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ON PROPERTIES OF M-IDEALS

JUAN CARLOS CABELLO AND EDUARDO NIETO

ABSTRACT. Given $r, s \in]0,1]$, consider a Banach space X which satisfies the following inequality

$$(*) \qquad \|f+g\| \geq r\|f\| + s\|g\|$$

for every f in X^* and g in the annihilator of X in X^{***}. It is well known that if $r = s = 1$, then X is a WCG Asplund space, satisfying property (u) of Pełczyński and property (A), i.e., every isometric isomorphism of X^{**} is the bitranspose of an isometric isomorphism of X. The aim of this work is to show that, to have the above-mentioned properties, it is not necessary to suppose that $r = s = 1$. We prove, e.g., that $r + s > 1$ implies the Asplundness, $r = 1$ implies property (u) (with $k_u(X) \leq 1/s$), and $s = 1$ implies X is WCG satisfying property (A). Also many examples are given. For instance, a renormed James space J satisfies $(*)$ for $s = 1$ and the renorming of c_0 by Johnson and Wolfe does not have property (A) and satisfies $(*)$ for $r = 1$.

1. Introduction. A Banach space X is an *M-ideal* in its bidual, in short, M-ideal, if the equality $\|\varphi\| = \|\pi\varphi\| + \|\varphi - \pi\varphi\|$ holds for every $\varphi \in X^{***}$, where π is the canonical projection of X, the natural projection from X^{***} onto X^*. The class of M-ideals has been carefully investigated by Å. Lima, G. Godefroy and the "Berlin school", among others. As a consequence of these efforts, P. Harmand, D. Werner and W. Werner have published a recent monograph [**15**] which is considered the most systematic and complete study about this class. The spaces $c_0(I)$, I any set, equipped with their canonical norm belong to this class, which also contains, e.g., certain spaces $\mathcal{K}(E,F)$ of compact operators between reflexive spaces, see, e.g., [**3, 14, 18** and **27**] or [**15**, Chapter VI]. M-ideals are known to enjoy many interesting isometric and isomorphic properties, e.g., they are weakly compactly generated (WCG) [**8**] and Asplund spaces [**20**], have properties (u) (with constant one) and (V) of Pełczyński [**11**] and [**12**], satisfy the

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uniqueness property U of Phelps [**15**] and are proximinal subspaces in their biduals [**1**] and [**2**], and isometric isomorphisms of their biduals are bitransposes of isometric isomorphisms of them [**14**]. For more complete information, see [**15**, Chapter III].

A large family of generalizations arose after the notion of M-ideal. We are particularly interested in the ones given in terms of the canonical projection π of X. In particular, we remark on the notions of HB-subspace and SU-property introduced by J. Hennefeld [**16**] and E. Oja [**24**], respectively. Indeed, see [**24**], SU-property is equivalent to property U, i.e., any element $f \in X^*$ admits a unique norm preserving an extension to X^{**}. It is known that if a Banach space X is an HB-subspace in their bidual, then X is an Asplund space [**26**] satisfying property U [**16**]. Nevertheless, it seems to not be known which of the remaining properties under consideration remain true.

Recently, Godefroy, Kalton and Saphar have introduced the notion of a strict u-ideal [**10**]; actually, if a Banach space X contains no copy of l_1, then X is a strict u-ideal if and only if X has property (u) with constant one [**10**]. In the mentioned paper, the authors prove that if X is a strict u-ideal and $l_1 \not\subseteq X$, then X is an Asplund space such that every isometric isomorphism of X^{**} is the bitranspose of an isometric isomorphism of X, X^* contains no proper norming subspaces, but X is not necessarily proximinal in X^{**}. Nevertheless, it seems to not be known if X is WCG.

We will introduce in this paper some generalizations. One of them is the concept of U^*-space, which is nothing but the dual notion of SU-property. We prove that if X is a U^*-space, then every isometric isomorphism of X^{**} is the bitranspose of an isometric isomorphism of X; moreover, if in addition X is an Asplund space, then X is WCG. The remaining generalizations arise in studying the relation between the above considered properties and coefficients r and s, and the inequality

$$\|\varphi\| \geq r\|\pi\varphi\| + s\|\varphi - \pi\varphi\|, \quad \forall\ \varphi \in X^{***}.$$

For instance, if $r + s > 1$, then X is an Asplund space and X^* contains no proper norming subspaces. If $r = 1$, then X has property U of Phelps and property (u) of Pełczyński with $k_u(X) \leq 1/s$, but is not necessarily proximinal in its bidual. If $s = 1$, then X is an Asplund U^*-space, but there are Banach spaces without properties (u) and U satisfying the $M(r, 1)$-inequality as can be seen below.

All Banach spaces in this paper are real and infinite-dimensional. If X is a Banach space, π_X will denote the canonical projection of X. If there is no ambiguity, we write π instead of π_X. The closed unit ball and the unit sphere of X are denoted by B_X and S_X, respectively. The closed ball in X with center a and radius r is denoted by $B_X(a, r)$.

The concepts such as "closed", "dense", etc., are related to the norm topology unless otherwise stated. Given a subset S of a Banach space, the symbols $\overline{S}$, $\operatorname{span} S$ and $\operatorname{co} S$ are used to denote the closure, linear span and convex hull of S, respectively.

Given a closed subspace Z of a Banach space Y, we write, for each $y \in Y$,

$$P_Z(y) = \{z \in Z : \|z - y\| = \|y + Z\|\},$$

that is, the set of the best approximations of y in Z. If $P_Z(y)$ contains exactly (at least) one element for every $y \in Y$, then Z is said to be a Chebyshev (proximinal) subspace of Y.

A series $\sum x_n$ in a Banach space X is called weakly unconditionally Cauchy (wuC) if there exists $C \geq 0$ such that

$$\sup_{|\varepsilon_n| \leq 1} \left\| \sum_{n=1}^{N} \varepsilon_n x_n \right\| \leq C, \quad \forall\, N \in \mathbf{N}.$$

A Banach space X has property (u) if for every x^{**} in the sequential closure of X in (X^{**}, w^*), $B_a(X)$, there exists a series wuC $\sum x_n$ in X such that

$$x^{**} = w^*\text{-}\sum_{n=1}^{+\infty} x_n.$$

If X has property (u) and $x^{**} \in B_a(X)$, we denote its u-constant $k_u(x^{**})$, see [**10**, p. 22], to be the infimum of all C. By the closed graph theorem, there is a constant K such that

$$k_u(x^{**}) \leq K \|x^{**}\|, \quad \forall\, x^{**} \in B_a(X).$$

We will denote $k_u(X)$ the least such constant K.

A bounded set C of a Banach space X is called dentable if, for every $\varepsilon > 0$, there exists $x_\varepsilon \in C$ such that $x_e \notin \overline{\operatorname{co}}\,(C \backslash B_X(x_\varepsilon, \varepsilon))$.

We will say that a Banach space X is an *M-ideal*, respectively *canonical u-ideal/U^*-space*, in its bidual, in short, *M-ideal*, respectively *canonical u-ideal/U^*-space*, if, for every $\varphi \in X^{***}$, we have

$$\|\varphi\| = \|\varphi - \pi\varphi\| + \|\pi\varphi\|,$$

respectively, $\|(I - 2\pi)(\varphi)\| \leq \|\varphi\| / \|\varphi - \pi\varphi\| < \|\varphi\|$ whenever $\pi\varphi \neq 0$.

Note that the notion of canonical u-ideal coincides with the notion of strict u-ideal [**10**] whenever the Banach space X contains no copy of l_1.

Finally we introduce the key concept in this paper.

Given $r, s \in]0, 1]$, we will say that a Banach space X *satisfies the $M(r, s)$-inequality* if the following condition holds

$$\|\varphi\| \geq r\|\pi\varphi\| + s\|\varphi - \pi\varphi\|, \quad \forall\, \varphi \in X^{***}.$$

It is clear that, if $r = s = 1$, respectively $s = 1$, then X is an M-ideal, respectively U^*-space.

2. The $M(r, s)$-inequality. We shall now prove some results which will be fundamental in the sequel. First we assert the good stable behavior of several generalizations.

Proposition 2.1. *Let X be a Banach space satisfying the $M(r, s)$-inequality, respectively a canonical u-ideal/U^*-space. Then every closed subspace or quotient of X also satisfies the $M(r, s)$-inequality, respectively is a canonical u-ideal/U^*-space.*

Proof. It is similar to the one given in [**15**, p. 111]. □

Proposition 2.2. *Let X be an* HB*-subspace, respectively canonical u-ideal/U^*-space. Then $l_p(X)$ is an* HB*-subspace, respectively canonical u-ideal/U^*-space, for $1 < p < +\infty$.*

Proof. In the first place, we claim: *If Q is a norm one projection on a Banach space Z such that*

$$B_Z \subseteq \mathrm{co}\,(B_{Q(Z)} \cup \mathrm{Ker}\, Q),$$

then

$$B_{l_p(Z)} \subseteq \mathrm{co}\,(B_{l_p(Q(Z))} \cup l_p(\mathrm{Ker}\, Q)).$$

Indeed, given $\varphi = (x_n) \in B_{l_p(Z)}$, there are sequences $(\alpha_n), (y_n)$ and (z_n) in $[0,1]$, $B_{Q(Z)}$ and $\mathrm{Ker}\, Q$, respectively, such that

$$x_n = \alpha_n y_n + (1-\alpha_n) z_n, \quad \forall\, n \in \mathbf{N}.$$

It is clear that

$$\begin{aligned}\lambda &= \left[\sum_{n=1}^{+\infty} (\alpha_n \|y_n\|)^p\right]^{1/p} \\ &= \left[\sum_{n=1}^{+\infty} \|Qx_n\|^p\right]^{1/p} \\ &\le \|\varphi\|_p \le 1.\end{aligned}$$

If $\lambda = 0$, then the claim is trivial.

