ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 1, Spring 1998

ON PROPERTIES OF MULTIPLIERS OF CAUCHY TRANSFORMS

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ABSTRACT. In this paper we prove that the lengths of images of certain rectifiable arcs under a multiplier f of fractional analytic Cauchy-Stieltjes transforms on the disk are uniformly bounded by a constant depending on the multiplier norm of f. As a consequence of this result, we also prove that $|f'(z)|^2$ is integrable with respect to area measure on every Stolz angle. Finally, we prove that our results are sharp in two different senses.

1. Introduction. Let $\Delta = \{z : |z| < 1\}$ and let $\Gamma = \{z : |z| = 1\}$. Let $\mathcal{M}$ denote the set of complex-valued Borel measures on Γ. For each $\alpha > 0$, let $\mathcal{F}_\alpha$ denote the family of functions f having the property that there exists a measure $\mu \in \mathcal{M}$ such that

$$f(z) = \int_\Gamma \frac{1}{(1-\bar{x}z)^\alpha}\, d\mu(x) \tag{1}$$

for $|z| < 1$. In (1) and throughout this paper, each logarithm means the principal branch. $\mathcal{F}_\alpha$ is a Banach space with respect to the norm defined by

$$\|f\|_{\mathcal{F}_\alpha} = \inf\{\|\mu\|\} \tag{2}$$

where μ varies over all measures in $\mathcal{M}$ for which (1) holds and where $\|\mu\|$ denotes the total variation norm of μ. For $\alpha = 0$, let $\mathcal{F}_0$ denote the family of functions f having the property that there exists a measure $\mu \in \mathcal{M}$ such that

$$f(z) = f(0) + \int_\Gamma \log \frac{1}{(1-\bar{x}z)}\, d\mu(x) \tag{3}$$

Received by the editors on August 17, 1995.
1991 AMS *Mathematical Subject Classification.* Primary 30E20, Secondary 30D99.
Key words and phrases. Multipliers, fractional Cauchy transform.

for $|z| < 1$. $\mathcal{F}_0$ is a Banach space with respect to the norm defined by $\|f\|_{\mathcal{F}_0} = \inf\{\|\mu\|\} + |f(0)|$ where μ varies over all measures in M for which (3) holds. A function f is called a multiplier of $\mathcal{F}_\alpha$ provided $fg \in \mathcal{F}_\alpha$ for every $g \in \mathcal{F}_\alpha$. Let M_α denote the set of multipliers of $\mathcal{F}_\alpha$. M_α is a Banach space with respect to the norm defined by

$$\|f\|_{M_\alpha} = \sup\{\|fg\|_{F_\alpha} : g \in \mathcal{F}_\alpha, \|g\|_{\mathcal{F}_\alpha} \leq 1\}. \tag{4}$$

Some properties of M_α were derived in [**3**]. It was proved that, if $0 < \alpha < \beta$, then $M_\alpha \subset M_\beta$ [**3**]. Further, let $\alpha > 0$ and let $f \in M_\alpha$, then $f \in H^\infty$ and $\|f\|_{H^\infty} \leq \|f\|_{M_\alpha}$ [**3**]. Let $V(f,\theta) = \int_0^1 |f'(re^{i\theta})|\,dr$, the radial variation of f in the direction θ. In [**3**] it was proved that, if $f \in M_\alpha$, $\alpha > 0$, then there is a constant A depending only on α such that $V(f,\theta) \leq A\|f\|_{M_\alpha}$. In [**2**] all of these results were extended to the case $\alpha = 0$. When $\alpha = 1$ and $f \in M_1$, the uniform boundedness of $V(f,\theta)$ was proved in [**5**]. We prove that these results on the uniform boundedness of $V(f,\theta)$ for $f \in M_\alpha$, $\alpha \geq 0$, are in a certain sense sharp results that cannot be improved. If $\alpha \geq 0$ and $f \in M_\alpha$, then it follows from the previously mentioned fact that the lengths of the images under f of each diameter are uniformly bounded. In a private conversation, T.H. MacGregor raised the question with one of the authors, of whether the lengths of the images of all chords of Δ of fixed length $l < 2$ under a multiplier would be uniformly bounded.

