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GENERATORS OF MAXIMAL IDEALS IN THE RING OF INTEGER-VALUED POLYNOMIALS

SCOTT T. CHAPMAN AND WILLIAM W. SMITH

ABSTRACT. Let $\text{Int}(\mathbf{Z}) = \{f(X) \in \mathbf{Q}[X] | f(z) \in \mathbf{Z} \text{ for all } z \in \mathbf{Z}\}$ represent the ring of integer-value polynomials over $\mathbf{Z}$. The maximal spectrum of $\text{Int}(\mathbf{Z})$ consists of all ideals of the form $\mathcal{M}_{p,\alpha} = \{f(X) \in \text{Int}(\mathbf{Z}) \mid |f(\alpha)|_p < 1\}$, where p is a prime integer in $\mathbf{Z}$, $\alpha \in \hat{\mathbf{Z}}_p$ and $|\cdot|_p$ is the usual p-adic valuation (see [**1**]). It is well known that the polynomials

$$\binom{X}{n} = \frac{X(X-1)\cdots(X-n+1)}{n!},$$

known as the binomial polynomials, form a basis of $\text{Int}(\mathbf{Z})$ as a free $\mathbf{Z}$-module. We use a theorem of Lucas to prove the following result. Let $\alpha \in \hat{\mathbf{Z}}_p$ be of the form $\alpha = \sum_{i=0}^{\infty} a_i p^i$. Then $\binom{X}{n} \in \mathcal{M}_{p,\alpha}$ if and only if $n = n_0 + n_1 p + \cdots + n_t p^t$ where $0 \leq a_i < n_i$ for some i. We use this result to produce generating sets for the $\mathcal{M}_{p,\alpha}$.

The ring of integer-valued polynomials over $\mathbf{Z}$, denoted $\text{Int}(\mathbf{Z})$, has been the focus of much recent research. Brizolis has shown, among other things, that $\text{Int}(\mathbf{Z})$ is a Prüfer domain [**2**]. This paper is based on his characterization of the maximal ideals of $\text{Int}(\mathbf{Z})$ [**1, 4**]. Let $\mathbf{Z}$ represent the integers, $\mathbf{Z}^+$ the positive integers, $\mathbf{N}$ the nonnegative integers, $\mathbf{Q}$ the rationals, $\hat{\mathbf{Z}}_p$ the p-adic integers, and

$$\text{Int}(\mathbf{Z}) = \{f(X) \in \mathbf{Q}[X] \mid f(z) \in \mathbf{Z} \text{ for all } z \in \mathbf{Z}\}$$

the ring of integer-valued polynomials over $\mathbf{Z}$. For each prime p in $\mathbf{Z}$ and each α in $\hat{\mathbf{Z}}_p$, set

$$\mathcal{M}_{p,\alpha} = \{f(X) \in \text{Int}(\mathbf{Z}) \mid |f(\alpha)|_p < 1\}$$

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where $|\cdot|_p$ represents the usual p-adic valuation. Then each $\mathcal{M}_{p,\alpha}$ is a maximal ideal of $\mathrm{Int}(\mathbf{Z})$ and every maximal ideal of $\mathrm{Int}(\mathbf{Z})$ is of this form. Moreover, $\mathcal{M}_{p,\alpha} = \mathcal{M}_{q,\beta}$ if and only if $p = q$ and $\alpha = \beta$ [**1**]. We focus on the relationship between these ideals and the well-known binomial basis of $\mathrm{Int}(\mathbf{Z})$. Recall that the set of polynomials defined by

$$\binom{X}{0} = 1$$

and

$$\binom{X}{n} = \frac{X(X-1)\cdots(X-n+1)}{n!}$$

for $n \geq 1$ forms a *Polya basis* for $\mathrm{Int}(\mathbf{Z})$ as a free $\mathbf{Z}$-module (i.e., a basis of the form $\{f_n(x)\}_{n=0}^{\infty}$ where the degree of $f_n(x)$ is n, see Polya [**8**]). 4In this note we show for a fixed prime $p \in \mathbf{Z}$ and $\alpha \in \hat{\mathbf{Z}}_p$ with $\alpha = \sum_{i=0}^{\infty} a_i p^i$ that

$$\binom{X}{n} \in \mathcal{M}_{p,\alpha}$$

if and only if $n = n_0 + n_1 p + \cdots + n_t p^t$ where $0 \leq a_i < n_i$ for some i. Important in the argument will be the extension of a theorem of Lucas concerning congruences of binomial coefficients to the p-adic integers. We will apply our result to produce generating sets given in terms of the binomial polynomials for each maximal ideal of $\mathrm{Int}(\mathbf{Z})$.