If $\lambda = 1$, then $\|(x_n)\|_p = \|(Qx_n)\|_p$, and since, for every $n \in \mathbf{N}$, $\|Qx_n\| \le \|x_n\|$, we have that $\|x_n\| = \|Qx_n\|$ so, by assumption, $Qx_n = x_n$, that is, $\varphi \in B_{l_p(Q(Z))}$.

Otherwise, it is enough to take $\varphi_1 \in B_{l_p(Q(Z))}$ and $\varphi_2 \in l_p(\mathrm{Ker}\, Q)$ defined by

$$\varphi_1(n) = \lambda^{-1} \alpha_n y_n$$

and

$$\varphi_2(n) = (1-\lambda)^{-1}(1-\alpha_n) z_n, \quad \forall\, n \in \mathbf{N}.$$

Hence, $\varphi = \lambda\varphi_1 + (1-\lambda)\varphi_2$.

On the other hand, it is straightforward to prove that X is an HB-subspace, respectively U^*-space, if and only if $B_{X^{***}} \subseteq \mathrm{co}\,(X^\perp \cup B_{X^*})$ and $\|I - \pi_X\| \le 1$, respectively $B_{X^{***}} \subseteq \mathrm{co}\,(B_{X^\perp} \cup X^*)$. Therefore, by the claim, denoting $Y = l_p(X)$ and taking $Q = \pi_X$, respectively $Q = I - \pi_X$, and since

$$\pi_Y(\varphi_n) = (\pi_X \varphi_n), \quad \forall\, (\varphi_n) \in Y^{***},$$

we have that

$$B_{Y^{***}} \subseteq \mathrm{co}\,(Y^\perp \cup B_{Y^*}),$$

respectively,

$$B_{Y^{***}} \subseteq \mathrm{co}\,(B_{Y^\perp} \cup Y^*).$$

Also, it is clear that $\|I - \pi_X\| \leq 1$ implies that $\|I - \pi_Y\| \leq 1$. The proof for canonical u-ideals is similar. □

Remark. The $M(r,s)$-inequality is not stable by taking l_p-sums, as can be seen below (see Example 3.7).

The next lemma will be needed further on, and its proof has been suggested to us by E. Oja.

Lemma 2.3. *Let $r, s \in \,]0,1]$. If P is a projection on a Banach space Z, then the following assertions are equivalent*:

1. *For all $x \in Z$,*

$$\|x\| \geq r\|Px\| + s\|x - Px\|.$$

2. *For all $x^*, y^* \in Z^*$,*

$$\|rP^*x^* + s(y^* - P^*y^*)\| \leq \max\{\|x^* + \mathrm{Ker}\,P^*\|, \|y^* + P^*(Z^*)\|\}.$$

Proof. Denote by T the operator from Z to $Z \oplus_1 Z$ defined by

$$Tx = (rPx, s(x - Px)), \quad \forall\, x \in Z.$$

1) $\Rightarrow$ 2). In this case the operator T has norm ≤ 1, so its adjoint also has norm ≤ 1, that is,

$$\|rP^*x^* + s(y^* - P^*y^*)\| \leq \max\{\|x^*\|, \|y^*\|\}, \quad \forall\, x^*, y^* \in Z^*.$$

Starting from here, the assertion on the norm of $x^* + \mathrm{Ker}\,P^*$ and $y^* + P^*(Z^*)$ is obvious.

2) $\Rightarrow$ 1). By assumption, $\|T^*\| \leq 1$, so for every $x \in Z$ we have

$$r\|Px\| + s\|x - Px\| = \|Tx\| \leq \|x\|. \qquad \square$$

It is well known that M-ideals can be characterized by intersection properties of balls [**1**, Theorem A], cf. [**15**, Theorem I.2.2]. In our context, we can establish the following result.

Proposition 2.4. *Let X be a nonreflexive Banach space satisfying the $M(r,s)$-inequality. Then, for every $x^{**} \in X^{**}$, $\varepsilon > 0$, $n \in \mathbf{N}$ and $x_1, x_2, \dots, x_n \in X$ with $\|x_i\| \le \|x^{**} + X\|$, there is a $z \in X$ such that*

$$\|rx_i + sx^{**} - z\| \le \|x^{**} + X\| + \varepsilon.$$

Proof. Indeed, since $\operatorname{Im} \pi^* = X^{\perp\perp}$, by the above lemma, for every $x^{**} \in X^{**}$, $x_i \in X$, $i = 1, \dots, n$, with $\|x_i\| \le \|x^{**} + X\|$, then

$$\|rx_i + s(x^{**} - \pi^* x^{**})\| \le \|x^{**} + X\|.$$

Hence,

$$X^{\perp\perp} \cap \bigcap_{i=1}^{n} B_{X^{(\mathrm{iv})}}(rx_i + sx^{**}, \|x^{**} + X\|) \neq \varnothing.$$

This means, by a result of Å. Lima [**19**, Corollary 1.3] that, for all $\varepsilon > 0$, there is a $z \in X$ satisfying

$$z \in \bigcap_{i=1}^{n} B_{X^{**}}(rx_i + sx^{**}, \|x^{**} + X\| + \varepsilon). \qquad \square$$

The case $r = s = 1$ of the following result is proved in [**13**, Lemma 4.1], [**20**, Theorem 2.4] and [**21**, Proposition 2.7], and our proof follows from them with some modifications.

Proposition 2.5. *Let X be a nonreflexive Banach space satisfying the $M(r,s)$-inequality. Then*

1. *If $r + s > 1$, then X^* contains no proper norming subspaces and X is an Asplund space.*

2. *If Y is a closed subspace of X such that there exists a space Z with Banach-Mazur distance $d(Y, Z^*) < r + s/2$, then Y is reflexive.*

3. *For all* $x^{**} \in X^{**}$*, there is a net* (x_α) *in* X w^**-converging to* x^{**} *such that*

$$\overline{\lim_\alpha} \|rx + s(x^{**} - x_\alpha)\| \leq \max\{\|x\|, \|x^{**} + X\|\}, \quad \forall\, x \in X.$$

Proof. 1) Let us recall that the characteristic $r(M)$ of a closed subspace M of X^* is defined by

$$r(M) = \inf\{\lambda > 0 : \lambda B_{X^*} \subseteq \overline{B_M}^{w^*}\}.$$

Obviously, $0 \leq r(M) \leq 1$. In fact, M is a norming subspace if and only if $r(M) = 1$.

With a similar argument to the one given in [**13**, Lemma 4.1], we obtain, for every proper subspace M of X^*, that $r(M) \leq 1/(r+s)$. Therefore, if $r+s > 1$, then X^* contains no proper norming subspaces.

On the other hand, if Y is a separable subspace of X, by Proposition 2.1, again Y satisfies the $M(r,s)$-inequality, and so, no proper subspace of Y^* is norming; therefore, Y^* is separable, that is, X is an Asplund space.

For the proof of assertion 2), it is enough to adapt [**20**, Lemma 2.3] as follows.

Lemma 2.6. *Let* $r, s \in\ ]0,1]$ *be such that* $r + s/2 > 1$ *and* $c \in\ ]1/(r+s/2), 1[$*. Suppose that, for every* $\varepsilon > 0$*, there are sequences* (x_n) *in* B_X*,* (f_m) *in* B_{X^*} *such that*

1. $f_m(x_n) \geq c$ *when* $m \geq n$*.*
2. $|f_m(x_n)| \leq \varepsilon$ *when* $m < n$*.*

Then X *does not satisfy the thesis of Proposition* 2.4 *for* $n = 2$*. In particular,* X *does not satisfy the* $M(r,s)$*-inequality.*

3) It is a consequence of the following lemma, which is a revisited version of [**28**, Proposition 2.3].

Lemma 2.7. *Let* $r, s \in\ ]0,1]$*. If* Z *is a closed subspace of a Banach space* Y *such that* $Z^\perp$ *is the kernel of a norm one projection* P*, then the following assertions are equivalent:*

1.
$$\|y^*\| \geq r\|Py^*\| + s\|y^* - Py^*\|, \quad \forall\, y^* \in Y^*.$$

2. *For all* $y \in B_Y$*, there is a net* (z_α) *in* Z *such that* $(z_\alpha) \to y$ *in the* $\sigma(Y, P(Y^*))$*-topology and*

$$\overline{\lim_\alpha} \|rz + s(y - z_\alpha)\| \leq \max\{\|z\|, \|y + Z\|\}, \quad \forall\, z \in Z.$$

Proof. 1) $\Rightarrow$ 2). By Lemma 2.3, the proof of (i) $\Rightarrow$ (ii) in [**28**, Proposition 2.3] may be adapted without problems in our case.