In this paper we prove a general result for some rectifiable curves which provides a positive answer for MacGregor's question. Furthermore, as a consequence of our result, we prove that if $f \in M_\alpha$, $\alpha \geq 0$, then $|f'(z)|^2$ is integrable with respect to area measure on every Stolz angle $S(\theta)$ with vertex at $e^{i\theta}$. Finally, we also generalize the result [**3, 5**] on radial variation to higher derivatives.

Throughout the paper we use A_1, A_2, etc., to denote certain absolute constants. Also, we use A, B, C, etc., to denote constants depending on various parameters or assumptions. The meaning of A, B, C, etc., may change within an argument or even within a line.

2. Main results. For each $\beta \in (0,\pi/2)$ and for any real θ, let $S_\beta(\theta)$ denote the convex hull of $\{z : |z| \leq \sin\beta\} \cup \{e^{i\theta}\}$.

Lemma 1. *Let*

$$g(z,x,\theta)=\alpha\frac{(e^{i\theta}-z)^{\alpha-1}(e^{i\theta}\bar{x}-1)}{(1-\bar{x}z)^{\alpha+1}}$$

where $|x|=1$. Then there exists a constant $C_1>0$ depending only on β and α such that

$$|g(z,x,\theta)|\le C_1\frac{|e^{i\theta}-x|}{|e^{i\theta}-z|^2} \tag{5}$$

and

$$|g(z,x,\theta)|\le C_1\frac{|e^{i\theta}-z|^{\alpha-1}}{|e^{i\theta}-x|^{\alpha}}, \tag{6}$$

when $z\in S_\beta(\theta)$.

Proof. We first prove (5). Since $z\in S_\beta(\theta)$ there exists a positive constant A such that $|e^{i\theta}-z|\le A(1-|z|)$ and A depends only on β. We easily infer from this that, for each $|x|=1$ and each $z\in S_\beta(\theta)$, we have

$$\frac{1}{|1-\bar{x}z|}\le\frac{A}{|e^{i\theta}-z|}. \tag{7}$$

The absolute value of the function g may be rewritten as

$$|g(z,x,\theta)|=\alpha\left(\frac{|e^{i\theta}-z|}{|1-\bar{x}z|}\right)^{\alpha+1}\frac{1}{|e^{i\theta}-z|^2}|e^{i\theta}-x|. \tag{8}$$

Now (7) and (8) give (5) with $C_1=\alpha A^{\alpha+1}$. To prove (6), first suppose $|e^{i\theta}-z|\ge(1/2)|e^{i\theta}-x|$ and $z\in S_\beta(\theta)$. This, together with (7), implies that

$$\frac{1}{|1-\bar{x}z|}\le\frac{2A}{|e^{i\theta}-x|}. \tag{9}$$

We infer from (9) that

$$\begin{aligned}|g(z,x,\theta)|&\le\alpha|e^{i\theta}-z|^{\alpha-1}\frac{2^{\alpha+1}A^{\alpha+1}}{|e^{i\theta}-x|^{\alpha+1}}|e^{i\theta}-x|\\&=2^{\alpha+1}\alpha A^{\alpha+1}\frac{|e^{i\theta}-z|^{\alpha-1}}{|e^{i\theta}-x|^{\alpha}}.\end{aligned} \tag{10}$$

Now suppose $|e^{i\theta} - z| \le (1/2)|e^{i\theta} - x|$ and $z \in S_\beta(\theta)$. Then

$$|1 - \bar{x}z| = |z - x| \ge |e^{i\theta} - x| - |z - e^{i\theta}| \ge \frac{1}{2}|e^{i\theta} - x|. \tag{11}$$

Hence, in this case, (11) gives

$$\begin{aligned} |g(z,x,\theta)| &\le \alpha 2^{\alpha+1} \frac{|e^{i\theta} - z|^{\alpha-1}|e^{i\theta} - x|}{|e^{i\theta} - x|^{\alpha+1}} \\ &= \alpha 2^{\alpha+1} \frac{|e^{i\theta} - z|^{\alpha-1}}{|e^{i\theta} - x|^{\alpha}}. \end{aligned} \tag{12}$$

Now let $C_1 = \max\{\alpha A^{\alpha+1}, 2^{\alpha+1}\alpha A^{\alpha+1}, \alpha 2^{\alpha+1}\}$. Then (7), (8), (10) and (12) give (5) and (6). □

We next prove our main results. Corollaries 1 and 2 to Theorem 1 give the positive results mentioned in the introduction.

