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OSCILLATION OF THE SOLUTIONS OF IMPULSIVE DIFFERENTIAL EQUATIONS AND INEQUALITIES WITH A RETARDED ARGUMENT

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ABSTRACT. Sufficient conditions for oscillation of all solutions of a class of impulsive differential equations and inequalities with a retarded argument and fixed moments of impulse effect are found.

1. Introduction. The impulsive differential equations are an adequate mathematical apparatus for simulation of processes and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnologies, industrial robotics, economics, etc. Due to this reason, in recent years they have been an object of active research. In the monographs [**2**–**4**] a number of properties of their solutions are studied and an extensive bibliography is given.

We shall note that, in spite of the great number of investigations of the impulsive differential equations, their oscillation theory has not yet been elaborated unlike the oscillation theory of the differential equations with a deviating argument (see the monographs [**6**–**8**]).

The first work in which the oscillatory behavior of impulsive differential equations with a deviating argument and fixed moments of impulsive effect is investigated is [**5**]. Moreover, we shall note the work [**1**] in which the Sturmian theory for impulsive differential equations is considered.

In the present work sufficient conditions for oscillation of all solutions of a class of impulsive differential equations and inequalities with a retarded argument and fixed moments of impulse effect are found.

2. Preliminary notes. Let $\mathbf{N}_m = \{1, 2, \dots, m\}$ and h_i be positive constants, $i \in \mathbf{N}_m$, $\bar{h} = \max\{h_i : i \in \mathbf{N}_m\}$, $h = \min\{h_i : i \in \mathbf{N}_m\}$, $\{\tau_k\}_{k=1}^{\infty}$ be a monotone increasing, unbounded sequence of positive numbers, and let $\{b_k\}_{k=1}^{\infty}$ be a sequence of real numbers.

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Consider the impulsive differential equation and inequalities with a retarded argument

$$(1)\quad \begin{gathered} x'(t)+a(t)x(t)+p(t)f(x(t-h_1),x(t-h_2),\dots,x(t-h_m))\le 0,\\ t\ne\tau_k\\ \Delta x(\tau_k)=x(\tau_k+0)-x(\tau_k-0)=b_kx(\tau_k-0), \end{gathered}$$

where $x(\tau_k-0)=x(\tau_k)$;

$$(2)\quad \begin{gathered} x'(t)+a(t)x(t)+p(t)f(x(t-h_1),x(t-h_2),\dots,x(t-h_m))\ge 0,\\ t\ne\tau_k\\ \Delta x(\tau_k)=b_kx(\tau_k), \end{gathered}$$

and

$$(3)\quad \begin{gathered} x'(t)+a(t)x(t)+p(t)f(x(t-h_1),x(t-h_2),\dots,x(t-h_m))= 0,\\ t\ne\tau_k\\ \Delta x(\tau_k)=b_kx(\tau_k), \end{gathered}$$

with initial function

$$(4)\qquad x(t)=\varphi(t),\quad t\in[-\bar h,0]$$

where $\varphi\in C([-\bar h,0],\mathbf{R})$.

Introduce the following conditions:

H1. $a\in C_{\mathrm{loc}}(\overline{\mathbf{R}}^+,\mathbf{R}^+)$, $\mathbf{R}^+=(0,\infty)$, $\overline{\mathbf{R}}^+=[0,\infty)$.

H2. $p\in C_{\mathrm{loc}}(\overline{\mathbf{R}}^+,\mathbf{R}^+)$.

H3. $f\in C_{\mathrm{loc}}(\mathbf{R}^m,\mathbf{R})$, $f(u_1,u_2,\dots,u_m)u_1>0$ for $u_1\ne 0$ and $\operatorname{sgn}u_1=\operatorname{sgn}u_2=\cdots=\operatorname{sgn}u_m$.

