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CONTRACTIVE PROJECTIONS AND ISOMETRIES IN SEQUENCE SPACES

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ABSTRACT. We characterize one-complemented subspaces of finite codimension in strictly monotone one-p-convex, $2 < p < \infty$, sequence spaces. Next we describe, up to isometric isomorphism, all possible types of one-unconditional structures in sequence spaces with few surjective isometries. We also give a new example of a class of real sequence spaces with few surjective isometries.

1. Introduction. This paper is divided into three parts. Throughout we consider real sequence spaces with one-unconditional basis.

First we study images of contractive projections–we prove (Theorem 1) that in strictly monotone and one-p-convex, $2 < p < \infty$, (or, dually, one-q-concave, $1 < q < 2$) sequence spaces every one-complemented subspace of finite codimension n contains all but at most $2n$ basic vectors. Calvert and Fitzpatrick [**11**] showed that if any such hyperplane is one-complemented, then the space is isometric to ℓ_p or c_0.

Characterizations of contractive projections are important in approximation theory, and they have been studied in various function spaces, cf. survey [**15**], spaces with Lorentzian metric [**16**] and sequence spaces. Existence of a norm-one projection onto a subspace is also related to the existence of a linear selection for the metric projection onto its complement–this connection and related references are discussed in [**4**]. For the fuller discussion of existing (extensive) literature we refer to [**13, 3, 4**].

Theorem 1 applies to a rich class of spaces including, e.g., ℓ_p, $1 < p < \infty$, $p \neq 2$, $\ell_p(\ell_r)$ where $2 < p$, $r < \infty$ or $1 < p$, $r < 2$, as well as a wide class of Orlicz and Lorentz spaces. It generalizes the analogous result known for classical sequence spaces: see [**7, 32, 33, 4**] for ℓ_1, [**5, 3**] for ℓ_p, $1 < p < \infty$, $p \neq 2$, [**31**] for ℓ_p^n, $1 \leq p < \infty$, $p \neq 2$. The analogous result is not true in c_0 [**7, 4**] or ℓ^∞ [**2**].

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Our method of proof is quite different, and we believe simpler, than those used before.

Next we investigate all (up to isometric equivalence) one-unconditional structures in a given sequence space. This is an isometric version of the question of uniqueness of unconditional basis, which has been studied since the late sixties, cf. [**9**] for various sequence spaces and [**19**] for detailed discussion and references.

In the complex case the situation is well understood. Kalton and Wood proved [**21**, Theorem 6.1] that all one-unconditional bases in a complex Banach space are isometrically equivalent, cf. also [**29**, discussion on p. 452 and Corollary 3.13] and [**18**]. Lacey and Wojtaszczyk [**22**] observed that this does not hold in real L_p-spaces; they give a complete description of the two possible types of one-unconditional structure in L_p, cf. also [**6**]. As far as we know, very little work has been done since then in real Banach spaces, except [**29**].

In Theorem 4 below we establish that in real sequence spaces which have few surjective isometries there are two types of isometrically nonequivalent one-unconditional structure. Corollary 5 formulates some additional assumptions which yield the uniqueness of one-unconditional basis.

It now becomes of interest to describe the spaces satisfying assumptions of Theorem 4, i.e., spaces with few surjective isometries. This is a problem that has been studied for its own right by many authors starting with Banach [**1**] who characterized isometries in ℓ_p^n. In the complex case the theory is well developed, see, e.g., the survey [**17**] and its references.

In the real case Braveman and Semenov [**10**], cf. also [**28**, Theorem 9.8.3], proved that symmetric sequence spaces have few (in our sense) isometries. Skorik [**30**] showed an analogous result for a special class of real sequence spaces. Also recently there has been some interest in isometries of finite-dimensional sequence spaces from a linear algebra point of view, see [**14, 12, 23**], but they did not enlarge a class of spaces with one-unconditional basis and which have few isometries.

In Section 4 we prove that there is another general class of spaces with only elementary surjective isometries. Namely, as an application of Theorem 1, we show (Theorem 10) that all surjective isometries between two strictly monotone sequence spaces which are both one-p-

convex, $2 < p < \infty$, or one-q-concave, $1 < q < 2$, are elementary. Our results are valid in both finite and infinite-dimensional spaces.

2. Norm-one complemented subspaces of finite codimension in sequence spaces. We say that a Banach space X is *one-p-convex*, respectively *one-q-concave*, if for every choice of elements $\{x_i\}_{i=1}^n$ in X the following inequality holds:

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\| \leq \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p} \quad \text{if } 1 \leq p < \infty,$$

or, respectively,

$$\left\| \left(\sum_{i=1}^{n} |x_i|^q \right)^{1/q} \right\| \geq \left(\sum_{i=1}^{n} \|x_i\|^q \right)^{1/q} \quad \text{if } 1 \leq q < \infty,$$

cf. [**24**, Definition 1.d.3]).