We open with a brief review of some of the basic properties of maximal ideals in $\mathrm{Int}(\mathbf{Z})$. Recall that if I is an ideal in $\mathrm{Int}(\mathbf{Z})$, then for $a \in \mathbf{Z}$ the set $I(a) = \{f(a) \mid f(X) \in I\}$ is an ideal of $\mathbf{Z}$. By the strong Skolem property [**10**], if I and J are finitely generated ideals of $\mathrm{Int}(\mathbf{Z})$ with $I(a) = J(a)$ for each $a \in \mathbf{Z}$, then $I = J$. Using this condition, since $\alpha \in \hat{\mathbf{Z}}_p \backslash \mathbf{Z}$ implies that $\mathcal{M}_{p,\alpha}(a) = \mathbf{Z}$ for each $a \in \mathbf{Z}$ (see [**1**]) we have that $\mathcal{M}_{p,\alpha}$ is not finitely generated (although the same argument does not apply if $\alpha \in \mathbf{Z}$, since $\mathcal{M}_{p,\alpha}(\alpha) = p\mathbf{Z}$, it remains that none of the maximal ideals $\mathcal{M}_{p,\alpha}$ are finitely generated [**1**, Theorem 1.2.2]). Also, if $\alpha \in \hat{\mathbf{Z}}_p$, then $\mathcal{M}_{p,\alpha} \cap \mathbf{Z} = p\mathbf{Z}$ and $\mathcal{M}_{p,\alpha} \cap \mathbf{Z}[X] = (p, X - a)$ for some $a \in \mathbf{Z}$ (see [**1**]). Finally, if α is algebraic over $\mathbf{Q}$ then $\mathcal{M}_{p,\alpha}$ is a height 2 prime; otherwise, $\mathcal{M}_{p,\alpha}$ is of height 1 (see [**4**]).

Now, for each $\mathcal{M}_{p,\alpha}$, set $\mathrm{Int}(\mathbf{Z})/\mathcal{M}_{p,\alpha} = F_{p,\alpha}$. Our previous remarks imply, since

$$\mathbf{Z} \subseteq \mathbf{Z}[X] \subseteq \mathrm{Int}(\mathbf{Z})$$

and

$$p\mathbf{Z} \subseteq (p, X - a) \subseteq \mathcal{M}_{p,\alpha},$$

that

$$\mathbf{Z}/p\mathbf{Z} \subseteq \mathbf{Z}[X]/(p, X - a) \subseteq F_{p,\alpha}.$$

Thus $\mathbf{Z}_p \subseteq F_{p,\alpha}$. If $f(X) \in \operatorname{Int}(\mathbf{Z})$, then let $\overline{f(X)}$ represent the image of $f(X)$ under the natural map from $\operatorname{Int}(\mathbf{Z})$ to $F_{p,\alpha}$. Since

$$\left\{\binom{X}{n} \mid n \geq 0\right\}$$

forms a basis for $\operatorname{Int}(\mathbf{Z})$ over $\mathbf{Z}$, the set

$$\left\{\overline{\binom{X}{n}} \mid n \geq 0 \quad \text{and} \quad \binom{X}{n} \notin \mathcal{M}_{p,\alpha}\right\}$$

spans $F_{p,\alpha}$ as a vector space over $\mathbf{Z}_p$. It is a matter of interest to determine for a given prime $p \in \mathbf{Z}$ and $\alpha \in \hat{\mathbf{Z}}_p$ exactly which $\binom{X}{n}$ are in $\mathcal{M}_{p,\alpha}$. We shall require a theorem of Lucas [**7**, pp. 417–420] for the characterization. An alternate proof of this theorem can be found in [**5**].

