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PONTRYAGIN REFLEXIVE GROUPS ARE NOT DETERMINED BY THEIR CONTINUOUS CHARACTERS

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ABSTRACT. A theorem of Glicksberg states that, for an abelian group G, two locally compact topologies with the same set of continuous characters must coincide. In [**12**] it is asserted that this fact also holds for two Pontryagin reflexive topologies. We prove here that this statement is not correct, and we give some additional conditions under which it is true. We provide some examples of classes of groups determined by their continuous characters.

Let G be an abelian topological group. By a character of G we mean a homomorphism of G into the group $T := R/Z$, which can also be identified with the unit circle of the complex plane. By τ, we denote the topology of G, and by τ_ω the weak topology induced by the set $G^\wedge$ of all continuous characters on G. In the literature τ_ω is also called Bohr topology. For brevity, we write G_ω instead of G endowed with τ_ω. If G is a locally compact abelian group (LCA), the following facts hold:

A) G and G_ω have the same compact subsets.

B) The topology of G is determined by the set of all continuous characters it produces, i.e., if τ' is another locally compact topology on G such that (G,τ) and (G,τ') have the same continuous characters, then $\tau=\tau'$.

The results A) and B) were proved by Glicksberg [**5**]. Varoupoulos independently also proved B) [**11**, Section 2].

It is a natural question to extend these results to a class of groups larger than that of LCA groups. In this respect Venkataramann asserts that A) and B) also hold for reflexive groups [**12**, Theorem 1.1 and 1.2]. However, Remus and Trigos have proved in [**8**] that Theorem 1.1

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of [**12**] does not hold, and they ask if Theorem 1.2 of [**12**] could be true in spite of the fact that the proof given relies on Theorem 1.1. We give below (proposition) a family of spaces which are counterexamples to Theorem 1.2. To obtain such a family we consider topological vector spaces in their additive structure, which constitute a qualified class of abelian topological groups.

2. An extension of a theorem of Glicksberg. Denote by X a locally convex vector space, by X^* the set of all continuous linear functionals on X and by $\sigma(X, X^*)$ the weakest topology in X that makes continuous all the elements of X^*. Let τ_b and τ_c be the topologies in X^* of uniform convergence on the bounded subsets of X and on the compact subsets of X, respectively. We call X_b^* and X_c^* the vector space X^* endowed with τ_b and τ_c. The polar set of a subset M of X is denoted by $M^0 := \{f \in X^* : |f(x)| \leq 1, \text{ for all } x \in M\}$, and if N is a subset of X_b^*, $N^0 := \{h \in (X_b^*)^* : |h(x)| \leq 1 \text{ for all } x \in N\}$. A neighborhood basis of zero in τ_b, respectively in τ_c, is given by the polar sets of all bounded, respectively compact, subsets of X.

The space X can be treated as a topological group, and in this case $X^\wedge$ will stand for the set of all continuous characters on X and $X_b^\wedge$ and $X_c^\wedge$ will have analogous meaning. We say that X is *reflexive as a space* (usual sense of reflexivity) if the canonical embedding from X into $(X_b^*)_b^*$ is a topological isomorphism. It is *reflexive as a group* or *Pontryagin reflexive* if the canonical embedding from X into $(X_c^\wedge)_c^\wedge$ is a topological isomorphism of groups.

Proposition. *Let X be an infinite dimensional reflexive Banach space. Then X_b^* and X_c^* are reflexive additive groups, which admit the same set of continuous characters. Nevertheless, the topologies τ_b and τ_c are distinct.*

We will need the following results for the proof.

Lemma 1. *Let X be a topological vector space, and let $\rho : X_c^* \to X_c^\wedge$ be the mapping defined by $\rho(f) = e^{2\pi i f}$. Then ρ is a topological isomorphism of groups.*

Proof. It can be seen in [**1**, Proposition 2.3]. □

Lemma 2. *If X is a reflexive topological vector space, then $(X_b^*)^* = (X_c^*)^*$.*

Proof. [**9**, Lemma 1.1]. □

Lemma 3. *Let X be a reflexive topological vector space. Then X is reflexive as a group.*

Proof. Combine Theorem 1 of [**9**] with Lemma 1. □

Lemma 4. *If X is a Banach space, then X is reflexive as a group.*

Proof. It is Theorem 2 of [**9**] plus Lemma 1. □

Proof of the proposition. If X is an infinite dimensional reflexive Banach space, so also is X_b^*, and applying Lemma 3 we obtain that X_b^* is a reflexive group. On the other hand, by Lemma 1, the group X_c^* is topologically isomorphic to $X_c^\wedge$. Now $X_c^\wedge$ is reflexive by Lemma 4, taking into account that the dual of a reflexive group is again reflexive.