Theorem 1. *Suppose $f \in M_\alpha$ for $\alpha \ge 0$. Let $z(t)$, $0 \le t \le \eta$ denote a rectifiable arc γ parametrized by arc length, contained in a Stolz angle $S_\beta(\theta)$ at $e^{i\theta}$ such that $z(0) = e^{i\theta}$. Further, suppose there exists a positive constant α such that $at \le |z(t) - e^{i\theta}|$ for all $t \in [0, \eta]$. Then*

$$\int_\gamma |f'(z)|\,|dz| \le C\|f\|_{M_\alpha} \tag{13}$$

where the constant C depends on α, a and the angular opening β.

Proof. Since $\|f\|_{M_\alpha} \le A_1\|f\|_{M_0}$ for each $\alpha > 0$, where the constant A_1 depends only on α [**2**], we may and do assume that $\alpha > 0$. Let $f \in M_\alpha$. For each θ we can select a measure $\mu \in \mathcal{M}$ such that

$$(e^{i\theta} - z)^{-\alpha} f(z) = \int_\Gamma (1 - \bar{x}z)^{-\alpha}\, d\mu(x) \tag{14}$$

and $\|\mu\| \le 2\|f(\cdot)(e^{i\theta} - \cdot)^{-\alpha}\|_{\mathcal{F}_\alpha} \le 2\|f\|_{M_\alpha}$. It follows from (14) that

$$f'(z) = \int_\Gamma g(z,x)\, d\mu(x), \tag{15}$$

where

$$g(z,x) = g(z,x,\theta) = \alpha \frac{(e^{i\theta}-z)^{\alpha-1}(e^{i\theta}\bar{x}-1)}{(1-\bar{x}z)^{\alpha+1}}.$$

By (15), we have

$$\begin{aligned} \int_\gamma |f'(z)|\,|dz| &= \int_0^\eta |f'(z(t))|\,dt \\ &= \int_0^\eta \left| \int_\Gamma g(z(t),x,\theta)\,d\mu(x)\right| dt \\ &\le \int_\Gamma \left(\int_0^\eta |g(z(t),x,\theta)|\,dt \right) d|\mu|(x). \end{aligned} \tag{16}$$

Set $I(x) = \int_0^\eta |g(z(t),x,\theta)|\,dt$. Then, by (16), we have

$$\begin{aligned} \int_\gamma |f'(z)|\,|dz| &\le \int_\Gamma I(x)\,d\mu(x) \\ &\le \sup_{|x|=1} I(x)\|\mu\| \\ &\le 2 \sup_{|x|=1} I(x)\|f\|_{M_\alpha}. \end{aligned} \tag{17}$$

To complete this proof it is enough to show that $I(x) \le B_1$, where B_1 depends only on α and the angular opening β of the Stolz region.

Let us write

$$I(x) = \int_0^\eta |g(z(t),x,\theta)|\,dt = L_1 + L_2, \tag{18}$$

where $L_k = \int_{T_k} |g(z(t),x,\theta)|\,dt$, $k=1,2$, with $T_1 = \{t : |z(t)-e^{i\theta}| > |e^{i\theta}-x|\}$, and $T_2 = \{t : |z(t)-e^{i\theta}| \le |e^{i\theta}-x|\}$. In the case when T_1 is nonempty we estimate L_1 using (5) from Lemma 1, as follows

$$\begin{aligned} L_1 &\le C_1 \int_{T_1} \frac{|e^{i\theta}-x|}{|e^{i\theta}-z(t)|^2}\,dt \\ &\le \frac{C_1|e^{i\theta}-x|}{a^2} \int_{T_1} \frac{dt}{t^2}. \end{aligned} \tag{19}$$

But, recalling that the curve is parametrized by arc length, we have $|z(t) - e^{i\theta}| \le t$ and so $\inf T_1 = \inf\{t > 0 : |z(t) - e^{i\theta}| > |e^{i\theta} - x|\} \ge \inf\{t > 0 : t > |e^{i\theta} - x|\} = |e^{i\theta} - x|$. Therefore, by (19), we have

$$L_1 \le \frac{C_1|e^{i\theta} - x|}{a^2} \int_{|e^{i\theta}-x|}^{\infty} \frac{dt}{t^2} = \frac{C_1}{a^2}. \tag{20}$$