Theorem 1 (Lucas). *Let p be a prime integer and $a, n \in \mathbf{N}$ with p-adic representations*

$$a = a_0 + a_1 p + \cdots + a_k p^k$$

and

$$n = n_0 + n_1 p + \cdots + n_k p^k$$

where $0 \leq a_i,\ n_i \leq p - 1$ for all i. Then

$$\binom{a}{n} \equiv \prod_{i=0}^{k} \binom{a_i}{n_i} \pmod{p}.$$

Corollary 2. *Let p be a prime integer, $n \in \mathbf{N}$ and $\alpha \in \hat{\mathbf{Z}}_p$ with*

$$\alpha = \sum_{i=0}^{\infty} a_i p^i \quad \textit{and} \quad n = n_0 + n_1 p + \cdots + n_k p^k.$$

Then in $\hat{\mathbf{Z}}_p$,

$$\binom{\alpha}{n} \equiv \prod_{i=0}^{k} \binom{a_i}{n_i} \qquad (\text{mod } p).$$

Proof. If $b_t = \sum_{i=0}^{t} a_i p^i$, then $\lim_{t\to\infty} b_t = \alpha$. Since $\binom{X}{n} \in \mathbf{Q}[X]$, it is a continuous function of $\hat{\mathbf{Z}}_p \to \hat{\mathbf{Q}}_p$ and thus

$$\lim_{t\to\infty} \binom{b_t}{n} = \binom{\alpha}{n}.$$

By Lucas' theorem,

$$\binom{b_t}{n} \equiv \prod_{i=0}^{t} \binom{a_i}{n_i} \qquad (\text{mod } p).$$

Since $\binom{a_i}{n_i} = 1$ whenever $i > k$, $\left\{\binom{b_t}{n}\right\}_{t\geq 0}$ is eventually constant modulo p and thus

$$\binom{\alpha}{n} \equiv \prod_{i=0}^{k} \binom{a_i}{n_i} \qquad (\text{mod } p). \qquad \square$$

Another immediate observation based on the Lucas theorem is that the values $\{\binom{a}{n}\}_{a\in\mathbf{N}}$ modulo p are periodic with a period which can be taken to be p^t where $n < p^t$. This is easily seen since $a + p^t$ and a will have the same first k digits in the p-adic representation where $n = n_0 + \cdots + n_k p^k$. Moreover, since any polynomial in $\text{Int}(\mathbf{Z})$ of degree n can be expressed as $f(X) = \sum_{i=0}^{n} c_i \binom{X}{i}$, it follows that the values $\{f(b)\}_{b\in\mathbf{N}}$ are periodic modulo p with period p^m for a suitable m. We state this in slightly more general terms as a corollary. A stronger result can be found in [**6**, Proposition 2.7 and 7.2] in terms of these sequences modulo p^v for $v \geq 1$.

Corollary 3. *Let* I *be a finitely generated ideal of* $\text{Int}(\mathbf{Z})$. *The sequence of values* $\{I(b)\}_{b\in\mathbf{N}}$ *is periodic modulo* p *with period* p^m *for some* $m \in \mathbf{N}$.

Corollary 2 leads us immediately to the desired characterization.

Proposition 4. *Let p be a prime integer, n a positive integer and $\alpha \in \hat{\mathbf{Z}}_p$ with $\alpha = \sum_{i=0}^{\infty} a_i p^i$ and $n = \sum_{i=0}^{k} n_i p^i$. The following statements are equivalent*:

1. $\binom{X}{n} \in \mathcal{M}_{p,\alpha}$,
2. $0 \le a_i < n_i$ *for some* i.

Proof. Let p, α and n be as above. By Corollary 2,

$$\binom{\alpha}{n} \equiv \prod_{i=0}^{k} \binom{a_i}{n_i} \pmod{p}.$$

Since $0 \le n_i$, $a_i \le p-1$, we have that

$$p \Big| \binom{a_i}{n_i} \Longleftrightarrow \binom{a_i}{n_i} = 0 \Longleftrightarrow 0 \le a_i < n_i.$$

Hence,

$$\binom{\alpha}{n} \equiv 0 \pmod{p}$$

if and only if $0 \le a_i < n_i$ for some i. □

We consider some special cases of the last proposition.