Theorem 1. *Let X be a strictly monotone sequence space,* $\dim X = d \geq 3$, *with a one-unconditional basis $\{e_i\}_{i=1}^d$. Suppose that*

(a) *X is one-p-convex, $2 < p < \infty$, or*

(b) *X is one-q-concave, $1 < q < 2$, and smooth at each basic vector.*

Then any one-complemented subspace F of codimension n in X contains all but at most $2n$ basic vectors of X.

Remark. Notice that Theorem 1 states only necessary and not sufficient conditions for the subspace to be one-complemented (unlike the theorem of Baronti and Papini [**3**] for ℓ_p). Also Baronti and Papini [**3**] prove that in ℓ_p every one-complemented subspace of finite codimension is an intersection of one-complemented hyperplanes. The analogous statement is not true in general, cf. [**8**] and [**4**, Example 6.4].

For the proof of Theorem 1 we will need the following observation which we state in the form of the lemma for easy reference.

Lemma 2. *Let X be a one-p-convex, $2 < p < \infty$, sequence space with a one-unconditional basis, and let $P : X \xrightarrow{\text{onto}} F$ be a projection. Assume* *that there exist disjoint elements* $x, y \in X$ *such that* $\operatorname{supp} P_y \supset \operatorname{supp} x$, $Px = x$ *and* $\operatorname{card}(\operatorname{supp} x) < \infty$. *Then* $\|P\| > 1$.

Proof of Lemma 2. Let us assume, for contradiction, that $\|P\| \leq 1$ and take x, y with $\|x\| = \|y\| = 1$. By one-p-convexity of X, and since x and y are disjoint, we get for all $\tau \in \mathbf{R}$:

$$\|P(x+\tau y)\| \leq \|x+\tau y\| = \|(|x|^p + |\tau y|^p)^{1/p}\| \leq (1+|\tau|^p)^{1/p}. \tag{1}$$

Since $p > 2$, X is one-2-convex [**24**, Proposition 1.d.5], and for any $\tau \in \mathbf{R}$, we get:

$$\begin{aligned}(2)\quad \|(|P(x+\tau y)|^2 + |P(x-\tau y)|^2)^{1/2}\| \\ &\leq (\|P(x+\tau e_1)\|^2 + \|P(x-\tau e_1)\|^2)^{1/2} \\ &\leq \sqrt{2}(1+|\tau|^p)^{1/p},\end{aligned}$$

by (1). On the other hand,

$$\begin{aligned}(3)\quad \|(|P(x+\tau y)|^2 + |P(x-\tau y)|^2)^{1/2}\| \\ &\geq \Big\| \sum_{i \in \operatorname{supp} x} (2(|x_i|^2 + \tau^2 |(Py)_i|^2)^2)^{1/2} e_i \Big\| \\ &\geq \sqrt{2} \Big\| \sum_{i \in \operatorname{supp} x} |x_i| \sqrt{1+\tau^2((Py)_i/x_i)^2} e_i \Big\| \\ &\geq \sqrt{2}\sqrt{1+\eta\tau^2}\|x\| \\ &= \sqrt{2}\sqrt{1+\eta\tau^2}\end{aligned}$$

where

$$\eta = \min_{i \in \operatorname{supp} x} \left\{ \left(\frac{(Py)_i}{x_i} \right)^2 \right\}.$$

Notice that $\eta > 0$, since $\operatorname{supp} x \subset \operatorname{supp} Py$.

Combining (2) and (3), we get $\sqrt{1+\eta\tau^2} \leq (1+|\tau|^p)^{1/p}$ which gives us the desired contradiction when $|\tau| < \eta^{1/(p-2)}$. □

Proof of Theorem 1. We first prove part (a) of the theorem. Let F be a one-complemented subspace of codimension n, say $F = \cap_{j=1}^n \ker f_j$

for some $f_j \in X^*$, and the contractive projection $P : X \to F$ is given by $P = Id_X - \sum_{j=1}^n f_j \otimes v^j$ for some linearly independent $v^j \in X$ with $f_j(v^k) = \delta_{jk}$ (where δ_{jk} denotes Kronecker delta).

Assume that $e_i \notin F$ if $i \in I$. If $\operatorname{card} I < n$ there is nothing to prove so without loss of generality $\{1, 2, \dots , n\} \subset I$ and $f_j(e_i) = \delta_{ij}$, $i, j \leq n$.