If T_2 is nonempty, to estimate L_2 we first note that $\sup\{t > 0 : |z(t) - e^{i\theta}| \le |e^{i\theta} - x|\} \le \sup\{t > 0 : at \le |e^{i\theta} - x|\} = |e^{i\theta} - x|/a$. Hence, by (6) of Lemma 1, we have in the case, $0 < \alpha < 1$,

$$\begin{aligned} L_2 &\le C_1 \int_0^{\sup T_2} \frac{|z(t) - e^{i\theta}|^{\alpha-1}}{|e^{i\theta} - x|^{\alpha}}\, dt \\ &\le C_1 \int_0^{|e^{i\theta}-x|/a} \frac{a^{\alpha-1}t^{\alpha-1}}{|e^{i\theta} - x|^{\alpha}}\, dt \\ &= \frac{C_1}{a}. \end{aligned} \tag{21}$$

In the case of $1 \le \alpha < \infty$, we have by (6)

$$\begin{aligned} L_2 &\le C_1 \int_0^{|e^{i\theta}-x|/a} \frac{|z(t) - e^{i\theta}|^{\alpha-1}}{|e^{i\theta} - x|^{\alpha}}\, dt \\ &\le C_1 \int^{|e^{i\theta}-x|/a} \frac{t^{\alpha-1}}{|e^{i\theta} - x|^{\alpha}}\, dt \\ &= \frac{C_1}{a}. \end{aligned} \tag{22}$$

Now (13) follows from (17), (18), (20), (21) and (22), and the proof is complete. □

Remark. If γ is a chord of the unit disk of the length l, $0 < l \le 2$, then each of the halves of γ satisfies the assumption of Theorem 1 with $\beta = \arccos(l/2)$, $a = 1$ and $e^{i\theta}$ being one of the endpoints of the chord. Moreover, the constant C_1 in Lemma 1 can be taken to be $(C/\cos\beta)^{\alpha+1}$, where C is an absolute constant. Therefore, the proof of Theorem 1 gives the following corollary.

Corollary 1. *Let γ denote a chord of length $l \leq 2$ in Δ. If $f \in M_\alpha$ for $\alpha > 0$, then*

$$\int_\gamma |f'(z)|\,|dz| \leq \left(\frac{C_3}{l}\right)^{\alpha+1} \|f\|_{M_\alpha} \tag{23}$$

where C_3 is an absolute constant.

Corollary 2. *If $f \in M_\alpha$ for $\alpha \geq 0$, then, for fixed $\beta \in (0, \pi/2)$,*

$$\iint_{S_\beta(\theta)} |f'(z)|^2\, dA(z) \leq C\|f\|_\infty \|f\|_{M_\alpha} \tag{24}$$

where $dA(z)$ denotes area measure and C depends only on α and β.

Proof. There is no loss of generality in assuming that $\theta = 0$. Integrating in polar coordinates, we obtain

$$\begin{aligned}\iint_{S_\beta(\theta)} &|f'(z)|^2\, dA(z) \\ &\leq \int_{-\beta}^{\beta} \int_0^{\cos s + \sqrt{\sin^2\beta - \sin^2 s}} |f'(1 - te^{is})|^2 t\, dt\, ds.\end{aligned} \tag{25}$$

For each fixed $\beta \in (0, \pi/2)$ there is a constant A depending only on β such that for $z \in S_\beta(0)$ we have $|1 - z| \leq A(1 - |z|)$. Hence we have

$$\begin{gathered}|f'(z)| \leq \frac{\|f\|_\infty}{1 - |z|} \leq \frac{A\|f\|_\infty}{|1 - z|}, \\ z \in S_\beta(0).\end{gathered} \tag{26}$$

Now (25) and (26) give

$$\begin{aligned}\iint_{S_\beta(\theta)} &|f'(z)|^2\, dA(z) \\ &\leq A\|f\|_\infty \int_{-\beta}^{\beta} \int_0^{\cos s + \sqrt{\sin^2\beta - \sin^2 s}} |f'(1 - te^{is})|\, dt\, ds.\end{aligned} \tag{27}$$