Corollary 5. *Let p be a prime integer and $\alpha = \sum_{i=0}^{\infty} a_i p^i \in \hat{\mathbf{Z}}_p$.*

1. *If $\alpha = -1$, then $\mathcal{M}_{p,\alpha}$ contains no binomial polynomials.*
2. *If $\alpha \neq -1$, then $\mathcal{M}_{p,\alpha}$ contains infinitely many binomial polynomials.*
3. *If $\alpha = 0$, then $\mathcal{M}_{p,0}$ contains every nonconstant binomial polynomial.*
4. *Let $a \in \mathbf{Z}^+$. There exists an $m \in \mathbf{N}$ such that $\binom{X}{n} \in \mathcal{M}_{p,a}$ for all $n \ge m$.*

Proof. (1) If $\alpha = -1$, then $\alpha = \sum_{i=0}^{\infty} (p-1)p^i$ which is to say that $a_i = p-1$ for all i. Hence, whatever the integer n, n_i is never such that $a_i < n_i$ and, by Proposition 4, $\binom{X}{n} \notin \mathcal{M}_{p,\alpha}$.

(2) If $\alpha \neq -1$, then $a_i \neq p-1$ for some i and hence there are infinitely many positive integers n such that $a_i < n_i$. For each such n, Proposition 4 implies that $\binom{X}{n} \in \mathcal{M}_{p,\alpha}$.

(3) If $\alpha = 0$, then for each $n \in \mathbf{Z}^+$ there is an i such that $a_i < n_i$ and the result follows.

(4) First, it is obvious that one can take $m = a+1$, since for $n \geq m$ we then have $\binom{a}{n} = 0$ and hence $\binom{X}{n} \in \mathcal{M}_{p,\alpha} = \{f(x) | p | f(a)\}$. In fact, m may also be chosen by writing $a = \sum_{i=0}^{t} a_i p^i$ and then for $n > p^{t+1}$, $\binom{X}{n} \in \mathcal{M}_{p,\alpha}$ by Proposition 4. □

Corollary 6. *Let $\alpha \in \hat{\mathbf{Z}}_p$. For every $n \in \mathbf{Z}^+$ there exist infinitely many $a \in \mathbf{Z}$ and for every $a \in \mathbf{Z}$, $a \neq \alpha + 1$, there exist infinitely many $n \in \mathbf{Z}^+$ such that*

$$\binom{X-a}{n} \in \mathcal{M}_{p,\alpha}.$$

Proof. For the first statement, it is clear that $\binom{X-a}{n} \in \mathcal{M}_{p,\alpha}$ if and only if $\binom{X}{n} \in \mathcal{M}_{p,\beta}$ where $\beta = \alpha - a$. Write $\alpha = \sum_{i=0}^{\infty} a_i p^i$. If $n \neq 0$, then some $n_i \neq 0$ and any $a = \sum_{j=0}^{k} b_j p^j$ with corresponding "digit" $b_i = a_i$ and $b_j \leq a_j$ for $j \neq i$ will yield $\binom{X-a}{n} \in \mathcal{M}_{p,\alpha}$.

For the second statement, set $\beta = \alpha - a$. Since $\beta \neq -1$, $\mathcal{M}_{p,\beta}$ contains infinitely many binomial polynomials. For each $\binom{X}{n} \in \mathcal{M}_{p,\beta}$, $\binom{X-a}{n} \in \mathcal{M}_{p,\alpha}$ and the result follows. □

Hence, given a prime p and p-adic integer α, we can construct a sequence of integers $\{z_i\}_{i=1}^{\infty}$ such that, for each $n \geq 1$,

$$\binom{X-z_n}{n} \in \mathcal{M}_{p,\alpha}.$$

Proposition 7. *Let p, α and $\{z_i\}_{i=1}^{\infty}$ be as above.*

1)
$$\left\{1, \binom{X-z_1}{1}, \binom{X-z_2}{2}, \dots\right\}$$

is a Polya basis for Int $(\mathbf{Z})$.

2)

$$\mathcal{M}_{p,\alpha} = \left(p, \binom{X - z_1}{1}, \binom{X - z_2}{2}, \dots\right).$$

Proof. Note that the leading coefficient of $\binom{X-z_n}{n}$ is $1/n!$. Hence, 1) follows from a simple inductive argument.