H4. There exist constants $L>0$ and $\alpha_1,\alpha_2,\dots,\alpha_m$, $\alpha_i\ge 0$, $i\in\mathbf{N}_m$, such that $\sum_{i=1}^m\alpha_i=1$ and

$$|f(u_1,u_2,\dots,u_m)|\ge L|u_1|^{\alpha_1}|u_2|^{\alpha_2}\cdots|u_m|^{\alpha_m}.$$

H5. There exist constants l_1 and l_2 such that

$$\lim_{k\to\infty}(\tau_k-kl_1)=l_2.$$

H6. There exists a constant $M > 0$ such that, for any $k \in \mathbf{N}$, the inequalities $0 < b_k < M$ are valid.

H7. $\tau_{k+1} - \tau_k \geq T > \bar{h}$ for $k \in \mathbf{N}$.

Denote by $i[a,b]$ the number of the impulse moments in the interval $[a,b]$, $0 < a < b < \infty$.

Let us construct the sequence

$$\{t_i\}_{i=1}^{\infty} = \{\tau_i\}_{i=1}^{\infty} \cup \{\tau_{is}\}_{i=1,s=1}^{\infty,\ m}$$

where $\tau_{is} = \tau_i + h_s$, $i \in \mathbf{N}$, $s \in \mathbf{N}_m$ and $t_i < t_{i+1}$, $i \in \mathbf{N}$.

Definition 1. By a *solution* of equation (3) with initial function (4), we mean any function $x : [-\bar{h}, \infty) \to \mathbf{R}$ for which the following conditions are valid:

1. If $-\bar{h} \leq t \leq 0$, then $x(t) = \varphi(t)$.

2. If $0 \leq t \leq t_1 = \tau_1$, then x coincides with the solution of the problem

$$x'(t) + a(t)x(t) + p(t)f(x(t-h_1), \dots, x(t-h_m)) = 0$$

with initial condition (4).

3. If $t_i < t \leq t_{i+1}$, $t_i \in \{\tau_i\}_{i=1}^{\infty} \backslash \{\tau_{is}\}_{i=1,s=1}^{\infty,\ m}$, then x coincides with the solution of the problem

$$x'(t) + a(t)x(t) + p(t)f(x(t-h_1), \dots, x(t-h_m)) = 0$$
$$x(t_i + 0) = (1 + b_{k_i})x(t_i)$$

where the number k_i is determined from the equality $t_i = \tau_{k_i}$.

4. If $t_i < t \leq t_{i+1}$, $t_i \in \{\tau_{is}\}_{i=1,s=1}^{\infty,\ m} \backslash \{\tau_i\}_{i=1}^{\infty}$, then x coincides with the solution of the problem

$$x'(t) + a(t)x(t+0) + p(t)f(x(t-h_1+0), \dots, x(t-h_m+0)) = 0$$
$$x(t_i + 0) = x(t_i).$$

5. If $t_i < t \leq t_{i+1}$, $t_i \in \{\tau_i\}_{i=1}^{\infty} \cap \{\tau_{is}\}_{i=1,s=1}^{\infty,\ m}$, then x coincides with the solution of the problem

$$x'(t) + a(t)x(t+0) + p(t)f(x(t-h_1+0), \dots, x(t-h_m+0)) = 0$$
$$x(t_i + 0) = (1 + b_{k_i})x(t_i).$$

Remark 1. The definition of a solution of the problem (1), (4) ((2), (4)) is analogous to Definition 1.

Definition 2. The nonzero solution x of the equation (3) is said to be *nonoscillating* if there exists a point $t_0 \geq 0$ such that $x(t)$ has a constant sign for $t \geq t_0$. Otherwise, the solution x is said to *oscillate*.

3. Main results.

Lemma 1. *Let condition* H5 *hold. Then there exists a constant* $l \in \mathbf{N}$ *such that the number of the impulse moments in each of the intervals* $[a, a+h]$, $a > 0$ *is not greater than* l.

