, Theorem 2.3] and assume that $\overline{U}(B(X))$ is not conditionally weakly compact. Then by

Rosenthal's dichotomy there is a sequence $(\bar{f}_n) \subset \overline{U}(B(X))$ which is equivalent to the l_1-basis. Choose $(x_n)_{n\in\mathbf{N}} \subset B(X)$ such that, for all $n \in \mathbf{N}$: $\bar{f}_n = \overline{U}(x_n)$. We define

$$f_n : \Omega \longrightarrow Y^*, \\ \omega \longmapsto U(\omega)(x_n).$$

Since U is bounded, $f_n \in \mathcal{L}_\infty^{w^*}(\mu, Y^*)$ and $\sup_{n\in\mathbf{N}} \|f_n\|_\infty < \infty$. By the result of M. Talagrand's, see [**22**, p. 20], there are subsets $\Omega_1, \Omega_2 \in \Sigma$ of Ω and for $n \in \mathbf{N}$ there is $g_n \in \mathrm{co}\,(f_n, f_{n+1}, \dots)$, such that

i) $\mu(\Omega_1 \cup \Omega_2) = \mu(\Omega)$,

ii) for al $\omega \in \Omega_1$ there exists $j \in \mathbf{N}$: $(g_n^\rho(\omega))_{n\geq j} \sim l_1$-basis,

iii) for all $\omega \in \Omega_2$: $(g_n^\rho(\omega))_{n\in\mathbf{N}}$ is a weak Cauchy sequence.

The assumption concerning U and Rosenthal's dichotomy imply that $\Omega_1 = \varnothing$. Thus (g_n^ρ) is a weak Cauchy sequence, see, e.g., [**22**, p. 20]. But this contradicts the assumption that $(\bar{f}_n)$ is equivalent to the l_1-basis. □

Theorem 3.11. *Let $1 < p < \infty$. Then*

a) *$l_1 \subset L_p(\mu, X)$ is isomorphic if and only if $l_1 \subset X$ is isomorphic.*

b) *$l_1 \subset L_p^{w^*}(\mu, X^*)$ is isomorphic if and only if $l_1 \subset X^*$ is isomorphic.*

Proof. The proof to a) is omitted, since it runs in a parallel way to b).

$\Rightarrow$. We assume $l_1 \not\subset X^*$, but we have an isomorphic embedding $T : l_1 \hookrightarrow L_p^{w^*}(\mu, X^*)$. Since $l_1 \subset L_1$ is complemented, we may extend T to $T{:}L_1 \to L_p^{w^*}(\mu, X^*)$. The embedding $\imath : L_p^{w^*}(\mu, X^*) \hookrightarrow L_1^{w^*}(\mu, X^*)$ is uniformly integrable, thus $\hat{T} : L_1 \to L_1^{w^*}(\mu, X^*)$ is uniformly integrable. Since $l_1 \not\subset X^*$, we also have $l_\infty \not\subset X^*$. Let ρ be a lifting of $\mathcal{L}_\infty$. Then by Theorem 2.5 for a given $\varepsilon > 0$, there is a w^*-measurable $U : \Omega \to L(L_1, X^*)$ such that $\overline{U} : L_1 \to L_\infty^{w^*}(\mu, X^*)$, $\|\hat{T} - \overline{U}\| < \varepsilon$ and $\rho(U) = U$. If we apply Theorem 3.10, we can demonstrate that $\overline{U}$ is conditionally weakly compact, since $l_1 \not\subset X^*$. Hence, $\hat{T}$ is also conditionally weakly compact, since the ideal of conditionally weakly compact operators is closed. Finally we can conclude using Theorem

3.9 that T is also weakly conditionally compact, which contradicts the Rosenthal dichotomy.

$\Leftarrow$. This implication is obvious, since X^* embeds isometrically in $L_p^{w^*}(\mu, X^*)$. □

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