2) $\Rightarrow$ 1). It follows as (iv) $\Rightarrow$ (i) in [**21**, Proposition 2.7]. □

Remark. R. Haller and E. Oja have informed us that they have independently proved Lemma 2.7 in a forthcoming paper.

Now let us collect some consequences.

Corollary 2.8. *Let* X *be a nonreflexive Banach space satisfying the* $M(r, s)$*-inequality for* $r + s > 1$*. Then*

1. X *does not contain an isomorphic copy of* l_1*.*

2. *If* Z *is a Banach space such that* $X \subsetneqq Z \subseteq X^{**}$*, then there are no norm one projections from* Z *onto* X*.*

3. *Every subspace or quotient of* X *which is isometric to a dual space is reflexive.*

Proof. 1) It follows from the first assertion of Proposition 2.5 and the fact that l_1 is not an Asplund space, and this property is inherited by subspaces.

2) Let Q be a norm one projection. It is clear that, for every $z \in Z \backslash X$,

$$Qz \in \bigcap_{x \in X} B_X(x, \|z - x\|).$$

So, by [**13**, Lemma 2.4], X^* contains a proper norming subspace, which is a contradiction with Proposition 2.5.

3) If Y is a subspace or quotient, by Proposition 2.1, Y again satisfies the $M(r,s)$-inequality. Therefore, if Y is isometric to a dual Banach space, then there is a norm one projection, which is a contradiction to the above assertion. □

Corollaries 2.9 and 2.10 below were proved for the case $r = s = 1$ by G. Godefroy and P. Saphar in [**13** Proposition 4.3 and Corollary 4.4]. They follow from Proposition 2.5.

Corollary 2.9. *Let X be a separable Banach space satisfying the $M(r,s)$-inequality for $r+s>1$. Let (T_n) be a sequence of finite rank operators on X such that*

1. $\sup_n \|T_n\| < r+s$,
2. $T_n T_k = T_k T_n$ *for all* $n,k \in \mathbf{N}$,
3. $\lim_n \|T_n x - x\| = 0$ *for all* $x \in X$.

Then we have that

$$\lim_n \|T_n^* x^* - x^*\| = 0, \quad \forall\, x^* \in X^*.$$

Corollary 2.10. *Let X be a Banach space satisfying the $M(r,s)$-inequality for $r+s>1$, and let (e_n) be a basic sequence in X. If the basis constant of (e_n) is strictly less than $r+s$, then (e_n) is shrinking.*

The next results will be a key tool in the construction of examples of Banach spaces satisfying the $M(r,s)$-inequality.

Proposition 2.11. *Let X be a Banach space with shrinking basis (e_n) and $r,s \in \,]0,1]$. For $n \in \mathbf{N}$, we denote*

$$P_n x = \sum_{i=1}^{n} x_i e_i \quad \textit{and} \quad P^n x = x - P_n x = \sum_{i=n+1}^{+\infty} x_i e_i,$$

where $x = \sum_{n=1}^{+\infty} x_i e_i$. If, for all $n \in \mathbf{N}$, $x \in B_X$ and $x^{**} \in B_{X^{**}}$,

$$\overline{\lim_m} \|rP_n x + sP^{m**} x^{**}\| \le 1,$$

then X satisfies the $M(r,s)$-inequality.

Proof. Let $\varphi \in X^{***}$ and $\varepsilon > 0$. Then there are $x^{**} \in B_{X^{**}}$ and $x \in B_X$ such that

$$\|\varphi - \pi\varphi\| - \varepsilon < (\varphi - \pi\varphi)(x^{**})$$

and

$$\|\pi\varphi\| - \varepsilon < \pi\varphi(x).$$

For $n, m \in \mathbf{N}$ large enough, we have that

$$\|\pi\varphi\| - \varepsilon < \pi\varphi(P_n x)$$

and

$$\|rP_n x + sP^{m**}x^{**}\| < 1 + \varepsilon,$$

and it is clear that, for every $m \in \mathbf{N}$,

$$(\varphi - \pi\varphi)(x^{**}) = (\varphi - \pi\varphi)(P^{m**}x^{**}),$$

actually $P_m^{**}x^{**} \in X$. Hence,

$$\begin{aligned}
\|\varphi\| &\geq \frac{1}{1+\varepsilon}|\varphi(rP_n x + sP^{m**}x^{**})| \\
&= \frac{1}{1+\varepsilon}|s(\varphi - \pi\varphi)(P^{m**}x^{**}) + r\pi\varphi(P_n x) + s\pi\varphi(P^{m**}x^{**})| \\
&\geq \frac{s}{1+\varepsilon}(\varphi - \pi\varphi)(P^{m**}x^{**}) + \frac{r}{1+\varepsilon}\pi\varphi(P_n x) \\
&\quad - \frac{s}{1+\varepsilon}|\pi\varphi(P^{m**}x^{**})| \\
&> \frac{s}{1+\varepsilon}\|\varphi - \pi\varphi\| + \frac{r}{1+\varepsilon}\|\pi\varphi\| \\
&\quad - \frac{2\varepsilon}{1+\varepsilon} - \frac{s}{1+\varepsilon}|\pi\varphi(P^{m**}x^{**})|.
\end{aligned}$$

Now, since (e_n) is shrinking, $|\pi\varphi(P^{m**}x^{**})| < \varepsilon$ for m large enough, indeed $\pi\varphi(P^{m**}x^{**}) \in \overline{P^m(B_X)}^{w^*}$. □

Remark. The case $r = 1$ of Proposition 2.11 is contained in [**23**, Corollary 3].

Corollary 2.12. *Let X be a Banach space with shrinking basis (e_n). If, for all $n \in \mathbf{N}$ and $x \in B_X$,*

$$\overline{\lim_m} \sup_{y \in B_X} \|rP_n x + sP^m y\| \leq 1,$$

in particular, if $\|rP_n x + sP^m y\| \leq 1$ for all $x, y \in B_X$ and all $n, m \in \mathbf{N}$ with $n < m$, then X satisfies the $M(r,s)$-inequality.

The first example shows a break with the classical case [**15**, Proposition II.1.1], since the $M(r,s)$-inequality does not imply proximinality.

Example 2.13. Let $\sum a_n$ be a convergent series of positive real numbers. Put $a := \sum_{n=1}^{+\infty} a_n$ and suppose $0 < a < 1$. For every $x = (x_n) \in c_0$, define

$$\|x\| := \sup \left\{ |x_n| + \sum_{k=1}^{n} |x_k| a_k : n \in \mathbf{N} \right\}.$$

Then

1. $(c_0, \|\cdot\|)$ is not proximinal in $(l_\infty, \|\cdot\|)$. In particular, $(c_0, \|\cdot\|)$ is not an M-ideal.

2. $(c_0, \|\cdot\|)$ satisfies the $M(1, 1-a)$-inequality.

Proof. 1) Let $e = (1, 1, \dots) \in l_\infty$. If $x \in c_0$, then

$$\|x - e\| \geq \|x - e\|_\infty \geq 1.$$

Let $\varepsilon > 0$. There exists $m \in \mathbf{N}$ such that $\sum_{k=m+1}^{+\infty} a_k < \varepsilon$. Put $x := \sum_{k=1}^{m} e_k$. Then

$$\|x - e\| \leq 1 + \sum_{k=m+1}^{+\infty} a_k < 1 + \varepsilon.$$

Hence, $\|e + c_0\| = 1$. If there is $x \in c_0$ such that $\|x - e\| = \|e + c_0\| = 1$, then

$$1 \geq |1 - x_n| + \sum_{k=1}^{n} |1 - x_k| a_k, \quad \forall\, n \in \mathbf{N}.$$

Letting $n \to +\infty$, $1 \geq 1+\sum_{n=1}^{+\infty} |1-x_n|a_n$, and this is a contradiction.

2) It is easy to show that the assumption in Corollary 2.12 is satisfied. □

Remark. It remains an open question if there are Banach spaces satisfying the $M(r,1)$-inequality without being proximinal.

Our second example shows that the $M(r,s)$-inequality does not imply property U.

Example 2.14. Let $0 < \gamma < 1$. Denote $Z = \mathbf{R} \times c_0$ with the norm

$$\|(\alpha, x)\| = \max\{|\alpha| + \gamma\|x\|, \|x\|\}, \quad (\alpha, x) \in \mathbf{R} \times c_0,$$

where $\|x\|$ is the usual norm in c_0. Then Z satisfies the $M(1-\gamma,1)$-inequality without having property U.

Proof. Note $e_1 = (1, (0, 0, \dots))$ and

$$e_{n+1} = (0, (\underbrace{0, \dots, 0}_{n-1}, 1, 0, \dots))$$

for all $n \in \mathbf{N}$. It is clear that (e_n) is a shrinking basis. Take $u = (\alpha, (x_j))$, $z = (\beta, (y_j)) \in B_Z$ and $n < m$. Observe that

$$\begin{aligned}
&\|(1-\gamma)P_n u + P^m z\| \\
&\quad = \max\left\{\begin{array}{c} (1-\gamma)(|\alpha| + \gamma\|(x_1, x_2, \dots, x_{n-1}, 0, \dots)\|), \\ (1-\gamma)|\alpha| + \gamma\|(0, 0, \dots, 0, y_m, y_{m+1}, \dots)\|, \\ (1-\gamma)\|(x_1, x_2, \dots, x_{n-1}, 0, \dots)\|, \\ \|(0, 0, \dots, 0, y_m, y_{m+1}, \dots)\| \end{array}\right\} \\
&\quad \leq \max\{\|u\|, \|z\|, (1-\gamma)|\alpha| + \gamma\|z\|\} \leq 1.
\end{aligned}$$

So, by Corollary 2.12, Z satisfies the $M(1, 1-\gamma)$-inequality.