Notice first that if $l \notin \cup_{i=1}^n \operatorname{supp} v^i$, then $P(e_l) = e_l - \sum_{i=1}^n f_i(e_l)v^i$ and so $(P(e_l))_l = 1$. Thus, by strict monotonicity of X, $P(e_l) = e_l$, i.e., $e_l \in F$. Therefore,

$$I \subset \bigcup_{i=1}^n \operatorname{supp} v^i. \tag{4}$$

Now take any $a = \sum_{i=1}^n a_i e_i$. Then

$$\begin{aligned} P(a) &= a - \sum_{j=1}^n f_j(a)v^j \\ &= \sum_{i=1}^n a_i e_i - \sum_{j=1}^n \sum_{i=1}^n a_i f_j(e_i) v^j \\ &= \sum_{i=1}^n a_i e_i - \sum_{i=1}^n a_i v^i. \end{aligned}$$

Hence there exists $a_0 \in \operatorname{span}\{e_1, \dots , e_n\}$ such that

$$\operatorname{supp} P(a_0) \backslash \{1, \dots , n\} = \bigcup_{i=1}^n \operatorname{supp} v^i \backslash \{1, \dots , n\}.$$

If $\operatorname{card}(\cup_{i=1}^n \operatorname{supp} v^i \backslash \{1, \dots , n\}) > n$, then $\operatorname{card}(\operatorname{supp} P(a_0) \backslash \{1, 2, \dots , n\}) \geq n+1$ and there exists $x \in F$ with $\operatorname{supp} x \subset \operatorname{supp} P(a_0) \backslash \{1, 2, \dots , n\}$, since $\operatorname{codim} F = n < n + 1$. Now x and a_0 satisfy assumptions of Lemma 2 which contradicts the fact that P is contractive.

Thus $\operatorname{card}(\cup_{i=1}^n \operatorname{supp} v^i \backslash \{1, \dots , n\}) \leq n$ and, by (4),

$$\operatorname{card} I \leq \operatorname{card}\left(\bigcup_{i=1}^n \operatorname{supp} v^i \right) \leq 2n,$$

which proves part (a) of the theorem.

We prove part (b) by duality. Consider contractive projection $P^* = Id_{X^*} - \sum_{j=1}^n v^j \otimes f_j$. X^* is one-p-convex for some $p > 2$ and strictly monotone so by part (a) we get that, say, $v^1, \dots, v^n \subset \operatorname{span}\{e_1^*, \dots, e_{2n}^*\}$. Thus, by (4), $I \subset \{1, \dots, 2n\}$ since X is strictly monotone. $\square$

Theorem 1 can be combined with our previous results about nonexistence of one-complemented hyperpanes in nonatomic function spaces which do not have any bands isometrically equal to L_2 [**26**, Theorem 2], cf. also [**20**, Theorem 4.3, **27**, Theorem 2.7].

Corollary 3. *Suppose that X is a separable strictly monotone function space on (Ω, μ) which is either one-p-convex for some $2 < p < \infty$ or one-q-concave for some $1 < q < 2$ and smooth at χ_A for every atom A of μ. Suppose further that, for some $g \in X^*$, $\ker g$ is one-complemented in X. Then g is of the form $\alpha\chi_A + \beta\chi_B$, where $\alpha, \beta \in \mathbf{R}$ and A, B are atoms of μ.*

The above statement exactly parallels (and extends) the theorem proved by Beauzamy and Maurey for L_p [**5**, Proposition 3.1], cf. also [**25**].

3. Isometries and one-unconditional bases of sequence spaces. An operator $T : X \to Y$ between two sequence spaces with one-unconditional bases $\{e_i\}_{i=1}^d$ and $\{f_i\}_{i=1}^d$, respectively $d \leq \infty$, will be called *elementary* if

$$T(e_i) = a_i f_{\sigma(i)}$$

for some $a_i \in \mathbf{R}$ and a permutation σ of $\{1, \dots, d\}$.

We will say that a pair of indices k, l is *interchangeable* in X if for any $x, z \in X$ $|x_k| = |z_l|, |x_l| = |z_k|$ and $|x_i| = |z_i|$ for all $i \neq k, l$ imply that $\|x\| = \|z\|$. Space X is rearrangement invariant if and only if every two indices are interchangeable.

Theorem 4. *Suppose that X and Y are separable spaces with one-unconditional bases $\{e_i\}_{i=1}^d$ and $\{f_i\}_{i=1}^d$, respectively, and suppose that* *all surjective isometries of one of the spaces X or Y onto itself are elementary. Suppose that $T : X \to Y$ is a surjective isometry.*

Then there exist a set $A \subset \{1, \dots , d\}$ and a one-to-one map $\sigma : A \to \sigma(A) \subset \{1, \dots , d\}$ such that for every $i \in A$

$$T(e_i) = \varepsilon_i f_{\sigma(i)}$$

where $\varepsilon_i = \pm 1$.