Since the curve $\gamma(t) = 1 - te^{is}$, $0 \le t \le \cos s + \sqrt{\sin^2\beta - \sin^2 s}$ satisfies the assumptions of Theorem 1 with $a = 1$ with each $s \in (-\beta, \beta)$; applying Theorem 1 to the inner integral on the righthand side of (26) we obtain (24), where the constant C depends only on α and β. □

Remark. Corollary 2 is sharp in the sense that the integrand in (24) can neither in general be replaced by $\phi(|f'(z)|)$ where $\phi(t)$, $0 \le t < \infty$ is a positive nondecreasing function with $\lim_{t\to\infty}\phi(t)/t^2 = \infty$ nor by $\psi(|z|)|f'(z)|^2$, where $\psi(t)$ is a measurable positive function on $[0,1]$ with $\lim_{\rho\to 1^-}\psi(\rho) = \infty$. The proofs of these two assertions are very similar to the proofs of Theorems 3 and 4 to follow, and we do not give details in the paper.

Theorem 2 is a technical result needed for the proofs of Theorems 3 and 4. These last mentioned results show that Theorem 1 is a sharp result in at least two senses.

Theorem 2. (i) *Let $k(z) = (1-z)^{-1}$, and let $k_r(z) = k(rz)$. Then, for each $\alpha > 0$, there is a constant D_1 depending only on α such that*

$$\|k_r\|_{M_\alpha} \le D_1 \frac{1}{1-r}, \quad 0 \le r < 1. \tag{28}$$

(ii) *Let $k^*(z) = -\log(1-z)$, and let $k_r^*(z) = k^*(rz)$. Then there is a constant D_2 independent of r such that*

$$\|k_r^*\|_{M_0} \le D_2 \log\frac{1}{1-r}, \quad \frac{1}{2} < r < 1. \tag{29}$$

Proof. To prove (i), note that $\|(k_r)'\|_{H^1(\Delta)} \le C/(1-r)$, $0 \le r < 1$, for some constant C independent of r. Since, by a generalization [**3**, Theorem 3.5] of a result of [**5**], we have

$$\|f\|_{M_\alpha} \le C(\|f'\|_{H^1(\Delta)} + |f(0)|), \tag{30}$$

with a constant C that does not depend on the function f, part (i) follows.

To prove (ii) it is enough to prove that there is a constant C independent of $r \in (1/2, 1)$ and of $x \in \Gamma$ such that

$$\|k_r^* k^*(x\cdot)\|_{\mathcal{F}_0} \le C \log \frac{1}{1-r}. \tag{31}$$

We observe that

$$\begin{aligned} \|k_r^* k^*(x\cdot)\|_{\mathcal{F}_0} &= \|[k_r^* k^*(x\cdot)]'\|_{\mathcal{F}_1} \\ &\le \|k_r^* k(x\cdot)\|_{\mathcal{F}_1} + \|k_r k^*(x\cdot)\|_{\mathcal{F}_1}. \end{aligned} \tag{32}$$

The first term in (32) may be estimated using (30) as follows

$$\|k_r^* k(x\cdot)\|_{\mathcal{F}_1} \le \|k_r^*\|_{M_1} \le C\|(k_r^*)'\|_{H^1(\Delta)} \le C \log \frac{1}{1-r}. \tag{33}$$

Since $k_r(z) = (1/2\pi) k * P_r(z)$, where P_r denotes the Poisson kernel, we have $\|k_r\|_{\mathcal{F}_1} = \|k\|_{\mathcal{F}_1} = 1$. Using this to estimate the second term, note first that

$$\begin{aligned} \|k_r k^*(x\cdot)\|_{\mathcal{F}_1} &\le \|k_r(k^*(x\cdot)+1)\|_{\mathcal{F}_1} + \|k_r\|_{\mathcal{F}_1} \\ &= \|k_r \cdot (k^*(x\cdot)+1)\|_{\mathcal{F}_1} + 1. \end{aligned} \tag{34}$$