Proof. In view of $\tau_k > 0$, $k = 1, 2, \dots$, it follows that $l_1 > 0$. Let $\varepsilon_0 < l_1/2$. Then there exists $k_0 \in \mathbf{N}$ such that, for any $k \geq k_0$, we have

$$l_2 - \varepsilon_0 < \tau_k - kl_1 < l_2 + \varepsilon_0,$$

i.e.,

$$l_2 - \varepsilon_0 + kl_1 < \tau_k < l_2 + \varepsilon_0 + kl_1.$$

Analogously, we obtain

$$l_2 - \varepsilon_0 + (k+1)l_1 < \tau_{k+1} < l_2 + \varepsilon_0 + (k+1)l_1$$

whence we deduce

$$\tau_{k+1} - \tau_k > [l_2 - \varepsilon_0 + (k+1)l_1] - [l_2 + \varepsilon_0 + kl_1] = l_1 - 2\varepsilon_0 > 0.$$

Consequently, if $a > \tau_{k_0}$, then in the interval $[a, a+h]$ there are at most $h/(l_1 - 2\varepsilon_0)$ impulse moments.

Finally, in each interval of the form $[a, a+h]$, $a > 0$, we have at most $l = k_0 + h/(l_1 - 2\varepsilon_0)$ impulse moments. □

Theorem 1. *Let the following conditions hold*:

1. *Conditions* H1–H6 *are met.*

2.
$$\liminf_{t\to\infty}\int_{t-h}^{t} a(s)\,ds \ge k > 0,$$
where $k = \text{const}$.

3.
$$\liminf_{t\to\infty}\int_{t-h}^{t} p(s)\,ds > \frac{(1+M)^{2l}}{Le^k}\max\left\{\frac{1}{e}, 2(1+M)^l[(1+M)^l-1]\right\}$$
where $e = \exp$.

Then the inequality (1) *has no positive solutions.*

Proof. Let x be a positive solution of the inequality (1) for $t \ge t_0 \ge 0$. It is clear that $x(t-h_i) > 0$, $i \in \mathbf{N}_m$, and
$$f(x(t-h_1), x(t-h_2), \dots, x(t-h_m)) > 0 \quad \text{for } t \ge t_0 + \bar{h}.$$

Let $t > T \ge t_0 + \bar{h}$. Multiply (1) by $e^{\int_T^t a(s)\,ds}$ and obtain
$$\left(x(t)e^{\int_T^t a(s)\,ds}\right)' + p(t)e^{\int_T^t a(s)\,ds} f(x(t-h_1), \dots, x(t-h_m)) \le 0. \tag{5}$$

Set
$$z(t) = x(t)e^{\int_T^t a(s)\,ds}, \quad t \ge T \tag{6}$$
and from (5) find that
$$z'(t) + p(t)e^{\int_T^t a(s)\,ds} f\Big(z(t-h_1)e^{-\int_T^{t-h_1} a(s)\,ds}, \dots, z(t-h_m)e^{-\int_T^{t-h_m} a(s)\,ds}\Big) \le 0, \quad t \ne \tau_k, \tag{7}$$
$$\Delta z(\tau_k) = z(\tau_k + 0) - z(\tau_k) = b_k x(\tau_k)e^{\int_T^{\tau_k} a(s)\,ds} = b_k z(\tau_k).$$

From (6) it follows that $z(t) > 0$ for $t > T$. Then $z(t-h_i) > 0$, $i \in \mathbf{N}_m$, and
$$f\Big(z(t-h_1)e^{-\int_T^{t-h_1} a(s)\,ds}, \dots, z(t-h_m)e^{-\int_T^{t-h_m} a(s)\,ds}\Big) > 0$$

for $t \geq T_1 = T + \bar{h}$.

From the above inequalities and from (7) it follows that z is a nonincreasing function in the set $(T_1, \tau_s) \cup [\cup_{i=s}^{\infty} (\tau_i, \tau_{i+1})]$, where $\tau_{s-1} < T_1 < \tau_s$.