By Lemma 2, combined with the fact that the mapping ρ of Lemma 1 is in particular an algebraic isomorphism, we obtain that $(X_b^*)^\wedge = (X_c^*)^\wedge$.

Finally X_b^* and X_c^* are nonisomorphic groups. For, suppose otherwise that $\phi : X_b^* \to X_c^*$ is a topological isomorphism; in particular, it has to be linear. Take U the unit ball of X_b^*, $\phi(U)$ is a zero neighborhood in X_c^*; then there exists a compact subset K of X such that $\phi(U) \supseteq K^0$. By Lemma 2, $(X_c^*)^* = (X_b^*)^*$, and by reflexivity of X, we have that both sets may be identified with X. Call $\phi^* : (X_c^*)_b^* \to (X_b^*)_b^*$ the dual mapping of ϕ. It is also a topological isomorphism.

Let us see that K^{00} is compact in $(X_c^*)_b^*$. By the bipolar theorem and the reflexivity of X, K^{00} can be identified with the $\sigma(X, X^*)$-closed convex hull of $K \cup \{0\}$, which is compact due to completeness of X [**6**, p. 241]. Thus, $(\phi(U))^0 \subseteq K^{00}$ is relatively compact in $(X_b^*)_b^*$.

Since τ_b and τ_c are compatible with the duality $(X_b^*, (X_b^*)^*)$, they have the same bounded sets; therefore $(X_b^*)_b^*$ and $(X_c^*)_b^*$ coincide. From the equality $U^0 = \phi^*((\phi(U))^0)$, we conclude that the closed unit ball of X is compact, which contradicts the fact that X is infinite dimensional. □

Glicksberg's theorem holds for reflexive groups if a further assumption is made. Following the terminology of [**8**], if a topological abelian group (G,τ) verifies A), we will say that G respects compactness. Now it is straightforward to prove the following:

Theorem. *Let G be an abelian group. If τ_1 and τ_2 are group topologies on G such that*

i) (G,τ_i) *respect compactness for* $i = 1, 2$.

ii) (G,τ_i) *are reflexive* (*or metrizable*) *for* $i = 1, 2$.

iii) *They admit the same set of continuous characters,*

then $\tau_1 = \tau_2$.

Some classes of groups determined by their continuous characters.

1. *Reflexive* (*or metrizable*) *nuclear groups.* Nuclear groups were introduced by Banaszczyk in [**1**]. Roughly speaking, they constitute the smallest class of abelian groups which includes locally compact abelian groups and nuclear vector spaces considered in their group structure, and is closed through the operations of taking subgroups, Hausdorff quotients and arbitrary products. In [**2**], it is proved that nuclear groups respect compactness.

2. *Banach spaces with the Schur property.* If B is a Banach space, B is reflexive as a group. If, moreover, B has the Schur property, every weakly compact set $K \subset B$ is compact. Indeed, by [**8**], weakly compact sets in the topology induced by all the continuous characters (Bohr topology) are weakly compact in the weak topology induced by all the continuous linear functionals. Take now a sequence $(x_n)_{n\in N}$ in K. Due to the Eberlein theorem [**3**], it has a weakly convergent

subsequence, say $(x_{i_n})_{n\in N}$. Since B has the Schur property, $(x_{i_n})_{n\in N}$ is also convergent in the ordinary sense. Thus, K is compact.

3. *Montel spaces.* In [**8**], it is proved that a reflexive linear space respects compactness if and only if it is a Montel space.

Remark 1. In the category of locally compact abelian groups, it is proved that Pontryagin duality is unique up to natural equivalence (see, for example, [**4**]).

The proof of this fact relies on assertion B). Thus, a generalization of B) can be interesting in order to determine for which other classes of groups there is essentially one duality theory.

Remark 2. V. Tarieladze has pointed out that our counterexample can be generalized to reflexive non-Montel vector spaces, i.e., if X is a reflexive, linear non-Montel space, X_b^* and X_c^* are distinct reflexive groups, which admit the same family of continuous characters.

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