For 2), notice that

$$g(x) = b_0 + b_1\binom{X - z_1}{1} + \cdots + b_n\binom{X - z_n}{n} \in \mathcal{M}_{p,\alpha}$$

if and only if $p|b_0$ (this follows since $\binom{X-z_i}{i} \in \mathcal{M}_{p,\alpha}$ for $i \geq 1$). The result now easily follows. □

Example 1. We demonstrate the construction of Proposition 7.

a) Consider part 3 of Corollary 5. Here $\alpha = 0$ and $z_n = 0$ for each $n \geq 1$. Hence,

$$\mathcal{M}_{p,0} = \left(p, \left\{\binom{X}{n}\right\}_{n\geq 1}\right).$$

b) Let $\alpha = 1 + p^2 + p^3 + \cdots + p^t + \cdots \in \hat{\mathbf{Z}}_p$, and suppose that $n = \sum_{i=0}^{k} n_i p^i \in \mathbf{N}$. From the proof of Corollary 6, we may choose $z_n = p^k$ where k is the highest power of p less than or equal to n. For instance, if $p = 3$, then

$$\mathcal{M}_{3,\alpha} = \left(3, \binom{X-1}{1}, \binom{X-1}{2}, \binom{X-3}{3}, \right.$$
$$\left. \dots, \binom{X-3}{8}, \binom{X-9}{9}, \binom{X-9}{10}, \dots\right).$$

c) Let $\alpha = 1 + p + p^2 + p^6 + \cdots + p^{m!} + \cdots$. Again, by the proof of Corollary 6, $z_n = p^k$ if $k = m!$ for some $m \in \mathbf{N}$ and $z_n = 0$ otherwise. For instance, if $p = 2$, then

$$\mathcal{M}_{2,\alpha} = \left(2, \binom{X-1}{1}, \binom{X-2}{2}, \binom{X-2}{3}, \binom{X-4}{4}, \right.$$
$$\left. \dots, \binom{X-4}{7}, \binom{X}{8}, \binom{X}{9}, \dots\right).$$

In the case where $\alpha = n \in \mathbf{Z}^+$, we can give a more explicit description of the generators of $\mathcal{M}_{p,n}$.

Proposition 8. *Let p be a prime integer and $n \in \mathbf{Z}^+$. Then*

$$\mathcal{M}_{p,n} = \left(p, 1 + (p-1)\binom{X}{n}, \left\{\binom{X}{m}\right\}_{m>n}\right).$$

Proof. As illustrated in part 4 of Corollary 5, $J \subseteq \mathcal{M}_{p,n}$ where

$$J = \left(p, \left\{\binom{X}{m}\right\}_{m>n}\right).$$

If $g(x) \in \mathcal{M}_{p,n}$, then (since the binomial polynomials form a basis of $\operatorname{Int}(\mathbf{Z})$) we can write

$$g(x) = a_0 + a_1\binom{X}{1} + \cdots + a_n\binom{X}{n} + h(x)$$

where $0 \leq a_i \leq p-1$ and $h(x) \in J$. There are only finitely many choices for $g(x) - h(x)$, label them $f_1(x), \ldots, f_t(x)$. Setting

$$I = (p, f_1(x), \ldots, f_t(x)),$$

we have that

$$\mathcal{M}_{p,n} = I + J.$$

By [**6**, Theorem 3.5] there exists a polynomial $f(x) \in \operatorname{Int}(\mathbf{Z})$ so that $I = (p, f(x))$. Since we are only concerned with having such an ideal I where $\mathcal{M}_{p,n} = I + J$ and $\binom{X}{m} \in J$ for $m > n$, we may assume that $f(x)$ has degree n or less. We now have

$$\mathcal{M}_{p,n} = \left(p, f(x), \left\{\binom{X}{m}\right\}_{m>n}\right)$$

and will verify that $f(x)$ can be replaced with the indicated polynomial $g(x) = 1 + (p-1)\binom{X}{n}$. The argument uses the fact that if two polynomials in $\operatorname{Int}(\mathbf{Z})$ have the same values at the integers $0, 1, \ldots, t$, then when written in terms of the binomial polynomial basis, they will have the same coefficients for

$$\binom{X}{0}, \binom{X}{1}, \dots, \binom{X}{t}.$$

This is easily seen as these coefficients are determined by a difference table involving these common values of the polynomials.

