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LOGARITHMIC TRANSFORMATIONS INTO l^1

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ABSTRACT. Throughout this paper we shall write l to denote l^1. Let t be a sequence in $(0,1)$ that converges to 1, and define the logarithmic matrix L_t by $a_{nk} = -t_n^{k+1}/[(k+1)\log(1-t_n)]$. The matrix L_t determines a sequence-to-sequence variant of the logarithmic power series method of summability introduced by Borwein in [**1**]. The purpose of this paper is to study these transformations as mappings into l. A necessary and sufficient condition for L_t to be l–l is proved. The strength of L_t in the l–l setting is investigated. Also it is shown that L_t is translative in the l–l sense for certain sequences.

1. Introduction and background. Since the appearance of the famous Knopp-Lorentz theorem in [**5**], there have been many studies of the general properties of l–l summability methods, but still there are relatively few results about specific l–l methods. The shortage of examples of l–l methods and the study made by Fridy in [**3**] have provided the present study.

The logarithmic power series method of summability [**1**], denoted by L, is the following sequence-to-function transformation if

$$\lim_{x\to 1^-}\left\{\frac{-1}{\log(1-x)}\sum_{k=0}^{\infty}\frac{1}{k+1}u_k x^{k+1}\right\} = A,$$

then u is L-summable to A. In order to consider this method as a mapping into l, we must modify it into a sequence-to-sequence transformation. This is achieved by replacing the continuous parameter x with a sequence t such that $0 < t_n < 1$ for all n and $\lim_n t_n = 1$. Thus, the sequence u is transformed into the sequence $L_t u$ whose nth term is given by

$$(L_t u)_n = \frac{-1}{\log(1-t_n)}\sum_{k=0}^{\infty}\frac{1}{k+1}u_k t_n^{k+1}.$$

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This transformation is determined by the matrix L_t whose nkth entry is given by

$$a_{nk} = \frac{-1}{\log(1-t_n)} \frac{1}{k+1} t_n^{k+1}.$$

The matrix L_t is called a logarithmic matrix. The L_t matrix is regular and, indeed, totally regular.

2. Basic notations and definitions. Let $A = (a_{nk})$ be an infinite matrix defining a sequence-to-sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k,$$

where $(Ax)_n$ denotes the nth term of the image sequence Ax. Let y be a complex number sequence. Throughout this paper we shall use the following basic notations and definitions.

$$\begin{aligned} l &= \left\{ y : \sum_{k=0}^{\infty} |y_k| < \infty \right\} \\ l(A) &= \{y : Ay \in l\} \\ d(A) &= \left\{ y : \sum_{k=0}^{\infty} a_{nk} y_k < \infty \text{ for each } n \geq 0 \right\} \\ G &= \{y : y_k = O(r^k) \text{ for some } r \in (0,1)\} \\ G_w &= \{y : y_k = O(r^k) \text{ for some } r \in (0,w), 0 < w < 1\} \\ c(A) &= \{y : y \text{ is summable by } A\}. \end{aligned}$$

Definition 1. If X and Y are complex number sequences, then the matrix A is called an X-Y *matrix* if the image Au of u under the transformation A is in Y whenever u is in X.

Definition 2. The summability matrix A is said to be l-*translative* for a sequence u in $l(A)$ provided that each of the sequences T_u and S_u is in $l(A)$, where $T_u = \{u_1, u_2, u_3, \dots\}$ and $S_u = \{0, u_0, u_1, \dots\}$.

Definition 3. The matrix A is l-*stronger* than the matrix B provided that $l(B) \subset l(A)$.

3. The main results. Our first main result gives a necessary and sufficient condition for L_t to be l–l.

Theorem 1. *The logarithmic matrix L_t is l-l if and only if* $1/\log(1-t) \in l$.

Proof. Since $0 < t_n < 1$, it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} |a_{nk}| &= \frac{1}{k+1} \sum_{n=0}^{\infty} \frac{-1}{\log(1-t_n)} t_n^{k+1} \\ &\leq \sum_{n=0}^{\infty} \frac{-1}{\log(1-t_n)}\end{aligned}$$

for every k. Thus, if $1/\log(1-t) \in l$, the Knopp-Lorentz theorem [**5**] guarantees that L_t is an l-l matrix. Conversely, if $1/\log(1-t) \notin l$, then, considering the sum of the first column of L_t, we have

$$\sum_{n=0}^{\infty} |a_{n,0}| = \sum_{n=0}^{\infty} \frac{-t_n}{\log(1-t_n)} = \infty,$$

so the condition of the Knopp-Lorentz theorem [**5**] fails to hold, and hence L_t is not an l-l matrix. □

Corollary 1. *If $0 < t_n < w_n < 1$ and L_t is an l-l matrix, then L_w is also an l-l matrix.*

Proof. Since the hypothesis implies that

$$\frac{-1}{\log(1-t_n)} > \frac{-1}{\log(1-w_n)},$$

the assertion easily follows by Theorem 1. □

Corollary 2. *If L_t is an l-l matrix, then* $\arcsin(1-t) \in l$.