Introduce the notation

$$w(t) = \frac{z(t-h)}{z(t)}, \qquad t \geq T_2 = T_1 + h.$$

Let us renumber the points of jump so that

$$t - h < \tau_1 < \tau_2 < \cdots < \tau_\lambda < t$$

where, by Lemma 1, $\lambda \leq l$, $\lambda \in \mathbf{N}$. Then

$$\begin{aligned} z(t-h) \geq z(\tau_1) = \frac{z(\tau_1 + 0)}{1 + b_1} &\geq \cdots \\ &\geq \frac{z(t)}{\prod_{i=1}^{\lambda}(1 + b_i)} \geq \frac{z(t)}{(1+M)^l}, \end{aligned}$$

i.e.,

$$w(t) = \frac{z(t-h)}{z(t)} \geq \frac{1}{(1+M)^l}. \tag{8}$$

We shall prove that the function w is bounded from above for sufficiently large t.

From condition 3 of Theorem 1 it follows that there exists a constant $N > 0$ such that

$$\int_{t-h}^{t} p(s)\, ds \geq N > \frac{(1+M)^{2l}}{Le^k} \max\left\{\frac{1}{e}, 2(1+M)^l[(1+M)^l - 1]\right\}$$

for sufficiently large t, $t \geq T_2$.

Let $t^* \geq T_3$ for some $T_3 \geq T_2$. Then there exists a point t, $t \geq T_2$, such that $t^* \in (t-h, t)$ and

$$\int_{t-h}^{t^*} p(s)\, ds \geq \frac{N}{2}, \qquad \int_{t^*}^{t} p(s)\, ds > \frac{N}{2}.$$

Integrate (7) from $t-h$ to t^* and obtain that

$$
\begin{aligned}
(9)\quad z(t-h)-z(t^*)+\sum_{\tau_k\in[t-h,t^*)} b_k z(\tau_k)
&\geq L\int_{t-h}^{t^*} p(s)e^{\int_T^s a(u)\,du} z^{\alpha_1}(s-h_1)z^{\alpha_2}(s-h_2)\\
&\qquad \cdots z^{\alpha_m}(s-h_m)\prod_{i=1}^{m} e^{-\alpha_i\int_T^{s-h_i} a(u)\,du}\,ds.
\end{aligned}
$$

Moreover,

$$
(10)\qquad
\begin{aligned}
e^{\int_T^s a(u)\,du}\prod_{i=1}^{m} e^{-\alpha_i\int_T^{s-h_i} a(u)\,du} &= \prod_{i=1}^{m} e^{\alpha_i\int_{s-h_i}^{s} a(u)\,du}\\
e^{\sum_{i=1}^{m}\alpha_i \liminf_{s\to\infty}\int_{s-h_i}^{s} a(u)\,du} &\geq e^{\sum_{i=1}^{m}\alpha_i k} \geq e^k.
\end{aligned}
$$

From the fact that z is a nonincreasing function in the set $(T_1,\tau_s)\cup[\cup_{i=s}^{\infty}(\tau_i,\tau_{i+1})]$, it follows that

$$
z(s-h_i) > \frac{z(s-h)}{(1+M)^{i[s-h_i,s-h]}} \geq \frac{z(s-h)}{(1+M)^l}.
$$

Then

$$
(11)\qquad \prod_{i=1}^{m} z^{\alpha_i}(s-h_i) \geq \prod_{i=1}^{m}\frac{z^{\alpha_i}(s-h)}{(1+M)^{\alpha_i l}} = \frac{z(s-h)}{(1+M)^l}.
$$

From (9), (10) and (11), we obtain that

$$
\begin{aligned}
(12)\qquad z(t-h)-z(t^*)+\sum_{\tau_k\in[t-h,t^*]} b_k z(\tau_k)
&\geq \frac{Le^k}{(1+M)^l}\int_{t-h}^{t^*} p(s)z(s-h)\,ds\\
&\geq \frac{Le^kN}{2(1+M)^l}\inf_{s\in[t-h,t^*]} z(s-h)\\
&\geq \frac{LNe^k}{2(1+M)^l}\,\frac{z(t^*-h)}{(1+M)^{i[t-2h,t^*-h]}}.
\end{aligned}
$$

Since $i[t-2h, t^*-h] \le l$, then from (12) it follows that

$$z(t^*-h) \le \frac{2(1+M)^{2l}}{LNe^k}\Big[z(t-h) + \sum_{\tau_k \in [t-h,t^*]} b_k z(\tau_k)\Big]. \tag{13}$$