The first term following the last inequality sign in (34) can be estimated as follows

$$\begin{aligned} \|k_r \cdot (k^*(x\cdot)+1)\|_{\mathcal{F}_1} &= \left\| \frac{k_r}{k_r^*+1}(k^*(x\cdot)+1)(k_r^*+1) \right\|_{\mathcal{F}_1} \\ &\le \left\| \frac{k_r}{k_r^*+1}(k^*(x\cdot)+1) \right\|_{\mathcal{F}_1} \|(k_r^*+1)\|_{M_1}. \end{aligned} \tag{35}$$

Since (33) implies $\|(k_r^*+1)\|_{M_1} \le C \log(1/(1-r))$, $0 \le r < 1$, to complete the proof it is enough to show that the $\mathcal{F}_1$ norm of $(k_r/(k_r^*+1))(k^*(x\cdot)+1)$ is bounded uniformly with respect to $0 \le r < 1$, and $|x| = 1$. To this end, we will use the following fact [**3, 5**]

$$\|f\|_{\mathcal{F}_1} = \sup\left\{ \left| \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{it}) \overline{h(e^{it})}\, dt \right| : 0 \le \rho < 1, h \in H^\infty(\Delta), \|h\|_\infty \le 1 \right\}, \tag{36}$$

and also the fact [**2**, Lemma 3.8] that the family

$$\left\{\frac{k}{k^*+1}(k^*(x\cdot)+1) : |x|=1\right\}$$

is bounded in $\mathcal{F}_1$. In particular, there is a constant C independent of x, $|x|=1$, such that for every $h \in H^\infty(\Delta)$ and for every ρ, $0 \le \rho < 1$, we have

$$\left|\int_0^{2\pi} \frac{k(\rho e^{it})}{k^*(\rho e^{it})+1}[k(x\rho e^{it})+1]\overline{h(e^{it})}\,dt\right| \le C\|h\|_\infty. \tag{37}$$

Let $P_r(t) = (1-r^2)/|1-re^{it}|^2$ denote the Poisson kernel for Δ. Since the function $k/(k^*+1)$ is harmonic in Δ, we may write for each $r \in [0,1)$, each $\rho \in [0,1)$, each $x \in \Gamma$, and each $h \in H^\infty(\Delta)$,

$$\begin{aligned}
&\left|\int_0^{2\pi} \frac{k(\rho re^{it})}{k^*(\rho re^{it})+1}\,[k(x\rho e^{it})+1]\overline{h(e^{it})}\,dt\right| \\
&\qquad = \left|\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{k(\rho e^{i(t-\theta)})}{k^*(\rho e^{i(t-\theta)})+1}P_r(\theta)\,d\theta\right]\right. \\
&\qquad\qquad\qquad \left.\cdot\,[k(x\rho e^{it})+1]\overline{h(e^{it})}\,dt\right| \\
&\qquad = \left|\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta)\int_0^{2\pi}\frac{k(\rho e^{i(t-\theta)})}{k^*(\rho e^{i(t-\theta)})+1}\right. \\
&\qquad\qquad\qquad \left.\cdot\,[k(x\rho e^{it})+1]\overline{h(e^{it})}\,dt\,d\theta\right| \\
&\qquad \le \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta)\left|\int_0^{2\pi}\frac{k(\rho e^{i\tau})}{k^*(\rho e^{i\tau})+1}\right. \\
&\qquad\qquad\qquad \left.\cdot\,[k(x\rho e^{i\theta}e^{i\tau})+1]\overline{h(e^{i\theta}e^{i\tau})}\,d\tau\right|d\theta \\
&\qquad \le \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta)C\|h(e^{i\theta}\cdot)\|_\infty\,d\theta \\
&\qquad = C\|h\|_\infty,
\end{aligned} \tag{38}$$

where (37) was used to obtain the last inequality. Since (38) holds for each $\rho \in [0,1)$, and each $h \in H^\infty(\Delta)$, we conclude, by (36), that the $\mathcal{F}_1$ norm of $(k_r/(k_r^*+1))(k^*(x\cdot)+1)$ is bounded uniformly with respect to $0 \le r < 1$ and $|x| = 1$. As explained above, this completes the proof of (29). $\square$