Proof. By Theorem 1 we have $1/\log(1-t) \in l$, and this yields $(1-t) \in l$ using the inequality $\log(1/1-t_n) < 1/(1-t_n)$. Now observe that, for $0 < t_n < 1$, we have

$$\arcsin(1 - t_n) < \frac{1 - t_n}{\sqrt{1 - (1 - t_n)^2}},$$

and consequently $\arcsin(1 - t) \in l$. $\square$

Corollary 3. *Suppose $\alpha > -1$ and L_t is an l-l matrix; then $(1 - t)^{\alpha+1} \in l$.*

Proof. It is easy to see that

$$\frac{1}{k+1} \leq M_1 \binom{k+\alpha}{k}, \quad \text{for some } M_1 > 0,$$

and this yields

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{k+1} t_n^{k+1} &\leq M_1 \sum_{k=0}^{\infty} \binom{k+\alpha}{k} t_n^{k+1} \\ &= \frac{M_1 t_n}{(1 - t_n)^{\alpha+1}}. \end{aligned}$$

Now it follows that

$$-\log(1 - t_n) < \frac{M_1}{(1 - t_n)^{\alpha+1}},$$

and, consequently, we have

$$\frac{-1}{\log(1 - t_n)} > \frac{(1 - t_n)^{\alpha+1}}{M_1}.$$

The hypothesis that L_t is l-l implies that $1/\log(1-t) \in l$ by Theorem 1, and hence $(1 - t)^{\alpha+1} \in l$. $\square$

The following result gives a relationship between the logarithmic matrix L_t and the zeta matrix Z_w introduced by Chu in [**2**].

Theorem 2. *Suppose* $w_n = 1/t_n$ *and* L_t *is an* l-l *matrix*; *then the zeta matrix* Z_w *is also an* l-l *matrix.*

Proof. If L_t is l-l, then by Theorem 1, $1/\log(1-t) \in l$ and this gives us $(1-t) \in l$. Now $(1-t) \in l$ implies that $(w-1) \in l$, and the theorem follows by Theorem 5 [**2**]. □

Remark 1. The converse to Theorem 2 is not true. To see this, let

$$w_n = 1/t_n \quad \text{and} \quad t_n = 1 - (n+2)^{-2}.$$

Then, by Theorem 5 [**2**], Z_w is l-l, but by Theorem 1, L_t is not l-l.

Our next theorem has the form of an extension mapping theorem. It indicates that a mapping of L_t from G_w into l can be extended to a mapping of l into l.

Theorem 3. *The following statements are equivalent*:

(1) L_t *is an* l-l *matrix*;

(2) L_t *is a* G-l *matrix*;

(3) L_t *is a* G_w-l *matrix.*

Proof. Since G is a subset of l and G_w is a subset of G, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follow easily. The assertion that (3) $\Rightarrow$ (1) follows by Theorem 1.1 [**6**] and Theorem 1. □

Corollary 4. (1) *If* L_t *is a* G-G *matrix, then* L_t *is an* l-l *matrix.*

(2) *If* L_t *is a* G_w-G_w *matrix, then* L_t *is an* l-l *matrix.*

Proof. Since both G and G_w are subsets of l, the corollary follows easily by Theorem 3. □

The next result suggests that the logarithmic matrix L_t is l-stronger than the identity matrix. The result indicates that the L_t matrix is a rather strong method in the l-l setting.

Theorem 4. *If L_t is an l-l matrix and the series $\sum_{k=0}^{\infty} x_k$ has bounded partial sums, then it follows that $x \in l(L_t)$.*

Proof. Let

$$w_n^k = \frac{1}{k+1} t_n^{k+1}, \qquad S_k = \sum_{i=1}^{k} x_i,$$
$$S_0 = x_0 \quad \text{and} \quad |S_k| \le M.$$