On the other hand, it is straightforward to verify that $Z^{***} = \mathbf{R} \times l_\infty^*$ with the norm

$$\|(\alpha, \varphi)\| = \max\{|\alpha|, \|\varphi\| + (1-\gamma)|\alpha|\}, \qquad (\alpha, \varphi) \in \mathbf{R} \times l_\infty^*.$$

Now it is easy to prove that, for every $(\alpha, \varphi) \in \mathbf{R} \times l_\infty^*$ holds

$$P_{Z^\perp}(\alpha, \varphi) = \{0\} \times (B_{l_\infty^*}(\varphi, \max\{\|\varphi + c_0^\perp\|, \gamma|\alpha|\}) \cap c_0^\perp).$$

Therefore, Z does not have property U. □

3. The $M(1, s)$-inequality and property (u). By a theorem of G. Godefroy and D. Li [**11**, Theorem 1], cf. [**15**, p. 133], and by [**15**, p. 11], M-ideals have properties (u) of Pełczyński and U of Phelps. The aim of this section is to show that, for these properties, it is not necessary to suppose that $s = 1$. More precisely,

Theorem 3.1. *Let X be a nonreflexive Banach space satisfying the $M(1, s)$-inequality. Then*

1. *X has property U of Phelps.*
2. *X has property (u) of Pełczyński with constant $k_u(X) \leq 1/s$.*

Proof. 1) As $P_{X^\perp}(\varphi) = \{\varphi - \pi\varphi\}$ for all $\varphi \in X^{***}$, $X^\perp$ is Chebyshev, but this is equivalent to property U [**25**, Theorem 1.1]. Assertion 1 also follows [**24**].

Now we proceed to show that X satisfies property (u). The proof follows essentially the lines of the proof of the main result in [**11**]. Some extra difficulties are however to be overcome, and this is done in the next lemmas.

The first lemma is a revisited version of [**11**, Lemma 2], which is crucial to prove that M-ideals have property (u).

Lemma 3.2. *Let X be a Banach space satisfying the $M(1, s)$-inequality and $x^{**} \in X^{**}$. Then $x^{**} = h_1 - h_2$ on B_{X^*}, where h_1, h_2 are positive lower semi-continuous functions on (B_{X^*}, w^*) such that*

$$h_1(x^*) + h_2(x^*) \leq 1/s, \quad \forall\, x \in B_{X^*}.$$

Proof. We only give the main ideas of the proof. It is straightforward to prove that X satisfies the $M(1,s)$-inequality if and only if

$$B_{X^{***}} \subseteq \operatorname{co}\left(\frac{1}{s}B_{X^\perp} \cup B_{X^*}\right).$$

It is clear that $K = \operatorname{co}((1/s)B_{X^\perp} \cup B_{X^*})$ is w^*-compact.

Fix $x^{**} \in X^{**}$ and define $h_{x^{**}} : K \to \mathbf{R}_0^+$ by

$$h_{x^{**}}(\varphi) = \begin{cases} \varphi(x^{**}) & \text{if } \varphi \in (1/s)B_{X^\perp}, \varphi(x^{**}) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $\hat{h_{x^{**}}} : K \to \mathbf{R}$ by

$$\hat{h_{x^{**}}}(\varphi) = \inf\{a(\varphi) : a \in A(K), h_{x^{**}} \leq a\}, \quad \forall \varphi \in K,$$

where $A(K)$ denotes the set of all affine and w^*-continuous functions on K.

Denote

$$S = \operatorname{co}(\{(k,r) : 0 \leq r \leq h_{x^{**}}(k), k \in \frac{1}{s}B_{X^\perp}, k(x^{**}) \geq 0\} \cup K \times \{0\}).$$

By a Hahn-Banach argument, we have that

$$(\varphi, \hat{h_{x^{**}}}(\varphi)) \in S, \quad \forall \varphi \in K.$$

A standard procedure, see, for instance, [**15**, Lemma I.2.5] and [**11**, Lemma 2] allows us to assert that

$$(\varphi - \pi\varphi)(x^{**}) = \hat{h_{x^{**}}}(\varphi) - \hat{h_{x^{**}}}(-\varphi), \quad \forall \varphi \in B_{X^{***}}.$$

Hence, if we consider the functions g_1 and g_2 from K to $\mathbf{R}$ given by

$$g_1(\varphi) = \frac{1/s + \varphi(x^{**})}{2} - \hat{h_{x^{**}}}(\varphi),$$
$$g_2(\varphi) = \frac{1/s - \varphi(x^{**})}{2} - \hat{h_{x^{**}}}(-\varphi),$$

for all $\varphi \in K$, it is easy to show that g_1, g_2 are positive and lower semi-continuous functions on (K, w^*) such that

$$\pi(\varphi(x^{**}) = g_1(\varphi) - g_2(\varphi) \quad \forall \varphi \in K$$
$$\text{and} \quad g_1 + g_2 \leq 1/s.$$

By Saint-Raymond's lemma [**15**, Lemma I.2.8], there are h_1, h_2 positive and lower semi-continuous functions on (B_{X^*}, w^*) such that

$$x^{**} = h_1 - h_2 \quad \text{and} \quad h_1 + h_2 \leq 1/s. \qquad \square$$

Now we want to draw attention to a careful reading of the proof of [**15**, Theorem I.2.10] which allows us to assert that:

Lemma 3.3. *Let Z be a separable Banach space such that, for every $z^{**} \in Z^{**}$, there are positive lower semi-continuous functions $h_1, h_2 : (B_{Z^*}, w^*) \to \mathbf{R}$ satisfying*

$$z^{**} = h_1 - h_2 \quad \textit{and} \quad h_1 + h_2 \leq C.$$

Then, for each $z^{**} \in Z^{**}$ and $\varepsilon > 0$, there is a sequence (z_n) in Z such that

$$z^{**} = w^*\text{-}\sum_{n=1}^{+\infty} z_n,$$

$$\sup_{|\varepsilon_n| \leq 1} \left\| \sum_{n=1}^{N} \varepsilon_n z_n \right\| \leq (1+\varepsilon) C \|z^{**}\|, \quad \forall N \in \mathbf{N}.$$

Let us now conclude the proof of the theorem.

2) Fix $x^{**} \in B_a(X)$ and $(y_n) \overset{w^*}{\to} x^{**}$, and write $Z = \overline{\operatorname{span}}\{y_n : n \in \mathbf{N}\}$.

According to Proposition 2.1 and Lemma 3.2, Z satisfies the hypothesis of Lemma 3.3, and, therefore, for every $\varepsilon > 0$, there is a sequence

(z_n) in X such that

$$x^{**} = w^* - \sum_{n=1}^{+\infty} z_n,$$

$$\sup_{|\varepsilon_n| \leq 1} \left\| \sum_{n=1}^{N} \varepsilon_n z_n \right\| \leq \frac{1}{s}(1+\varepsilon)\|x^{**}\|, \quad \forall N \in \mathbf{N}. \qquad \square$$

It is known, see, e.g., [**15**, p. 133], that a Banach space X with property (u) has property (V), i.e., every subset K of X^* satisfying

$$\lim_n \sup_{x^* \in K} |x^* x_n| = 0$$

for every wuC-series $\sum x_n$ in X is relatively weakly compact, whenever X contains no isomorphic copy of l_1. Now we can extend [**15**, Corollary III.3.7] by simply adapting its proof to the new more general situation with the help of previously stated results.

Corollary 3.4. *Let X be a nonreflexive Banach space satisfying the $M(1,s)$-inequality. Then*

1. *Every subspace of X has property (V). In particular, X contains a copy of c_0, X is not* wsc *(weakly sequentially complete) and X fails the Radon-Nikodým property.*

2. *X^* is* wsc *and contains a complemented copy of l_1.*

3. *X is not complemented in X^{**}.*

4. *X^{**}/X is not separable.*

5. *Every subspace or quotient of X which is isomorphic to a dual space is reflexive.*

6. *Every operator from X to a space not containing c_0, in particular, every operator from X to X^*, is weakly compact.*

Example 2.14 and the next example show that condition $r = 1$ cannot be dropped in Theorem 3.1.

Example 3.5. For $\delta > 0$, let J_δ be the space of all null sequences (α_n) in $\mathbf{R}$ satisfying

$$\sup\left\{(\delta\alpha_{k_1} - \alpha_{k_2})^2 + \sum_{i=2}^{n}(\alpha_{k_i} - \alpha_{k_{i+1}})^2 + (\alpha_{k_{n+1}} - \delta\alpha_{k_1})^2\right\}^{1/2} < +\infty,$$

where the supremum is taken over all $n \in \mathbf{N}$ and all finite increasing sequences $k_1 < k_2 < \cdots < k_{n+1}$ in $\mathbf{N}$, with norm $\|\cdot\|_\delta$ defined by this supremum. Then

1. For every δ, $(J_\delta, \|\cdot\|_\delta)$ is isomorphic to the James space.

2. For $\delta > \sqrt{2}$, $(J_\delta, \|\cdot\|_\delta)$ satisfies the $M(t,1)$-inequality for all $t > 0$ such that

$$(*) \qquad \max\left\{\frac{(1+\delta t)^2}{2}, \frac{(1+\delta t)^2 + (1+t)^2 + 2(\delta t)^2}{2\delta^2}\right\} < \frac{1}{2}.$$

Proof. 1) It is trivial.