The complementary sets

$$B_X = \{1, \dots , d\}\backslash A \quad \text{and} \quad B_Y = \{1, \dots , d\}\backslash \sigma(A)$$

split into families of disjoint pairs $\mathcal{P}_X \subset 2^{B_X}$, $\mathcal{P}_Y \subset 2^{B_Y}$ so that there exists a one-to-one map $\tau : \mathcal{P}_X \overset{\text{onto}}{\longrightarrow} \mathcal{P}_Y$ and if $\tau(i, j) = (k, l)$, then

$$T(e_i) = \frac{\delta_i}{\|f_k + f_l\|}(f_k + \varepsilon_i f_l)$$
$$T(e_j) = \frac{\delta_i}{\|f_k + f_l\|}(f_k - \varepsilon_i f_l)$$

where $\delta_i, \varepsilon_i = \pm 1$.

Moreover,

(a) *all pairs $(i, j) \in \mathcal{P}_X$ and $(k, l) \in \mathcal{P}_Y$ are interchangeable in X or Y, respectively.*

(b) *If all isometries of Y, respectively X, onto itself are elementary, then the set A, respectively $\sigma(A)$, depends only on the spaces X, Y and not on the isometry T.*

The following fact is an immediate consequence of Theorem 4.

Corollary 5. *In the situation of Theorem* 4, *if we assume additionally that no two-dimensional subspace of one of the spaces X or Y is isometric to ℓ_2^2 and both spaces X and Y are either one-2-convex or one-2-concave, then every surjective isometry $T : X \to Y$ is elementary.*

Remark. Since all surjective isometries of rearrangement-invariant sequence spaces onto itself are elementary [**10**], Corollary 5 may be viewed as an isometric and sequence space version of the deep result of Kalton about (isomorphic) uniqueness of lattice structure in nonatomic 2-convex (or strictly 2-concave) Banach lattices which embed complementably in a strictly 2-convex, respectively strictly 2-concave, rearrangement-invariant function space [**19**, Theorems 8.1 and 8.2].

Proof of Theorem 4. We use all the notation as introduced above.

Let us first see that the final remark follows readily from the main statement of the theorem.

(a) Let $y \in Y$ and $x = T^{-1}(y)$. Consider the element $\tilde{x} \in X$ such that $\tilde{x}_j = -x_j$ and $\tilde{x}_\nu = x_\nu$ for $\nu \neq j$. Then $\|\tilde{x}\| = \|x\|$ and so $\|y\| = \|T\tilde{x}\|$. But from the form of T we see that $(T\tilde{x})_k = \varepsilon_i y_l$, $(T\tilde{x})_l = \varepsilon_i y_k$ and $(T\tilde{x})_\nu = y_\nu$ for $\nu \neq k, l$. Hence (k, l) is interchangeable in Y. Proof for $(i, j) \in \mathcal{P}_X$ is similar.

(b) Assume that the set A depends on the isometry T, and use the notation A_T to emphasize that dependence. Assume that $i \in A_U \backslash A_T$ for some isometries U, T. Then

$$TU^{-1}(\varepsilon_i f_{\sigma(i)}) = T(e_i) = \frac{\delta_i}{\|f_k + f_l\|}(f_k + \varepsilon_i f_l),$$

which contradicts the fact that the isometry $TU^{-1} : Y \to Y$ is elementary.

Now let us return to the proof of the main statement of the theorem. It is clear that if T has the described form then so does T^{-1}. So we can assume without loss of generality that all isometries of, say, Y onto itself are elementary.

We will split the proof of the theorem into a series of lemmas.

Lemma 6. *For any $i \leq d$ there exist at most two indices k and l such that*

$$T(e_i) = \alpha_k f_k + \alpha_l f_l.$$

Lemma 7. *Suppose that for some $i, j, k, l \leq d$*

$$T(e_i) = \alpha_k f_k + \alpha_l f_l$$
$$T(e_j) = \varepsilon_k f_k,$$

where $\alpha_l \neq 0$, $\varepsilon_k = \pm 1$. *Then* $\alpha_k = 0$.

Lemma 8. *Suppose that for some* $i, j, k, l, m \leq d$,

$$T(e_i) = \alpha_k f_k + \alpha_l f_l$$
$$T(e_j) = \beta_k f_k + \beta_m f_m,$$

where $\alpha_k, \beta_k \neq 0$. *Then*

(a) $l = m$ *and* $\alpha_l, \beta_m \neq 0$,

(b) $\mathrm{sgn}\,(\alpha_k \alpha_l) = -\mathrm{sgn}\,(\beta_k \beta_m)$ *and* $|\alpha_k| = |\alpha_l| = |\beta_k| = |\beta_m| = 1/\|f_k + f_m\|$.