Then we have

$$\begin{aligned}
\left|\sum_{k=1}^{m} \frac{1}{k+1} t_n^{k+1} x_k\right| &= \left|\sum_{k=1}^{m} w_n^k x_k\right| \\
&= \left|\sum_{k=1}^{m} w_n^k (S_k - S_{k-1})\right| \\
&= \left|S_m w_n^m + \sum_{k=1}^{m-1} w_n^k S_k - \sum_{k=1}^{m} w_n^k S_{k-1}\right| \\
&= \left|w_n^m S_m + \sum_{k=1}^{m-1} S_k (w_n^k - w_n^{k+1})\right| \\
&< M.
\end{aligned}$$

This yields that

$$\left|\sum_{k=1}^{\infty} \frac{1}{k+1} t_n^{k+1} x_k\right| < M,$$

and consequently

$$|(L_t x)_n) < \frac{-M}{\log(1-t_n)}.$$

Thus, if L_t is an l-l matrix, then by Theorem 1, $1/\log(1-t) \in l$, so $x \in l(L_t)$. □

Remark 2. Theorem 4 indicates that, if L_t is l-l, then $l(L_t)$ contains the class of all sequences x such that $\sum_{k=0}^{\infty} x_k$ is conditionally convergent. This suggests how large the size of $l(L_t)$ is. In fact, we give a further indication of the size of $l(L_t)$ by showing that, if L_t is an l-l

matrix, then $l(L_t)$ also contains an unbounded sequence. To see this, consider the sequence x given by

$$x_k = (-1)^k (k+1)^2.$$

Then

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{1}{k+1} x_k t_n^{k+1} &= t_n \sum_{k=0}^{\infty} (-1)^k (k+1) t_n^k \\ &= \frac{t_n}{(1+t_n)^2}.\end{aligned}$$

Hence,

$$\begin{aligned}(L_t x)_n &= \frac{t_n}{-\log(1-t_n)(1+t_n)^2} \\ &< \frac{-1}{\log(1-t_n)}.\end{aligned}$$

Thus, if L_t is an l-l matrix, then, by Theorem 1, $1/\log(1-t) \in l$, so $x \in l(L_t)$.

Lemma. *The complex number sequence x is in the domain of the matrix L_t if and only if*

$$\limsup_k |x_k|^{1/k} \leq 1.$$

Proof. If x is in the domain of L_t, then we have

$$\sum_{k=0}^{\infty} a_{nk} x_k < \infty, \quad \text{for each } n \geq 0.$$

This yields that

$$\frac{-1}{\log(1-t_n)} \sum_{k=0}^{\infty} \frac{1}{k+1} x_k t_n^{k+1} < \infty, \quad \text{for } 0 < t_n < 1,$$

and hence the radius of convergence of the power series

$$(*) \qquad \sum_{k=0}^{\infty} \frac{1}{k+1} x_k z^{k+1}$$

is at least 1. Consequently, we have

$$\limsup_k |x_k|^{1/k} \leq 1.$$

Conversely, if $\limsup_k |x_k|^{1/k} \leq 1$, then it follows that the radius of convergence of the power series $(*)$ is at least 1. Since $0 < t_n < 1$ for all n, we have

$$\sum_{k=0}^{\infty} a_{nk} x_k < \infty, \quad \text{for each } n \geq 0.$$

Hence, x is in the domain of L_t. $\square$

Example 1. The L_t matrix is not l-stronger than the familiar Euler-Knopp matrix E_r for $r \in (0, 1)$. To see this, consider the sequence x given by

$$x_k = (-q)^k,$$
$$r = 1/q \quad \text{and} \quad s = 1 - 1/q,$$

where $q > 1$. Then we have

$$\begin{aligned} |(E_{1/q}x)_n| &= \left| \sum_{k=0}^{\infty} \binom{n}{k} \frac{1}{q^k} (s)^{n-k} (-q)^k \right| \\ &= |(-1+s)^n| \\ &= \frac{1}{q^n}. \end{aligned}$$

Since $q > 1$, we have $E_{1/q}x \in l$, and hence $x \in l(E_t)$ but $x \notin l(L_t)$ by the above lemma. Thus, L_t is not l-stronger than E_r.

Our next theorem gives a necessary and sufficient condition for $d(L_t)$ to be equal to $l(L_t)$.

Theorem 5. *The following statements are equivalent*:

(1) $l(L_t) = d(L_t)$;

(2) *There exist numbers* M *and* r *such that* $0 < r < 1$ *and*

$$\sum_{n=0}^{\infty} |a_{nk}| \leq M r^k,$$

for every integer k.