2) It follows from [**7**, Properties I and II, pp. 81–82] that the sequence (e_n), where

$$e_n = (\underbrace{0, \dots, 0}_{n-1}, 1, 0, \dots),$$

is a monotone shrinking basis. By [**7**, Proposition 6.21], we may identify J_δ^{**} with the space of all convergent sequences $\beta = (\beta_n)$ in $\mathbf{R}$ satisfying

$$\sup_{m\in\mathbf{N}}\left\|\sum_{i=1}^{m}\beta_i e_i\right\|_\delta < +\infty,$$

with norm $\|\beta\|_\delta$ defined by this supremum. In what follows we will use the following notation. Given $l \in \mathbf{N}$, we define $\beta^{(l)} = (\beta_n^{(l)})$, where

$$\beta_n^{(l)} = \begin{cases} \beta_n & \text{if } n \le l, \\ 0 & \text{if } n > l. \end{cases}$$

Now it is clear that, for $\beta = (\beta_n) \in J_\delta^{**}$,

$$\|\beta\|_\delta = \sup\left\{(\delta\beta_{k_1}^{(l)} - \beta_{k_2}^{(l)})^2 + \sum_{i=2}^{n}(\beta_{k_i}^{(l)} - \beta_{k_{i+1}}^{(l)})^2 + (\beta_{k_{n+1}}^{(l)} - \delta\beta_{k_1}^{(l)})^2\right\}^{1/2},$$

where the supremum is taken over $n, l \in \mathbf{N}$, and finite increasing sequences $k_1 < k_2 < \cdots < k_{n+1}$ in $\mathbf{N}$.

By Proposition 2.11, it is enough to prove that, for t verifying $(*)$, and $\alpha = (\alpha_n) \in J_\delta$ and $\beta = (\beta_n) \in J_\delta^{**}$ with $\|\alpha\|_\delta = \|\beta\|_\delta = 1$, and $n \in \mathbf{N}$,

$$\overline{\lim_m}\, \|tP_n\alpha + P^{m**}\beta\|_\delta \le 1.$$

It is clear that, for all $n, h \in \mathbf{N}$, we have

$$2(\delta\alpha_n - \alpha_{n+h})^2 \le 1, \qquad 2(\delta\beta_n - \beta_{n+h})^2 \le 1,$$
$$|2(\delta\alpha_n)^2|, |2(\delta\beta_n)^2| \le 1.$$

Let $0 < \varepsilon < 1$. Since $\|\beta\|_\delta = 1$, there are $m_0, n_0 \in \mathbf{N}$ and $j_1 < \cdots < j_{n_0+1}$ in $\mathbf{N}$ such that, for

$$s_0 := (\delta\beta_{j_1}^{(m_0)} - \beta_{j_2}^{(m_0)})^2 + \sum_{i=2}^{n_0}(\beta_{j_i}^{(m_0)} - \beta_{j_{i+1}}^{(m_0)})^2 + (\beta_{j_{n_0+1}}^{(m_0)} - \delta\beta_{j_1}^{(m_0)})^2,$$

we have

$$s_0 > 1 - \varepsilon.$$

We claim that, for every $l \in \mathbf{N}$ with $l > \max\{m_0, j_{n_0+1}\}$ and $h_p < \cdots < h_{p+q}$, a finite increasing sequence in $\mathbf{N}$ with $h_p > j_{n_0+1}$,

$$\sum_{i=p}^{p+q-1}(\beta_{h_i}^{(l)} - \beta_{h_{i+1}}^{(l)})^2 < 1/2 + \varepsilon.$$

Indeed, let $h_p < \cdots < h_{p+q}$ be a finite sequence with $h_p > j_{n_0+1}$. First, suppose that $m_0 < j_{n_0+1}$, and denote

$$k = \min\{i \in \{1, 2, \ldots, n_0 + 1\} : j_i > m_0\}.$$

Note that $k > 1$ since $k = 1$ implies $s_0 = 0$, and this is a contradiction.

If $k = 2$, then $s_0 = 2(\delta\beta_{j_1})^2$. Take $h_{p+q+1} \in \mathbf{N}$ with $h_{p+q+1} > \max\{l, h_{p+q}\}$, and consider the finite sequence

$$j_1 < h_p < \cdots < h_{p+q} < h_{p+q+1}.$$

Then we have that

$$(\delta\beta_{j_1} - \beta^{(l)}_{h_p})^2 + \sum_{i=p}^{p+q}(\beta^{(1)}_{h_i} - \beta^{(l)}_{h_{i+1}})^2 + (\beta^{(l)}_{h_{p+q+1}} - \delta\beta_{j_1})^2 \leq \|\beta\|^2_\delta = 1.$$

So,

$$(\delta\beta_{j_1} - \beta^{(l)}_{h_p})^2 + \sum_{i=p}^{p+q-1}(\beta^{(l)}_{h_i} - \beta^{(l)}_{h_{i+1}})^2 + (\beta^{(l)}_{h_{p+q}})^2 + (\delta\beta_{j_1})^2 \leq 1.$$

Hence,

$$\begin{aligned}\sum_{i=p}^{p+q-1}(\beta^{(l)}_{h_i} - \beta^{(l)}_{h_{i+1}})^2 &\leq 1 - (\delta\beta_{j_1})^2 \\ &= 1 - \frac{s_0}{2} \\ &< 1 - \frac{1}{2}(1-\varepsilon) \\ &= \frac{1}{2} + \frac{1}{2}\varepsilon.\end{aligned}$$

If $k > 2$, then

$$s_0 = (\delta\beta_{j_1} - \beta_{j_2})^2 + \sum_{i=2}^{k-2}(\beta_{j_i} - \beta_{j_{i+1}})^2 + (\beta_{j_{k-1}})^2 + (\delta\beta_{j_1})^2,$$

and taking the finite sequence

$$j_1 < \cdots < j_{k_1} < h_p < \cdots < h_{p+q} < h_{p+q+1}$$

with $h_{p+q+1} > l$, which gives that

$$(\delta\beta_{j_1} - \beta_{j_2})^2 + \sum_{i=2}^{k-2}(\beta_{j_i} - \beta_{j_{i+1}})^2 + (\beta_{j_{k-1}} - \beta^{(l)}_{h_p})^2 + \sum_{i=p}^{p+q-1}(\beta^{(l)}_{h_i} - \beta^{(l)}_{h_{i+1}})^2 + (\beta_{h_{p+q}})^2 + (\delta\beta_{j_1})^2 \leq \|\beta\|^2_\delta = 1,$$

we deduce that

$$\sum_{i=p}^{p+q-1}(\beta^{(l)}_{h_i} - \beta^{(l)}_{h_{i+1}})^2 \leq 1 - s_0 + (\beta_{j_{k-1}})^2 < \frac{1}{2\delta^2} + \varepsilon < \frac{1}{2} + \varepsilon.$$

Finally, suppose that $m_0 \geq j_{n_0+1}$. In this case

$$s_0 = (\delta\beta_{j_1} - \beta_{j_2})^2 + \sum_{i=2}^{n_0}(\beta_{j_i} - \beta_{j_{i+1}})^2 + (\beta_{j_{n_0+1}} - \delta\beta_{j_1})^2,$$

and taking the finite sequence,

$$j_1 < \cdots < j_{n_0+1} < h_p < \cdots < h_{p+q},$$

which gives that

$$(\delta\beta_{j_1} - \beta_{j_2})^2 + \sum_{i=2}^{n_0}(\beta_{j_i} - \beta_{j_{i+1}})^2 + (\beta_{j_{n_0+1}} - \beta^{(l)}_{h_p})^2 + \sum_{i=p}^{p+q-1}(\beta^{(l)}_{h_i} - \beta^{(l)}_{h_{i+1}})^2 + (\beta^{(l)}_{h_{p+q}} - \delta\beta_{j_1})^2 \leq \|\beta\|^2_\delta = 1,$$

we have that

$$\sum_{i=p}^{p+q-1}(\beta^{(l)}_{h_i} - \beta^{(l)}_{h_{i+1}})^2 \leq 1 - s_0 + (\beta_{j_{n_0+1}} - \delta\beta_{j_1})^2 < \frac{1}{2} + \varepsilon.$$

Fix $m \in \mathbf{N}$ such that $m \geq \max\{n, m_0, j_{n_0+1}\}$, and let us denote by $\gamma = (\gamma_n)$ the sequence

$$tP_n\alpha + P^{m^**}\beta = (t\alpha_1, t\alpha_2, \dots, t\alpha_n, 0, \dots, 0, \beta_{m+1}, \beta_{m+2}, \dots).$$

Given $l \in \mathbf{N}$ and a finite sequence $k_1 < k_2 < \cdots < k_{p+1}$ in $\mathbf{N}$, we denote by

$$S := (\delta\gamma_{k_1}^{(l)} - \gamma_{k_2}^{(l)})^2 + \sum_{i=2}^{p}(\gamma_{k_i}^{(l)} - \gamma_{k_{i+1}}^{(l)})^2 + (\gamma_{k_{p+1}}^{(l)} - \delta\gamma_{k_1}^{(l)})^2.$$