Theorem 3. *Let $\alpha \ge 0$, and let $\phi(t)$, $0 \le t < \infty$, be a positive nondecreasing function with $\lim_{t\to\infty}(\phi(t)/t) = \infty$. Then there is an analytic function $f \in M_\alpha$ such that*

$$\int_0^1 \phi(|f'(\rho)|)\, d\rho = \infty. \tag{39}$$

Proof. Since $M_0 \subset M_\alpha$ [**2**], $\alpha > 0$, it is enough to prove this theorem for $\alpha = 0$. Let

$$g_r(z) = \frac{\log(1/(1-rz))}{\log(1/(1-r))} = \frac{k_r^*}{\log(1/(1-r))}. \tag{40}$$

Note that, by our assumption on ϕ, and an easy computation, we have

$$\lim_{r\to 1^-} \int_0^1 \phi(|2^{-n} g_r'(\rho)|)\, d\rho = \infty$$

for each integer n. In particular, for each positive integer n, we can choose a number $r_n \in (1/2, 1)$ such that $\int_0^1 \phi(|2^{-n} g_{r_n}'(\rho)|)\, d\rho \ge n$. Since, by Theorem 2(ii), the family g_r, $1/2 \le r < 1$, is bounded in M_0, the series $\sum_{n=1}^\infty 2^{-n} g_{r_n}$ is convergent in M_0, and almost uniformly in Δ to a function $f \in M_0$. Since $g_r'(\rho)$ is positive for $\rho \in (0,1)$, and since the function ϕ is nondecreasing, we have for each positive integer n,

$$\int_0^1 \phi(|f'(\rho)|)\, d\rho \ge \int_0^1 \phi(|2^{-n} g_{r_n}'(\rho)|)\, d\rho \ge n. \tag{41}$$

Clearly (41) implies (39). $\square$

Corollary 3. *For each nonnegative number α and each $\rho > 1$ there is a function $f \in M_\alpha$ such that $\int_0^1 |f'(\rho)|^p\, d\rho = \infty$.*

Theorem 4. *Let $\alpha \ge 0$, and let ψ be a measurable positive function with $\lim_{\rho\to 1^-} \psi(\rho) = \infty$. Then there exists a function $f \in M_\alpha$ such that $\int_0^1 \psi(\rho)|f'(\rho)|\, d\rho = \infty$.*

Proof. As in the proof of Theorem 2, we may and do assume that $\alpha = 0$. Let g_r be the function from the proof of Theorem 3. Then again an easy computation using our assumption on ψ gives

$$\lim_{r\to 1^-} \int_0^1 \psi(\rho)|g_r'(\rho)|\,d\rho = \infty. \tag{42}$$

In particular, for each positive integer n, there is a number $r_n \in (1/2, 1)$ with

$$\int_0^1 \psi(\rho)|g_{r_n}'(\rho)|\,d\rho \geq n2^n. \tag{43}$$

Let $f = \sum_{n=1}^{\infty} 2^{-n} g_{r_n}$. The function f is in M_0. Since $g_r'(\rho) > 0$ for $0 < \rho < 1$, we have $|f'(\rho)| \geq |2^{-n} g_{r_n}'(\rho)|$, $0 < \rho < 1$. And, hence, for each positive integer n, we have

$$\int_0^1 \psi(\rho)|f'(\rho)|\,d\rho \geq \int_0^1 \psi(\rho)|2^{-n} g_{r_n}'(\rho)|\,d\rho \geq n.$$

Therefore, $\int_0^1 \psi(\rho)|f'(\rho)|\,d\rho = \infty$. □

Remark. It is known [**1, 5**] that there are multipliers f such that $\iint_\Delta |f'(z)|^2\,dA(z) = +\infty$. Our final theorem is known when $n = 0$ [**3**].

Theorem 5. *If $f \in M_\alpha$ for $\alpha > 0$, then there exists a constant C depending only on α such that*

$$\int_0^1 (1-r)^n |f^{(n+1)}(re^{i\theta})|\,dr \leq A\|f\|_{M_\alpha}$$

for all θ and $n = 0, 1, 2, \ldots$.

Proof. This can be proved with the same technique as in the case $n = 0$ [**3**]. It is only necessary to make careful use of the Leibnitz formula for the nth derivative of a product. We do not give the details. □

Remark. The previous result shows that, when an analytic function f is a multiplier of $\mathcal{F}_\alpha$, $\alpha > 0$, there are strong restrictions on the behavior of all derivatives $f^{(n)}$, $n = 1, 2, \ldots$.

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