Lemma 9. *Suppose that, for all* $n \leq d$ $\mathrm{card}\,\mathrm{supp}\,T(e_n) \leq 2$. *Let* $i, k, l \leq d$ *be such that*

$$T(e_i) = \alpha_k f_k + \alpha_l f_l,$$

where $\alpha_k, \alpha_l \neq 0$, $k \neq l$. *Then there exist a unique* $j \neq i$ *and* $\beta_k, \beta_l \neq 0$ *so that*

$$T(e_j) = \beta_k f_k + \beta_l f_l.$$

Proof of Lemma 6. Denote

$$T(e_j) = \sum_{m=1}^{d} \alpha_{j,m} f_m, \qquad T^{-1}(f_n) = \sum_{j=1}^{d} \beta_{n,j} e_j.$$

For any choice of signs $\varepsilon = (\varepsilon_j)_{j=1}^d$, $\varepsilon_j = \pm 1$, we define an isometry $S_\varepsilon : X \to X$ by $S_\varepsilon(e_j) = \varepsilon_j e_j$. By unconditional convergence, we get for every n:

$$\begin{aligned} TS_\varepsilon T^{-1}(f_n) &= T\left(\sum_{j=1}^{d} \beta_{n,j} \varepsilon_j e_j\right) \\ &= \sum_{j=1}^{d} \varepsilon_j \beta_{n,j} \left(\sum_{m=1}^{d} \alpha_{j,m} f_m\right) \\ &= \sum_{m=1}^{d} \left(\sum_{j=1}^{d} \varepsilon_j \beta_{n,j} \alpha_{j,m}\right) f_m. \end{aligned}$$

Since $TS_\varepsilon T^{-1}$ is elementary (as a surjective isometry of Y) we conclude that for every $n \leq d$ and $\varepsilon = (\varepsilon_j)_{j=1}^d$ there exists exactly one m such that

$$\sum_{j=1}^{d} \varepsilon_j \beta_{n,j} \alpha_{j,m} \neq 0. \tag{5}$$

Now fix $i \leq d$. Since T^{-1} is onto there exists $n \leq d$ with $\beta_{n,i} \neq 0$. By (5) we get

$$\begin{aligned} &\exists! k \text{ with } \quad \sum_{j=1}^{d} \beta_{n,j} \alpha_{j,k} \neq 0 \\ &\qquad \varepsilon_j = 1 \text{ for all } j \end{aligned} \tag{6}$$

$$\begin{aligned} &\exists! l \text{ with } \quad -\sum_{j \neq i} \beta_{n,j} \alpha_{j,l} + \beta_{n,i} \alpha_{i,l} \neq 0 \\ &\qquad \varepsilon_j = \begin{cases} -1 & j \neq i \\ 1 & j = i. \end{cases} \end{aligned} \tag{7}$$

Hence, for any $m \neq k, l$, $\alpha_{i,m} = 0$, i.e.,

$$T(e_i) = \alpha_{i,k} f_k + \alpha_{i,l} f_l,$$

which proves the lemma. □

Proof of Lemma 7. Consider T^{-1}. Then

$$\begin{aligned} T^{-1}(f_k) &= \varepsilon_k e_j \\ T^{-1}(f_l) &= \frac{1}{\alpha_l} e_1 - \varepsilon_k \frac{\alpha_k}{\alpha_l} e_j. \end{aligned}$$

And, since $\|T(e_i)\| = \|T^{-1}(f_l)\| = 1$, we get $|\alpha_l| \leq 1$ and $1/|\alpha_l| \leq 1$. Hence $|\alpha_l| = 1$.

Thus, if X is strictly monotone, $\alpha_k = 0$.

If X is not strictly monotone, denote by

$$M = \sup\{m : \|e_i + me_j\| = 1\},$$
$$N = \sup\{n : \|f_l + nf_k\| = 1\}.$$

We have

$$\begin{aligned} 1 &= \|T(\varepsilon_k e_i + \operatorname{sgn}(\alpha_k) M e_j\| \\ &= \|\varepsilon_k \alpha_k f_k + \varepsilon_k \alpha_l f_l + M\varepsilon_k \operatorname{sgn}(\alpha_k) f_k\| \\ &= \|f_l + (M + |\alpha_k|) f_k\|. \end{aligned}$$