If $l \leq m$ or $k_{p+1} \leq m$, then

$$\begin{aligned} S &= (\delta\alpha_{k_1}^{(n)} - \alpha_{k_2}^{(n)})^2 + \sum_{i=2}^{p}(\alpha_{k_i}^{(n)} - \alpha_{k_{i+1}}^{(n)})^2 + (\alpha_{k_{p+1}}^{(n)} - \delta\alpha_{k_1}^{(n)})^2 \\ &\leq t^2\|\alpha\|_\delta^2 = t^2 \\ &\leq 1 + \varepsilon. \end{aligned}$$

Assume that $l \geq m+1$ and $k_{p+1} \geq m+1$. If $k_1 \geq m+1$, then

$$\begin{aligned} S &= (\delta\beta_{k_1}^{(l)} - \beta_{k_2}^{(l)})^2 + \sum_{i=2}^{p}(\beta_{k_i}^{(l)} - \beta_{k_{i+1}}^{(l)})^2 + (\beta_{k_{p+1}}^{(l)} - \delta\beta_{k_1}^{(l)})^2 \\ &\leq \|\beta\|_\delta^2 = 1 \\ &\leq 1 + \varepsilon. \end{aligned}$$

If $n < k_1 \leq m$, and we denote $r = \min\{i \in \{1, \dots, p+1\} : k_i \geq m+1\}$, we have that

$$\begin{aligned} S &= (\beta_{k_r})^2 + \sum_{i=r+1}^{p+1}(\beta_{k_i}^{(l)} - \beta_{k_{i+1}}^{(l)})^2 + (\beta_{k_{p+1}}^{(l)})^2 \\ &\leq \frac{1}{\delta^2} + \frac{1}{2} + \varepsilon < 1 + \varepsilon. \end{aligned}$$

If $k_1 \leq n$, and we denote $s = \max\{i \in \{1, \dots, p+1\} : k_i \leq n\}$, in the case $s = 1$ and $r = 2$, we have that

$$\begin{aligned} S &= (\delta t\alpha_{k_1} - \beta_{k_2}^{(l)})^2 + \sum_{i=2}^{p}(\beta_{k_i}^{(l)} - \beta_{k_{i+1}}^{(l)})^2 + (\beta_{k_{p+1}}^{(l)} - \delta t\alpha_{k_1})^2 \\ &\leq \frac{(1+\delta t)^2}{2\delta^2} + \frac{1}{2} + \frac{(1+\delta t)^2}{2\delta^2} + \varepsilon \\ &\leq 1 + \varepsilon. \end{aligned}$$

If $s = 1$ and $r > 2$, then

$$\begin{aligned} S &= (\delta t \alpha_{k_1})^2 + (\beta_{k_r}^{(l)})^2 + \sum_{i=r}^{p} (\beta_{k_i}^{(l)} - \beta_{k_{i+1}}^{(l)})^2 + (\beta_{k_{p+1}}^{(l)} - \delta t \alpha_{k_1})^2 \\ &\le \frac{(\delta t)^2}{2\delta^2} + \frac{1}{2\delta^2} + \frac{1}{2} + \frac{(1+\delta t)^2}{2\delta^2} + \varepsilon \\ &\le 1 + \varepsilon. \end{aligned}$$

If $s > 1$ and $r = s + 1$, then

$$\begin{aligned} S &= (\delta t \alpha_{k_1} - t\alpha_{k_2})^2 + \sum_{i=2}^{s-1} (t\alpha_{k_i} - t\alpha_{k_{i+1}})^2 + (t\alpha_{k_s} - \beta_{k_{s+1}}^{(l)})^2 \\ &\quad + \sum_{i=s+1}^{p} (\beta_{k_i}^{(l)} - \beta_{k_{i+1}}^{(l)})^2 + (\beta_{k_{p+1}}^{(l)} - \delta t \alpha_{k_1})^2 \\ &\le t^2 + \frac{(1+t)^2}{2\delta^2} + \frac{1}{2} + \frac{(1+\delta t)^2}{2\delta^2} + \varepsilon \\ &\le 1 + \varepsilon. \end{aligned}$$

If $s > 1$ and $r > s + 1$, then

$$\begin{aligned} S &= (\delta t \alpha_{k_1} - t\alpha_{k_2})^2 + \sum_{i=2}^{s-1} (t\alpha_{k_i} - t\alpha_{k_{i+1}})^2 + (t\alpha_{k_s})^2 + (\beta_{k_r}^{(l)})^2 \\ &\quad + \sum_{i=r}^{p} (\beta_{k_i}^{(l)} - \beta_{k_{i+1}}^{(l)})^2 + (\beta_{k_{p+1}}^{(l)} - \delta t \alpha_{k_1})^2 \\ &\le t^2 + \frac{1}{2\delta^2} + \frac{1}{2} + \frac{(1+\delta t)^2}{2\delta^2} + \varepsilon \\ &\le 1 + \varepsilon. \end{aligned}$$

Therefore,

$$\overline{\lim_{m}} \, \|tP_n\alpha + P^{m**}\beta\| \le 1,$$

as required. □

Remark. The renorming of the James space J_δ shows that, in general, the Banach spaces satisfying the $M(r, s)$-inequality cannot be renormed to be M-ideals. Note that M-ideals contain c_0 and this is not true for the James space.

The next result of this section is new even for M-ideals.

Theorem 3.6. *Let $s \in]0,1]$. If X is a nonreflexive Banach space satisfying the $M(1,s)$-inequality, then every slice of B_X has diameter greater than or equal to $2s$. In particular, B_X is not dentable.*

Proof. Let $x^{**} \in X^{**}$. We can suppose, without loss of generality, that $\|x^{**} + X\| = 1$. Fix $\varepsilon > 0$ and $0 < \delta < \varepsilon/2$. Take $x \in S_X$ and $x^* \in S_{X^*}$ such that $x^*x > 1-\delta$. By Proposition 2.5, there exists a net (x_α) in X w^*-converging to x^{**} satisfying

$$\overline{\lim_\alpha} \|s(x^{**} - x_\alpha) \pm x\| \leq 1.$$

Denote by S the slice $\{y \in B_X : x^*y > 1-\varepsilon\}$. For a suitable $0 < \lambda < 1$ and α large enough, we have

$$|sx^*(x^{**} - x_\alpha)| < \delta,$$
$$\lambda(x \pm s(x^{**} - x_\alpha)) \in \overline{S}^{w^*}.$$

Therefore,

$$\begin{aligned}\operatorname{diam} S = \operatorname{diam} \overline{S}^{w^*} &\geq \lambda\|(x + s(x^{**} - x_\alpha)) - (x - s(x^{**} - x_\alpha))\| \\ &= 2\lambda s\|x^{**} - x_\alpha\| \\ &\geq 2\lambda s\|x^{**} + X\| \\ &= 2\lambda s.\end{aligned}$$

Now, letting $\lambda \to 1$, we can conclude that $\operatorname{diam} S \geq 2s$ so, by the Hahn-Banach theorem, B_X is not dentable. □

Remark. Notice again that the condition $r = 1$ is essential. In fact, since the bidual of the James space is a dual separable, by the Dunford-Pettis theorem, see, e.g., [**6**, Theorem 1], has the Radon-Nikodým property so every bounded subset of J_δ is dentable [**4**].

To end this section we show, following an idea of [**10**, Proposition 4.4], that there are separable canonical u-ideals, simultaneously U^*-spaces and HB-subspaces, which cannot be renormed to satisfy the $M(1,s)$-inequality.

Example 3.7. If X is a nonreflexive separable M-ideal, then $l_p(X)$, $1 < p < +\infty$, is a canonical u-ideal, U^*-space and HB-subspace, but cannot be renormed to satisfy the $M(1,s)$-inequality for any $s \in]0,1]$.

Proof. If X is an M-ideal, then it is a canonical u-ideal, U^*-space and HB-subspace and so, by Proposition 2.2, $l_p(X)$ is also a canonical u-ideal, U^*-space and HB-subspace.