Integrate (7) from t^* to t and obtain that

$$\begin{aligned} z(t^*) - z(t) + \sum_{\tau_k \in [t^*,t]} b_k z(\tau_k) &\ge \frac{LNe^k}{2(1+M)^l} \inf_{s \in [t^*,t]} z(t-h) \\ &\ge \frac{LNe^k}{2(1+M)^l} \frac{z(t-h)}{(1+M)^{i[t^*-h,t-h]}} \\ &\ge \frac{LNe^k}{2(1+M)^l} z(t-h). \end{aligned} \tag{14}$$

From (14) it follows that

$$z(t-h) \le \frac{2(1+M)^{2l}}{LNe^k}\Big[z(t^*) + \sum_{\tau_k \in [t^*,t]} b_k z(\tau_k)\Big]. \tag{15}$$

From (13) and (15) there follows the estimate

$$z(t^*-h) \le A^2 z(t^*) + A^2 \sum_{\tau_k \in [t^*,t]} b_k z(\tau_k) + A \sum_{\tau_k \in [t-h,t^*]} b_k z(\tau_k) \tag{16}$$

where $A = 2(1+M)^{2l}/(LNe^k)$.

Moreover,

$$\sum_{\tau_k \in [t^*,t]} z(\tau_k) \le z(t^*) \sum_{s=1}^{i[t^*,t]} (1+M)^{s-1}, \tag{17}$$

$$\sum_{\tau_k \in [t-h,t^*]} z(\tau_k) < z(t-h) \sum_{s=1}^{i[t-h,t^*]} (1+M)^{s-1}, \tag{18}$$

$$z(t^*-h) > \frac{z(t-h)}{(1+M)^{i[t^*-h,t-h]}}. \tag{19}$$

From (18) and (19) it follows that

$$\sum_{\tau_k\in[t-h,t^*]} z(\tau_k) < z(t^*-h)(1+M)^{i[t^*-h,t-h]}\sum_{s=1}^{i[t-h,t^*]}(1+M)^{s-1}. \tag{20}$$

From (16), (17) and (20), we obtain that

(21)

$$\begin{aligned} z(t^*-h) &\le A^2 z(t^*) + A^2 M \sum_{\tau_k\in[t^*,t]} z(\tau_k) + AM \sum_{\tau_k\in[t-h,t^*]} z(\tau_k) \\ &\le A^2 z(t^*) + A^2 M z(t^*)\frac{(1+M)^{i[t^*,t]}-1}{M} \\ &\quad + AMz(t^*-h)(1+M)^{i[t^*-h,t-h]}\frac{(1+M)^{i[t-h,t^*]}-1}{M} \\ &\le A^2 z(t^*) + A^2 z(t^*)[(1+M)^l - 1] \\ &\quad + z(t^*-h)[A(1+M)^{2l} - A(1+M)^l]. \end{aligned}$$

From (21) it follows that

$$z(t^*-h)[1-A(1+M)^{2l}+A(1+M)^l] \le A^2(1+M)^l z(t^*). \tag{22}$$

The function w is bounded from above if the coefficient at $z(t^*-h)$ in (22) is positive, i.e., if

$$\begin{aligned} 1-A(1+M)^{2l}+A(1+M)^l &> 0, \\ A(1+M)^{2l}-A(1+M)^l &< 1, \\ A(1+M)^l[(1+M)^l-1] &< 1, \\ \frac{2(1+M)^{2l}}{LNe^k}(1+M)^l[(1+M)^l-1] &< 1, \\ \frac{2(1+M)^{3l}[(1+M)^l-1]}{Le^k} &< N. \end{aligned}$$

The last inequality follows from condition 3 of Theorem 1.