Hence

$$M + |\alpha_k| \leq N.$$

Similarly,

$$N + |\alpha_k| \leq M.$$

Hence

$$M + 2|\alpha_k| \leq N + |\alpha_k| \leq M$$

and $|\alpha_k| = 0$. □

Proof of Lemma 8. (a) Consider T^{-1}. We have

$$T^{-1}(\alpha_k f_k + \alpha_l f_l) = e_i, \tag{8}$$
$$T^{-1}(\beta_k f_k + \beta_m f_m) = e_j. \tag{9}$$

By Lemma 6 there exist indices $p, q, r, s, t, u \leq d$ such that

$$T^{-1}(f_k) = \gamma_p e_p + \gamma_q e_q, \quad p \neq q, \tag{10}$$
$$T^{-1}(f_l) = \eta_r e_r + \eta_s e_s, \quad r \neq s, \tag{11}$$
$$T^{-1}(f_m) = \xi_t e_t + \xi_u e_u, \quad t \neq u. \tag{12}$$

From (8) we get

$$\alpha_k \gamma_p e_p + \alpha_k \gamma_q e_q + \alpha_l \eta_r e_r + \alpha_l \eta_s e_s = e_i. \tag{13}$$

So $i \in \{p, q, r, s\}$, say $i = p$.

If $i \notin \{r,s\}$, then $\eta_r \cdot \eta_s = 0$, say $\eta_s = 0$ and $q = r$. But then, by Lemma 7, $\gamma_q = 0$ and

$$T^{-1}(f_k) = \gamma_p e_i, \qquad T^{-1}(f_l) = \eta_r e_r,$$

where $|\gamma_p| = |\eta_r| = 1$ which contradicts (8).

Hence $i \in \{r,s\}$, say $i = r$.

By Lemma 7, $\gamma_q, \eta_s \neq 0$, and thus, by (13), $q = s$. That is, we have

$$i = p = r \quad \text{and} \quad q = s \neq i.$$

Similarly, from (9), (10) and (12) we get that

$$j = q = t \quad \text{and} \quad u = p \neq j.$$

Therefore,

$$i = p = r = u \quad \text{and} \quad j = q = s = t,$$

and $T^{-1}(\operatorname{span}\{f_k, f_l, f_m\}) \subset \operatorname{span}\{e_i, e_j\}$ which implies that $\dim\operatorname{span}\ \{f_k, f_l, f_m\} = 2$ and so $m = l$.

It follows immediately from Lemma 7 that $\alpha_l, \beta_m \neq 0$.

(b) Since $l = m$, denote $T(e_j) = \beta_k f_k + \beta_l f_l$. Then

$$T^{-1}(f_k) = -\beta_l M e_i + \alpha_l M e_j$$
$$T^{-1}(f_l) = \beta_k M e_i - \alpha_k M e_j,$$

where $M = (\alpha_l \beta_k - \alpha_k \beta_l)^{-1}$.

Denote by S the isometry of X such that $S(e_i) = -e_i$ and $S(e_j) = e_j$. Then

$$TST^{-1}(f_k) = T(\beta_l M e_i + \alpha_l M e_j) = M(\alpha_l \beta_k + \alpha_k \beta_l) f_k + 2M\beta_l \alpha_l f_l.$$

Since TST^{-1} is a surjective isometry of Y, it is elementary and since $2M\beta_l\alpha_l \neq 0$ we get $\alpha_l \beta_k + \alpha_k \beta_l = 0$, i.e.,

$$\alpha_l \beta_k = -\alpha_k \beta_l. \tag{14}$$

Moreover,

$$|2M\beta_l \alpha_l| = 1. \tag{15}$$

Combining (14) and (15) we obtain $|\alpha_k| = |\alpha_l|$ and $|\beta_k| = |\beta_l|$, and since $\|T(e_i)\| = \|T(e_j)\| = 1$ we have $|\alpha_k| = |\alpha_l| = |\beta_k| = |\beta_l| = 1/\|f_k + f_l\|$. □

Proof of Lemma 9. Lemma 8 implies that for any $j \neq i$ we have either $\operatorname{supp} T(e_j) = \operatorname{supp} T(e_i)$ or $\operatorname{supp} T(e_j) \cap \operatorname{supp} T(e_i) = \varnothing$. Hence, by surjectivity of T, there exists $j \neq i$ with $T(e_j) = \beta_k f_k + \beta_l f_l$ and, by Lemma 8, $\beta_k, \beta_l \neq 0$.