Proof. Suppose (1) is true. By the above lemma, we have

$$d(L_t) = \Big\{x : \limsup_k |x_k|^{1/k} \le 1\Big\}.$$

The assumption that (1) holds implies that L_t maps $d(L_t)$ into l and, by Corollary 9 of [**4**], it follows that (2) holds. Conversely, if (2) holds then by Corollary 9 of [**4**] L_t maps $d(L_t)$ into l. This yields $d(L_t) = l(L_t)$ and hence (1) holds. □

The next main result suggests that L_t is l-translative for certain sequences in $l(L_t)$.

Theorem 6. *Every l-l L_t matrix is l-translative for each L-summable sequence in $l(L_t)$.*

Proof. Let $x \in c(L) \cap l(L_t)$. Then we will show that

(1) $T_x \in l(L_t)$ and

(2) $S_x \in l(L_t)$,

where T_x and S_x are as in Definition 2. Let us first show that (1) holds. Note that

$$\begin{aligned}
|(L_t T_x)_n| &= \frac{-1}{\log(1-t_n)}\left|\sum_{k=0}^{\infty}\frac{1}{k+1}x_{k+1}t_n^{k+1}\right| \\
&= \frac{-1}{\log(1-t_n)}\left|\sum_{k=1}^{\infty}\frac{1}{k}x_k t_n^{k}\right| \\
&= \frac{-1}{\log(1-t_n)}\left|\sum_{k=1}^{\infty}\left(\frac{1}{k+1}+\frac{1}{k(k+1)}\right)x_k t_n^{k}\right| \\
&\le \frac{-1}{\log(1-t_n)}\left|\sum_{k=1}^{\infty}\frac{1}{k+1}x_k t_n^{k}\right| \\
&\quad - \frac{1}{\log(1-t_n)}\left|\sum_{k=1}^{\infty}\frac{x_k t_n^k}{k(k+1)}\right|.
\end{aligned}$$

The use of the triangle inequality is legitimate as the radii of convergence of the two power series is at least 1. Now let us define

$$A_n = \frac{-1}{\log(1-t_n)}\left|\sum_{k=1}^{\infty}\frac{1}{k+1}x_k t_n^k\right|$$

and

$$B_n = \frac{-1}{\log(1-t_n)}\left|\sum_{k=1}^{\infty}\frac{1}{k(k+1)}x_k t_n^k\right|.$$

So we have

$$|(L_t T_x)_n| \le A_n + B_n,$$

and if we show that both A and B are in l, then (1) holds. The condition $A \in l$ follows from the hypothesis that $x \in l(L_t)$ and $B \in l$ will be shown as follows. Observe that

$$\begin{aligned} B_n &= \frac{-1}{\log(1-t_n)}\left|\frac{1}{2}x_1 t_n + \frac{1}{6}x_2 t_n^2 + \sum_{k=3}^{\infty}\frac{1}{k(k+1)}x_k t_n^k\right| \\ &\le \frac{-|x_1|t_n}{2\log(1-t_n)} - \frac{|x_2|t_n^2}{6\log(1-t_n)} \\ &\quad - \frac{1}{\log(1-t_n)}\left|\sum_{k=3}^{\infty}\frac{1}{k(k+1)}x_k t_n^k\right|. \end{aligned}$$

Next define

$$C_n = \frac{-|x_1|t_n}{2\log(1-t_n)} - \frac{|x_2|t_n^2}{6\log(1-t_n)}$$

and

$$D_n = \frac{-1}{\log(1-t_n)}\left|\sum_{k=3}^{\infty}\frac{1}{k(k+1)}x_k t_n^k\right|.$$

By Theorem 1, the hypothesis that L_t is l-l implies that $C \in l$, and hence there remains only to show that $D \in l$. Note that

$$\begin{aligned} D_n &= \frac{-1}{\log(1-t_n)}\left|\sum_{k=3}^{\infty}\frac{x_k}{(k+1)}\left(\int_0^{t_n} t^{k-1}\,dt\right)\right| \\ &= \frac{-1}{\log(1-t_n)}\left|\int_0^{t_n} dt\left(\sum_{k=3}^{\infty}\frac{1}{(k+1)}x_k t^{k-1}\right)\right|. \end{aligned}$$

The interchanging of the integral and the summation is legitimate as the radius of convergence of the power series

$$\sum_{k=3}^{\infty} \frac{1}{k+1} x_k t^{k-1}$$

is at least 1 by the above lemma, and hence the power series converges absolutely and uniformly for $0 \leq t \leq t_n$.