Divide (7) by $z(t)>0$ for $t\ge T_3$, integrate the inequality obtained from $t-h$ to t and obtain that

$$\ln\frac{z(t-h)}{z(t)} + \sum_{i=1}^{l}\ln\frac{z(\tau_i+0)}{z(\tau_i)} \ge \frac{Le^k}{(1+M)^l}\int_{t-h}^{t} p(s)\frac{z(s-h)}{z(s)}\,ds,$$

i.e.,

$$\ln[(1+M)^l w(t)] \geq \frac{Le^k}{(1+M)^l} \int_{t-h}^{t} p(s)w(s)\,ds. \tag{23}$$

Introduce the notation $w_0 = \liminf_{t\to\infty} w(t)$. It is clear that $0 < w_0 < \infty$.

From (23) it follows that

$$\begin{aligned}\liminf_{t\to\infty} \int_{t-h}^{t} p(s)\,ds &\leq \frac{(1+M)^l}{Le^k} \frac{\ln[(1+M)^l w_0]}{w_0} \\ &\leq \frac{(1+M)^l}{Le^k} \frac{(1+M)^l}{e} = \frac{(1+M)^{2l}}{Le^{k+1}}.\end{aligned}$$

The last inequality contradicts condition 3 of Theorem 1. □

Example 1. Let us consider the following impulsive differential inequality with a retarded argument

$$\begin{gathered} x'(t) + x(t) + 2^9 \sqrt[3]{x^2(t-1)x(t-2)} \leq 0, \quad t \neq \tau_k, \\ x(\tau_k + 0) - x(\tau_k) = \frac{1}{k} x(\tau_k), \\ \varphi(t) \equiv 0, \quad t \in [-2, 0], \end{gathered}$$

where $m = 2$, $h_1 = 1$, $h_2 = 2$, $\bar{h} = 2$, $b_k = 1/k$, $\tau_k = k$, $k = 1, 2, \dots, a(t) \equiv 1$, $p(t) = 2^9$.

It can be checked immediately that $l = 3$. It is clear that conditions H1, H2 and H6 are fulfilled for $M = 1$. The condition H3 is satisfied too since

$$u_1 f(u_1, u_2) = u_1 \sqrt[3]{u_1^2 u_2} > 0$$

for $u_1 \neq 0$ and $\operatorname{sgn} u_1 = \operatorname{sgn} u_2$.

Condition H4 holds true for $L = 1$, $\alpha_1 = 2/3$, $\alpha_2 = 1/3$, and condition H5 is true for $l_1 = 1$, $l_2 = 0$. Later, condition 2 of Theorem 1 is fulfilled for $k = 2$. Finally, for the given values of the above constants, condition 3 of Theorem 1 is satisfied too.

According to Theorem 1 the impulsive differential inequality with a retarded argument written above has no positive solution.

Corollary 1. *Let the conditions of Theorem* 1 *hold. Then*:

1. *The inequality* (2) *has no negative solutions.*
2. *All solutions of equation* (3) *are oscillating.*

Theorem 2. *Let the following conditions hold*:

1. *Conditions* H1–H4 *and* H7 *are met.*
2. $b_k > -1$, $k \in \mathbf{N}$.
3. $$\limsup_{k\to\infty} \frac{1}{1+b_k}\int_{\tau_k}^{\tau_k+h} p(s)e^{\int_{s-h}^{s} a(u)\,du}\,ds > \frac{1}{L}.$$

Then:

1. *All solutions of equation* (3) *oscillate.*
2. *The inequality* (1) *has no positive solutions.*
3. *The inequality* (2) *has no negative solutions.*

Proof of 1. Let x be a nonoscillating solution of equation (3). Without loss of generality we may assume that $x(t) > 0$ for $t \geq t_1 \geq 0$. It is clear that $x(t-h_i) > 0$, $i \in \mathbf{N}_m$, and $f(x(t-h_1), \dots, x(t-h_m)) > 0$ for $t \geq t_1 + \bar{h} = t_2$.

Then from (3) it follows that x is a nonincreasing function in the set $(t_2, \tau_s) \cup [\cup_{i=s}^{\infty}(\tau_i, \tau_{i+1})]$, where $\tau_{s-1} < t_2 < \tau_s$.