Uniqueness of j is an immediate consequence of the fact that T preserves the dimension of subspaces. □

4. Isometries in one-p-convex sequence spaces.

Theorem 10. *Suppose X and Y are separable strictly monotone sequence spaces with one-unconditional bases and* $\dim X = \dim Y = d \geq 3$, $d \leq \infty$. *Suppose that*

(a) *X and Y are one-p-convex, $2 < p < \infty$, or*

(b) *X and Y are one-q-concave, $1 < q < 2$, and smooth at each basic vector.*

Then any isometry U from X onto Y is of the form

$$U\left(\sum_{k=1}^{d} a_k e_k\right) = \sum_{k=1}^{d} \varepsilon_k a_k f_{\sigma(k)}$$

where σ is a permutation of $\{1, \dots, d\}$ and $\varepsilon_k = \pm 1$ for $k = 1, \dots, d$.

Proof. We will prove the theorem with the assumption (a). Part (b) follows by duality.

For any $k \leq d$ the hyperplane $\{x_k = 0\}$ is one-complemented in X and so is $U\{x_k = 0\}$ in Y. By Theorem 1 there are at most two numbers $k_1, k_2 \leq d$ such that $U\{x_k = 0\} = \{\alpha_1 y_{k_1} + \alpha_2 y_{k_2} = 0\}$ for some $\alpha_1, \alpha_2 \in \mathbf{R}$. We will say that coordinates k and l are related if $\{k_1, k_2\} \cap \{l_1, l_2\} \neq \varnothing$.

For the proof of the theorem we need three technical lemmas.

Lemma 11. *Suppose* $U\{x_k = 0\} = \{\alpha_1 y_{k_1} + \alpha_2 y_{k_2} = 0\}$ *where* $\alpha_1, \alpha_2 \neq 0$, *and suppose that* l *is related to* k. *Then* $\{l_1, l_2\} \subset \{k_1, k_2\}$.

Lemma 12. *For any* $k \leq d$ *there is at most one coordinate* l $(\neq k)$ *related to* k.

Lemma 13. *For any* $k \leq d$ *there exist* $i, j \leq d$, $\kappa_i, \kappa_j \in \mathbf{R}$ *such that*

$$U(e_k) = \kappa_i f_i + \kappa_j f_j.$$

Moreover, if both $\kappa_i, \kappa_j \neq 0$ *then there exist (unique)* $l \neq k$ *and* $\lambda_i, \lambda_j \in \mathbf{R}$ *such that* $U(e_l) = \lambda_i f_i + \lambda_j f_j$.

Let us first see that Theorem 10 indeed follows from Lemma 13.

If, say, $\kappa_j = 0$, then $|\kappa_i| = 1$ since U is an isometry. So we need only to show that κ_i, κ_j cannot both be nonzero.

Assume for contradiction that $\kappa_i, \kappa_j \neq 0$. Then by Lemma 13 there exists $l \neq k$ such that $U(e_l) = \lambda_i f_i + \lambda_j f_j$ for some $\lambda_i, \lambda_j \in \mathbf{R}$. By one-$p$-convexity of Y, we get

$$\begin{aligned} 1 = \|\kappa_i f_i + \kappa_j f_j\| &\leq (\kappa_i^p + \kappa_j^p)^{1/p} < (\kappa_i^2 + \kappa_j^2)^{1/2} \\ 1 = \|\lambda_i f_i + \lambda_j f_j\| &\leq (\lambda_i^p + \lambda_j^p)^{1/p} \leq (\lambda_i^2 + \lambda_j^2)^{1/2}. \end{aligned}$$

Hence

$$\kappa_i^2 + \kappa_j^2 + \lambda_i^2 + \lambda_j^2 > 2.$$

So, say,

$$(\kappa_i^2 + \lambda_i^2)^{1/2} = \|(\kappa_i, \lambda_i)\|_2 > 1. \tag{16}$$

On the other hand, by one-2-convexity of X for any $(a, b) \in \mathbf{R}^2$ we have $\|ae_k + be_l\| \leq \|(a, b)\|_2$. But

$$\begin{aligned} \|ae_k + be_l\| &= \|(a\kappa_i + b\lambda_i)f_i + (a\kappa_j + b\lambda_j)f_j\| \\ &\geq \|(a\kappa_i + b\lambda_i)f_i\| = \|(a\kappa_i + b\lambda_i)\|. \end{aligned}$$

So

$$\|(a\kappa_i + b\lambda_i)| \leq \|(a, b)\|_2,$$

and this means that $\|(\kappa_i, \lambda_i)\|_2 \leq 1$ which contradicts (16) and ends the proof of the theorem. $\square$