Now we let

$$F(t) = \sum_{k=3}^{\infty} \frac{1}{k+1} x_k t^{k-1}.$$

Then we have

$$\frac{F(t)}{-\log(1-t)} = \frac{-1}{\log(1-t)} \sum_{k=3}^{\infty} \frac{1}{k+1} x_k t^{k-1},$$

and the hypothesis that $x \in c(L)$ implies that

$$\lim_{t \to 1^-} \frac{F(t)}{-\log(1-t)} = A \text{ (finite)}, \quad \text{for } 0 < t < 1. \tag{1}$$

We also have

$$\lim_{t \to 0} \frac{F(t)}{-\log(1-t)} = 0. \tag{2}$$

Now (1) and (2) yield that

$$\left| \frac{F(t)}{-\log(1-t)} \right| \leq M, \quad \text{for some } M > 0,$$

and hence

$$|F(t)| \leq -M \log(1-t).$$

So we have

$$\begin{aligned} D_n &= \frac{-1}{\log(1-t_n)} \left| \int_0^{t_n} F(t)\,dt \right| \\ &\leq \frac{-1}{\log(1-t_n)} \int_0^{t_n} |F(t)|\,dt \\ &\leq \frac{-M}{\log(1-t_n)} \int_0^{t_n} -\log(1-t)\,dt \\ &= -M(1-t_n) - \frac{M t_n}{\log(1-t_n)}. \end{aligned}$$

The hypothesis that L_t is l-l implies that both $1/\log(1-t)$ and $(1-t)$ are in l, and hence $D \in l$.

Next we will show that (2) holds. We have

$$
(*)\qquad \begin{aligned}
|(L_tS_x)_n| &= \frac{-1}{\log(1-t_n)}\left|\sum_{k=1}^{\infty}\frac{1}{k+1}x_{k-1}t_n^{k+1}\right| \\
&= \frac{-1}{\log(1-t_n)}\left|\sum_{k=0}^{\infty}\frac{1}{k+2}x_k t_n^{k+2}\right| \\
&= \frac{-1}{\log(1-t_n)}\left|\sum_{k=0}^{\infty}\left(\frac{1}{k+1}-\frac{1}{(k+1)(k+2)}\right)x_k t_n^{k+2}\right| \\
&\le E_n + F_n,
\end{aligned}
$$

where

$$
\begin{aligned}
E_n &= \frac{-1}{\log(1-t_n)}\left|\sum_{k=0}^{\infty}\frac{1}{k+1}x_k t_n^{k+2}\right| \\
F_n &= \frac{-1}{\log(1-t_n)}\left|\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+2)}x_k t_n^{k+2}\right|.
\end{aligned}
$$

The use of the triangle inequality in $(*)$ is justified as above. The hypothesis that $x \in l(L_t)$ implies that $E \in l$, and we can show that $F \in l$ as follows. Note that

$$
\begin{aligned}
F_n &= \frac{-1}{\log(1-t_n)}\left|\frac{x_0t_n^2}{2}+\sum_{k=1}^{\infty}\frac{1}{(k+1)(k+2)}x_k t_n^{k+2}\right| \\
&\le G_n + H_n,
\end{aligned}
$$

where

$$
G_n = \frac{-|x_0|t_n^2}{2\log(1-t_n)}
$$

and

$$
H_n = \frac{-1}{\log(1-t_n)}\left|\sum_{k=1}^{\infty}\frac{1}{(k+1)(k+2)}x_k t_n^{k+2}\right|.
$$

By Theorem 1, the hypothesis that L_t is l-l implies that $G \in l$, and hence there remains only to show that $H \in l$. Observe that

$$\begin{aligned} H_n &= \frac{-1}{\log(1-t_n)} \left| \sum_{k=1}^{\infty} \frac{x_k}{(k+1)} \left(\int_0^{t_n} t^{k+1}\, dt \right) \right| \\ &= \frac{-1}{\log(1-t_n)} \left| \int_0^{t_n} dt \left(\sum_{k=1}^{\infty} \frac{1}{(k+1)} x_k t^{k+1} \right) \right|. \end{aligned}$$

The interchanging of the integral and the summation is justified as above. Now, proceeding as in the proof of (1) above, we can easily show that $H \in l$ and consequently our assertion follows. □

Corollary 5. *Every l-l L_t matrix is l-translative for the sequence x such that $\sum_{k=1}^{\infty} x_k$ has bounded partial sums.*

Proof. By Theorem 4, $x \in l(L_t)$ and also it is easy to see that $x \in c(L)$. Thus, by Theorem 6, the assertion follows. □

Example 2. Every l-l L_t matrix is l-translative for the unbounded sequence x given by

$$x_k = (-1)^k (k+1)^2.$$

Since $x \in l(L_t)$, by Remark 2, and also x is L-summable to 0, the assertion follows by Theorem 6.

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