For $0 \le k \le n$, let $b_k = f(k)$ and, since $f(x) \in \mathcal{M}_{p,n}$, we can write $b_n = pc$ for some integer c. Let $h(x)$ be the integer-valued polynomial of degree n or less for which $h(k) = b_k$ for $0 \le k \le n-1$ and $h(n) = c$. Taking note that $g(k) = 1$ for $0 \le k \le n-1$ and $g(n) = p$, we have $g(x)h(x)$ and $f(x)$ have the same values for $0, 1, \dots, n$. Writing

$$h(x)g(x) = d_0 + \cdots + d_n\binom{X}{n} + \cdots + d_m\binom{X}{m},$$

we conclude from the earlier remark that $f(x) = d_0 + \cdots + d_n\binom{X}{n}$. Hence,

$$f(x) - h(x)g(x) \in \left(\left\{\binom{X}{m}\right\}_{m>n}\right).$$

It follows that

$$\begin{aligned}\left(p, f(x), \left\{\binom{X}{m}\right\}_{m>n}\right) &= \left(p, g(x), \left\{\binom{X}{m}\right\}_{m>n}\right)\\ &= \mathcal{M}_{p,n}. \qquad \square\end{aligned}$$

An analysis of the proof of Proposition 8 implies that when $p > n$, the polynomial $g(x) = x + (p-n)$ could have been used. The property needed is that $p \nmid g(k)$ for $0 \le k \le n-1$ and $g(n) = p$. The $h(x)$ in the proof is then constructed having the needed values modulo p. Hence, for $p > n$, the second generator can be chosen with coefficients in **Z**. That is,

$$\mathcal{M}_{p,n} = \left(p, x + (p-n), \left\{\binom{X}{m}\right\}_{m>n}\right).$$

Example 2. In Proposition 8, if $p = 11$ and $a = 5$, we obtain

$$\mathcal{M}_{11,5} = \left(11, x+6, \binom{X}{6}, \binom{X}{7}, \dots\right).$$

If $p = 5$ and $a = 11$, we obtain

$$\mathcal{M}_{5,11} = \left(5, 1 + 4\binom{X}{11}, \binom{X}{12}, \binom{X}{13}, \dots\right).$$

Notice that, in each case

$$\mathcal{M}_{11,5}(5) = 11 \quad \text{and} \quad \mathcal{M}_{5,11}(11) = (5)$$

(see the remarks prior to Theorem 1).

If a is a negative integer, then the method set up in Proposition 8 can still be used to construct a set of generators for $\mathcal{M}_{p,a}$. Choose any integer b such that $a - b > 0$. Then, under the automorphism φ_b of Int $(\mathbf{Z})$ defined by $\varphi_b(X) = X + b$, $\varphi_b(\mathcal{M}_{p,a}) \cong \mathcal{M}_{p,a-b}$. So, if

$$\mathcal{M}_{p,a-b} = \left(p, f(x), \left\{\binom{X}{n}\right\}_{n \geq \mathbf{N}}\right),$$

then

$$\mathcal{M}_{p,a} = \left(p, f(x+b), \left\{\binom{X+b}{n}\right\}_{n \geq \mathbf{N}}\right).$$

We close with an observation regarding the residue field $F_{p,\alpha}$. It is well known that $F_{p,\alpha} \cong \mathbf{Z}_p$, see [**1, 3, 4**]. This is easily seen as a consequence of part 2 of Proposition 7, or by Corollary 6 combined with the fact that since

$$\binom{X-a}{n} = a_0\binom{X}{0} + \cdots + a_{n-1}\binom{X}{n-1} + \binom{X}{n}$$

(where $a_k = (-1)^{n+k}\binom{a+n-k-1}{n-k}$) then $\binom{X-a}{n} \in \mathcal{M}_{p,\alpha}$ implies that

$$\overline{\binom{X}{n}} \in \left\langle \overline{\binom{X}{0}}, \dots, \overline{\binom{X}{n-1}} \right\rangle.$$

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Trinity University, Department of Mathematics, 715 Stadium Drive, San Antonio, Texas 78212-7200

The University of North Carolina at Chapel Hill, Department of Mathematics, Chapel Hill, North Carolina 27599-3250