We denote $Y = l_p(X)$. Let $0 < s \leq 1$ and (δ_n) a sequence in $\mathbf{R}^+$ such that $\sum_{n=1}^{+\infty} \delta_n < +\infty$. Suppose that Y satisfies the $M(1,s)$-inequality. In order to reach a contradiction, we shall show by induction that there exists a sequence (x_n) in X satisfying $\|x_n\| \geq s/2$ and

$$\|(x_1, x_2, \dots, x_n, 0, \dots)\|_p < C_n, \quad \forall\, n \in \mathbf{N},$$

where $C_n = \prod_{k=1}^{n}(1+\delta_k)$. Indeed, for an arbitrary $x_1 \in X$ with $\|x_1\| = s/2$, it is clear that

$$\|(x_1, 0, \dots)\|_p = s/2 < C_1.$$

Assume that we have found $x_1, x_2, \dots, x_{n-1}$ as above, and denote $S_{n-1} = (x_1, x_2, \dots, x_{n-1}, 0, \dots)$, X_n to the subspace of Y defined by

$$\{(\underbrace{0, \dots, 0}_{n-1}, x, 0, \dots) : x \in X\}.$$

Since X is a proximinal subspace of X^{**}, see, e.g., [**15**, Proposition II.1.1], we can take $e_n^{**} \in X_n^{\perp\perp}$ with

$$\|e_n^{**}\|_p = \|e_n^{**} + X_n\|_p = s.$$

By Proposition 2.5, there exists a sequence (z_k) in Y (or in X_n, by Proposition 2.1) w^*-converging to e_n^{**} such that

$$\overline{\lim_k} \|S_{n-1} + e_n^{**} - z_k\|_p \leq C_{n-1}.$$

Let $k \in \mathbf{N}$ be such that $\|S_{n-1} + e_n^{**} - z_k\|_p < C_n$. By Goldstine's theorem, it is easy to find a sequence (u_j) in X_n w^*-converging to $e_n^{**} - z_k$ such that

$$\|S_{n-1} - u_j\|_p < C_n$$

and

$$\varliminf_j \|u_j\| \geq \|e_n^{**} + X_n\|_p = s.$$

Now it suffices to take $x_n = u_j$ for j large enough. □

4. U^*-spaces. In this section we show that, for a Banach space X to enjoy the previously not considered known properties of M-ideals, it is enough to suppose that X is a U^*-space. Observe that if X satisfies the $M(r,1)$-inequality, then X is a U^*-space. The converse is not true, as we will see below.

The next result is crucial in what follows.

Proposition 4.1. *Let X be a U^*-space. Then*

1. *X does not contain an isomorphic copy of l_1.*
2. *If Q is a norm one projection on X^*, then $Q(X^*)$ is w^*-closed.*

Proof. 1) If a Banach space X contains an isomorphic copy of l_1, then $\|I - \pi\| = 2$ [**10**, Proposition 2.6], and this is a contradiction to the assumption on X.

2) First of all, we claim that $Q^{**}\pi Q^{**} = \pi Q^{**}$.

In fact, if $\varphi \in X^{***}$, then it is clear that $Q^{**}\pi Q^{**}\varphi \in X^*$, and so, $\pi Q^{**}\pi Q^{**} = Q^{**}\pi Q^{**}$. By the assumption on X and Q, if $\pi(Q^{**}\pi Q^{**}\varphi - Q^{**}\varphi) \neq 0$, then

$$\begin{aligned}\|\pi Q^{**}\varphi - Q^{**}\varphi\| &\geq \|Q^{**}\pi Q^{**}\varphi - Q^{**}\varphi\| \\ &> \|(Q^{**}\pi Q^{**}\varphi - Q^{**}\varphi) - \pi(Q^{**}\pi Q^{**}\varphi - Q^{**}\varphi)\| \\ &= \|Q^{**}\varphi - \pi Q^{**}\varphi\|,\end{aligned}$$

and this is a contradiction. Therefore,

$$\pi Q^{**}\varphi = \pi Q^{**}\pi Q^{**}\varphi = Q^{**}\pi Q^{**}\varphi.$$

Since

$$\pi = i_{X^*}(i_X)^*, \qquad Q^{**}(X^{***}) = Q(X^*)^{\perp\perp},$$

and

$$Q^{**} i_{X^*} = i_{X^*} Q,$$

(where i_X denotes the canonical embedding) we have that

$$\begin{aligned} i_{X^*}(i_X)^* Q^{**} &= \pi_X Q^{**} = Q^{**} \pi_X Q^{**} \\ &= Q^{**} i_{X^*}(i_X)^* Q^{**} \\ &= i_X Q(i_X)^* Q^{**}. \end{aligned}$$

So, since i_{X^*} is injective, we have that

$$\begin{aligned} (i_X)^*(Q(X^*)^{\perp\perp}) &= (i_X)^*(Q^{**}(X^{***})) \\ &= Q(i_X)^* Q^{**}(X^{***}) \subseteq Q(X^*). \end{aligned}$$

Therefore, $Q(X^*)$ is w^*-closed. □

Our next result is proved for M-ideals in [**14**, Proposition 4.2].

Theorem 4.2. *Let X be a nonreflexive U^*-space. If Y is a Banach space such that $\|I - \pi_Y\| \leq 1$, then every isometric isomorphism from X^{**} onto Y^{**} is the bitranspose of an isometric isomorphism from X onto Y.*

Proof. Let $\varphi \in X^{***}$ and $x^* \in X^*$ with $\pi_X \varphi \neq x^*$. Then

$$\|\varphi - x^*\| > \|\varphi - x^* - \pi_X \varphi + \pi_X x^*\| = \|\varphi - \pi_X \varphi\|.$$

Therefore,

$$P_{X^*}(\varphi) = \{\pi_X \varphi\}, \quad \forall\, \varphi \in X^{***}.$$

Of course,

$$\pi_Y \chi \in P_{Y^*}(\chi), \quad \forall\, \chi \in Y^{***}.$$

Now let $U : X^{**} \to Y^{**}$ be an isometric isomorphism. Since X and Y contain no copy of l_1 [**10**, Proposition 2.6], by [**10**, Lemma 5.6] and

[**9**, Corollary 5.5], U is w^*-continuous. In particular, $U^*(Y^*) = X^*$. It is clear that

$$\begin{aligned}\|U^*\chi - U^*\pi_Y\chi\| &= \|\chi - \pi_Y\chi\| \\ &= \|\chi + Y^*\| \\ &= \|U^*\chi + X^*\|, \quad \forall\, \chi \in Y^{***},\end{aligned}$$

and so,

$$U^*\pi_Y = \pi_X U^*.$$

Hence,

$$U^*(Y^\perp) = X^\perp.$$

Therefore, by the Hahn-Banach theorem, $U(X) = Y$. Now we can define $H : X \to Y$ by

$$Hx = i_Y^{-1} U i_X x, \quad \forall\, x \in X.$$

The operator H is continuous and H^{**} coincides with U on X. Since both operators are w^*-continuous, $H^{**} = U$. □

The above theorem is not true for the $M(r,s)$-inequality with $s < 1$, not even with $r = 1$ as shown by the following renorming of c_0 due to Johnson and Wolfe [**17**].

Example 4.3. Let $0 < \mu < 1$. We consider in c_0 the following norm:

$$\|x\| = \sup\left\{\frac{|x_1|}{\mu}, |x_1 - x_2|, |x_1 - x_3|, \dots\right\},$$

where $x = (x_1, x_2, \dots) \in c_0$. We denote $s := (1-\mu)/(1+\mu)$. Then

1. $X = (c_0, \|\cdot\|)$ satisfies the $M(1,s)$-inequality.

2. X is neither a canonical u-ideal nor an HB-subspace. In particular, X is not an M-ideal.

3. The isometric isomorphism V of X^{**} defined by

$$V(\beta_n) = (-\beta_1, \beta_2 - 2\beta_1, \dots, \beta_n - 2\beta_1, \dots) \qquad \forall\, (\beta_n) \in X^{**},$$

is not the bitranspose of any isometric isomorphism of X.

Proof. 1) It is easy to show that X satisfies the assumption in Corollary 2.12.

2) In this case, X satisfies the equality $\|I-\pi\| = 1+\mu$ [**17**]. Now it is enough to observe that $\|I-2\pi\| = 1$ implies that $\|I-\pi\| = 1$.

3) Consider $U : X^* \to X^*$ defined by $U(\lambda_n) = (\mu_n)$ where $\mu_1 = -\lambda_1 - 2\sum_{n=2}^{+\infty} \lambda_n$ and $\mu_n = \lambda_n$ for every $n \geq 2$. Then U is an involutive isometry of X^* [**17**]. Let V be the transpose of U so that V is the involutive isometry of X^{**} given by $V(\beta_n) = (\alpha_n)$, where $\alpha_1 = -\beta_1$ and $\alpha_n = \beta_n - 2\beta_1$ for every $n \geq 2$. Then clearly $V(c_0) \neq c_0$. □

Remark. Using [**23**, Corollary 3], assertion 1 of the last example was proved in [**24**, Example 4], where it was also observed that X is not an HB-subspace.

The next results are proved for M-ideals in [**8**, Theorem 3]. Our proof involves looking at Propositions 2.1 and 4.1, and it is based on the classical case.

Theorem 4.4. *Let X be a nonreflexive Asplund U^*-space. Then X is* WCG.

Proof. According to [**8**, Theorem 1], there are a nondecreasing "long sequence" of subspaces $\{M_\alpha : \omega \leq \alpha \leq \mu\}$ of X and a "long sequence" $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ of linear projections on X^* such that $M_\mu = X$, P_μ is identity, and for all $\omega < \alpha \leq \mu$, where μ denotes the first ordinal with cardinality $\operatorname{dens} X$, the following conditions hold.

1. $\|P_\alpha\| = 1$,
2. $\operatorname{dens} P_\alpha(X^*) \leq |\alpha|$,
3. $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$ if $\beta \leq \alpha$,
4. $\cup_{\beta<\alpha} P_{\beta+1}(X^*)$ is dense in $P_\alpha(X^*)$,

where $|\alpha|$ denotes the cardinality of the ordinal α. (A "long sequence" $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ of linear projections which shares the above properties is called a PRI). Moreover, from (vii) in [**8**, Theorem 1], $\operatorname{Ker} P_\alpha = M_\alpha^\perp$, so it is w^*-closed and, since by Proposition 4.1, $\operatorname{Im} P_\alpha$ is w^*-closed, then P_α is w^*-continuous. This means that $P_\alpha^*(X) \subseteq X$;

hence, defining $Q_\alpha x = P_\alpha^* x$ for all $x \in X$, we have $Q_\alpha^* = P_\alpha$. Now we can follow as in the proof of [**8**, Theorem 3]. □

Remark. It remains an open question whether there are Banach spaces X satisfying the $M(1,s)$-inequality without being WCG.