Proof of Lemma 11. Our assumption is

$$U\{x_k = 0\} = \{\alpha_1 y_{k_1} + \alpha_2 y_{k_2} = 0\}, \tag{17}$$

where $\alpha_1, \alpha_2 \neq 0$ and l is related to k. Without loss of generality, $l_1 = k_1$ and we have

$$U\{x_l = 0\} = \{\beta_1 y_{k_1} + \beta_2 y_{k_2} = 0\} \tag{18}$$

where $\beta_1 \neq 0$. If $\beta_2 = 0$ there is nothing to prove so let us assume $\beta_2 \neq 0$. Proposition 1 applies to the isometry U^{-1} gives us:

$$U\{y_{k_1} = 0\} = \{\mu_1 x_{m_1} + \mu_2 x_{m_2} = 0\} = H_{k_1} \tag{19}$$

$$U\{y_{k_2} = 0\} = \{\nu_1 x_{n_1} + \nu_2 x_{n_2} = 0\} = H_{k_2} \tag{20}$$

$$U\{y_{l_2} = 0\} = \{\theta_1 x_{t_1} + \theta_2 x_{t_2} = 0\} = H_{l_1}. \tag{21}$$

Denote $E_l = \{x_l = 0\}$, $E_k = \{x_k = 0\} \subset X$.

Since U^{-1} is an isometry, equations (17), (19) and (20) imply that $E_k \cap H_{k_1} = E_k \cap H_{k_2} = H_{k_1} \cap H_{k_2}$, i.e., the following systems of equations are equivalent:

$$\left\{ \begin{matrix} \mu_1 x_{m_1} + \mu_2 x_{m_2} = 0 \\ x_k = 0 \end{matrix} \right\} \equiv \left\{ \begin{matrix} \nu_1 x_{n_1} + \nu_2 x_{n_2} = 0 \\ x_k = 0 \end{matrix} \right\}$$
$$\equiv \left\{ \begin{matrix} \mu_1 x_{m_1} + \mu_2 x_{m_2} = 0 \\ \nu_1 x_{n_1} + \nu_2 x_{n_2} = 0 \end{matrix} \right\}.$$

Since these systems have rank 2, this implies that, say, $m_1 = n_1 = k$, $m_2 = n_2 \neq k$ and $\mu_2, \nu_2 \neq 0$.

Similarly, by considering equations (18), (19) and (21), we obtain $m_1 = t_1$, $m_2 = t_2$ and either $m_1 = l$ or $m_2 = l$. Hence, $k = m_1 = n_1 = t_1$ and $l = m_2 = n_2 = t_2$. But this means that $\operatorname{codim}(H_{k_1} \cap H_{k_2} \cap H_{l_2}) \leq 2$. Since U is an isometry, we have

$$\begin{aligned} \operatorname{codim}\{y_{k_1}, y_{k_2}, y_{l_2} = 0\} &= \operatorname{codim} U(H_{k_1} \cap H_{k_2} \cap H_{l_2}) \\ &= \operatorname{codim}(H_{k_1} \cap H_{k_2} \cap H_{l_2}) \leq 2. \end{aligned}$$

Hence $l_2 = k_2$. □

Proof of Lemma 12. If k is related to l then, for at least one of k, l, say k, $U\{x_k = 0\} = \{\alpha_1 y_{k_1} + \alpha_2 y_{k_2} = 0\}$ where $\alpha_1, \alpha_2 \neq 0$. Then by Lemma 11 $\{l_1, l_2\} \subset \{k_1, k_2\}$, so if t is related to l it is also related to k and $\{t_1, t_2\} \subset \{k_1, k_2\}$. But then $U\{x_k, x_l, x_t = 0\} \subset \{y_{k_1}, y_{k_2} = 0\}$ and so $t \in \{k, l\}$. □

Proof of Lemma 13. We have

$$U(e_k) \in \bigcap_{\nu \neq k} U\{x_\nu = 0\}.$$

By Lemma 12 there exists at most one coordinate l related to k and by Lemma 11 $\{k_1, k_2, l_1, l_2\} = \{i, j\}$ where $i \neq j$ if and only if there exists $l \neq k$ related to k. Moreover,

$$\text{if} \quad \nu \neq k, l, \quad \text{then} \quad \{\nu_1, \nu_2\} \cap \{i, j\} = \varnothing, \tag{22}$$

where $U\{x_\nu = 0\} = \{\alpha(\nu) y_{\nu_1} + \beta(\nu) y_{\nu_2} = 0\}$.

Since U is one-to-one and onto,

$$\bigcap_{\nu \neq k, l} U\{x_\nu = 0\} = \bigcap_{\mu \neq i, j} \{y_\mu = 0\}.$$

Hence $U(e_k), U(e_l) \in \operatorname{span}\{f_i, f_j\}$ which proves the first part of the lemma.

The second part follows immediately from the fact that $\operatorname{span}\{U(e_k), U(e_l)\} = \operatorname{span}\{f_i, f_j\}$ and condition (22). □

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