Since a U^*-space contains no isomorphic copy of l_1, X is a strict u-ideal if and only if X is a canonical u-ideal [**10**, Proposition 5.2]. Therefore, we can state the following.

Corollary 4.5. *Let X be a nonreflexive U^*-space which is a strict u-ideal or satisfies the $M(1,s)$-inequality. Then X contains a copy of c_0. In particular, every copy of c_0 is complemented in X.*

Proof. According to [**10**, Proposition 2.8] or Propositions 2.1 and 2.5, X is an Asplund space. By the above theorem, X is WCG. By a standard procedure, see, e.g., [**5**, p. 149], one can get that there exists a nonreflexive separable subspace Y of X, together with a norm one projection Q from X onto Y. By Proposition 2.1, Y is a strict u-ideal or satisfies the $M(1,s)$-inequality, and by [**10**, Theorem 5.4] and [**15**, p. 133] or Corollary 3.4, it contains an isomorphic copy of c_0. Hence, by Sobczyk's theorem, see, e.g., [**22**, Theorem 2.f.5], there is a projection $P : Y \to c_0$. Then $P \circ Q$ is a projection, showing that c_0 is complemented in X. □

The following examples clarify the relation between U^*-spaces and the $M(r,s)$-inequality.

Example 4.6. Let X and Y be two M-ideals. Given $0 < \gamma \leq 1$, we denote

$$\|(x,y)\| = \max\left\{\|x\|, \|y\|, \frac{\|x\| + \|y\|}{1+\gamma}\right\}, \qquad x \in X,\ y \in Y.$$

Then $Z = (X \times Y, \|\cdot\|)$ satisfies, simultaneously, the $M(1,\gamma)$-inequality and the $M(\gamma,1)$-inequality. Moreover, if $\gamma \neq 1$, then Z is not an M-ideal.

Proof. We will need the following technical lemma, whose proof is straightforward.

Lemma 4.7. *For every* $0 \le \gamma \le 1$, *consider the norm in* $\mathbf{R}^2$ *defined by*

$$|(a,b)|_\gamma = \max\{|a| + \gamma|b|, |b| + \gamma|a|\}, \quad a, b \in \mathbf{R}.$$

Then, for every $a, b, c, d \in \mathbf{R}_0^+$, *we have that*

$$|(a+b, c+d)|_\gamma \ge ||(a,c)|_\gamma, |(b,d)|_\gamma|_\gamma.$$

It is easy to prove that $Z^* = (X^* \times Y^*, \|\cdot\|^*)$, where

$$\|(x^*, y^*)\|^* = \max\{\|x^*\| + \gamma\|y^*\|, \|y^*\| + \gamma\|x^*\|\}, \qquad x^* \in X,\ y^* \in Y^*.$$

According to the above lemma and by the assumptions on X and Y, the projection π_Z $(= \pi_X \times \pi_Y)$ satisfies

$$\|(\varphi, \chi)\| \ge \max \left\{ \begin{matrix} \|\pi_Z(\varphi,\chi)\| + \gamma\|(I - \pi_Z)(\varphi,\chi)\|, \\ \gamma\|\pi_Z(\varphi,\chi)\| + \|(I - \pi_Z)(\varphi,\chi)\| \end{matrix} \right\}$$

for every $(\varphi, \chi) \in Z^{***}$. If $\gamma < 1$, then it is straightforward to prove that π_Z is not an L-projection [**14**, Proposition 3.1]. □

Example 4.8. Let $X = c_0 \oplus_{l_2} c_0$. Then X is a U^*-space failing the $M(r,s)$-inequality for all $r, s \in\,]0,1]$ with $r^2 + s^2 > 1$.

Proof. It is clear that

$$X^* = l_1 \oplus_{l_2} l_1, \qquad X^{**} = l_\infty \oplus_{l_2} l_\infty,$$

and

$$\pi = \pi_{c_0} \times \pi_{c_0}.$$

Suppose that X satisfies the $M(r,s)$-inequality for certain $r, s \in\,]0,1]$ with $r^2 + s^2 > 1$. Let $\varphi \in c_0^\perp$ and $\psi \in l_1$, and write

$$a = \|\pi_{c_0}\psi\|, \qquad b = \|\varphi - \pi_{c_0}\varphi\|.$$

By assumption, we have that

$$a^2 + b^2 \geq r^2 a^2 + s^2 b^2 + 2rsab,$$

and, of course, for appropriate a and b, that is, φ and ψ, the above inequality is not true. $\square$

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REFERENCES

1. E.M. Alfsen and E.G. Effros, *Structure in real Banach spaces*, *Part* I and II, Ann. of Math. **96** (1972), 98–173.

2. T. Ando, *Closed range theorems for convex sets and linear liftings*, Pacific J. Math. **44** (1973), 393–410.

3. C.-M. Cho and W.B. Johnson, *A characterization of subspaces X of l_p for which $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$*, Proc. Amer. Math. Soc. **93** (1985), 466–470.

4. W.J. Davis and R.R. Phelps, *The Radon-Nikodým property and dentables sets in Banach spaces*, Proc. Amer. Math. Soc. **45** (1974), 119–122.

5. J. Diestel, *Geometry of Banach spaces—Selected topics*, Lecture Notes in Math. **485**, Springer, New York, 1975.

6. J. Diestel and J.J. Uhl, *Vector measures*, Math. Surveys **15**, American Mathematical Soceity, Providence, Rhode Island, 1977.

7. D. Van Dulst, *Reflexive and superreflexive Banach spaces*, Math. Centre Tracts **102**, Amsterdam, 1978.

8. M. Fabian and G. Godefroy, *The dual of every Asplund space admits a projectional resolution of the identity*, Studia Math. **91** (1988), 141–151.

9. G. Godefroy and N.J. Kalton, *The ball topology and its applications*, Contemp. Math. **85** (1989), 195–237.

10. G. Godefroy, N.J. Kalton and P.D. Saphar, *Unconditional ideals in Banach spaces*, Studia Math. **104** (1993), 13–59.

11. G. Godefroy and D. Li, *Banach spaces which are M-ideals in their bidual have property* (u), Ann. Inst. Fourier **39** (1989), 361–371.

12. G. Godefroy and P. Saab, *Weakly unconditionally convergent series in M-ideals*, Math. Scand. **64** (1989), 307–318.

13. G. Godefroy and D. Saphar, *Duality in spaces of operators and smooth norms in Banach spaces*, Illinois J. Math. **32** (1988), 672–695.

14. P. Harmand and A. Lima, *Banach spaces which are M-ideals in their biduals*, Trans. Amer. Math. Soc. **283** (1984), 253–264.

15. P. Harmand, D. Werner and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Math. **1547**, Springer, New York, 1993.

16. J. Hennefeld, *M-ideals, HB-subspaces, and compact operators*, Indiana Univ. Math. J. **28** (1979), 927–934.

17. J. Johnson and J. Wolfe, *On the norm of the canonical projection of E^{***} onto $E^{\perp}$*, Proc. Amer. Math. Soc. **75** (1979), 50–52.

18. N.J. Kalton, *M-ideals of compact operators*, Illinois J. Math. **37** (1993), 147–169.

19. Å. Lima, *Intersection properties of balls and subspaces in Banach spaces*, Trans. Amer. Math. Soc. **227** (1977), 1–62.

20. ———, *On M-ideals and best approximation*, Indiana Univ. Math. J. **31** (1982), 27–36.

21. Å. Lima, E. Oja, T.S.S.R.K. Rao and D. Werner, *Geometry of operator spaces*, Michigan Math. J. **41** (1994), 473–490.

22. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces* I, Springer, New York, 1977.

23. E. Oja, *On the uniqueness of the norm-preserving extension of a linear functional in the Hahn-Banach theorem*, Izv. Akad. Nauk Est. SSR **33** (1984), 424–438 (Russian).

24. E. Oja, *Strong uniqueness of the extension of linear continuous functionals according to the Hahn-Banach theorem*, Math. Notes **43** (1988), 134–139.

25. R.R. Phelps, *Uniqueness of Hahn-Banach extensions and unique best approximation*, Trans. Amer. Math. Soc. **95** (1960), 238–255.

26. M.A. Smith and Sullivan, *Extremely smooth Banach spaces*, in *Banach spaces of analytic functions*, Lecture Notes in Math. **604**, 125–137. Springer, New York, 1977.

27. D. Werner, *New classes of Banach spaces which are M-ideals in their biduals*, Math. Proc. Cambridge Phil. Soc. **111** (1992), 337–354.

28. D. Werner, *M-ideals and the 'basic inequality',* J. Approx. Theory **76** (1994), 21–30.

Universidad de Granada, Facultad de Ciencias, Departamento de Analisis Matematico, 18071 Granada, Spain
E-mail address: jcabello@goliat.ugr.es