Multiply (3) by $e^{\int_T^t a(s)\,ds}$, set $z(t) = x(t)e^{\int_T^t a(s)\,ds}$ and analogously to the proof of Theorem 1 obtain

$$\begin{aligned} &z'(t) + p(t)e^{\int_T^t a(u)\,du} f\Big(z(t-h_1)e^{-\int_T^{t-h_1} a(u)\,du}, \dots, \\ &\qquad z(t-h_m)e^{-\int_T^{t-h_m} a(u)\,du}\Big) = 0, \quad t \neq \tau_k \\ &\Delta z(\tau_k) = b_k z(\tau_k). \end{aligned} \tag{24}$$

Integrate (24) from τ_k to $\tau_k + h$, $k \geq s$, and obtain

$$z(\tau_k+h) - z(\tau_k+0) + L\int_{\tau_k}^{\tau_k+h} p(s)e^{\int_{s-h}^{s} a(u)\,du}\prod_{i=1}^{m} z^{\alpha_i}(s-h_i)\,ds \leq 0. \tag{25}$$

But, for each $s \in [\tau_k, \tau_k + h]$ in the interval $[s - h_i, s - h]$ there is no point of jump. From this fact and from the nonincreasing character of the function z in the interval $[s - h_i, s - h]$, $s \in [\tau_k, \tau_k + h]$ it follows that

$$\prod_{i=1}^{m} z^{\alpha_i}(s - h_i) \geq z^{\sum_{i=1}^{m} \alpha_i}(s - h) = z(s - h). \tag{26}$$

From (25) and (26) there follow the inequalities

$$z(\tau_k + h) - z(\tau_k + 0) + L \int_{\tau_k}^{\tau_k + h} p(s) e^{\int_{s-h}^{s} a(u)\, du} z(s - h)\, ds \leq 0,$$

$$L z(\tau_k) \int_{\tau_k}^{\tau_k + h} p(s) e^{\int_{s-h}^{s} a(u)\, du}\, ds \leq (1 + b_k) z(\tau_k),$$

$$\frac{1}{1 + b_k} \int_{\tau_k}^{\tau_k + h} p(s) e^{\int_{s-h}^{s} a(u)\, du}\, ds \leq \frac{1}{L}.$$

The last inequality contradicts condition 3 of Theorem 2.

The proofs of Assertions 2 and 3 of the theorem are analogous to the proof of Assertion 1. □

Corollary 2. *Let the following conditions hold*:

1. *Conditions* H1–H4, H6 *and* H7 *are met.*
2. $$\limsup_{k \to \infty} \int_{\tau_k}^{\tau_k + h} p(s) e^{\int_{s-h}^{s} a(u)\, du}\, ds > \frac{1 + M}{L}.$$

Then:

1. *The inequality* (1) *has no positive solutions.*
2. *The inequality* (2) *has no negative solutions.*
3. *All solutions of equation* (3) *oscillate.*

Corollary 3. *Let the following conditions hold*:

1. *Conditions* H1–H4 *and* H7 *are met.*
2. $b_k > -1$, $k \in \mathbf{N}$.

3.

$$\liminf_{k\to\infty}\int_{\tau_k-h}^{\tau_k} a(u)\,du \geq s > 0, \quad s = \text{const}.$$

4.

$$\limsup_{k\to\infty}\frac{1}{1+b_k}\int_{\tau_k}^{\tau_k+h} p(s)\,ds > \frac{1}{Le^s}.$$

Then the assertions of Corollary 2 *are valid.*

Theorem 3. *Let the following conditions hold*:

1. *Conditions* H1–H6 *are met.*
2. $$\limsup_{k\to\infty}\int_{\tau_k}^{\tau_k+h} p(s)e^{\int_{s-h}^{s} a(u)\,du}\,ds > \frac{(1+M)^{2l}}{L}.$$

Then:

1. *All solutions of equation* (3) *oscillate.*
2. *The inequality* (1) *has no positive solutions.*
3. *The inequality* (2) *has no negative solutions.*

Proof of 1. Let $x(t)$ be a positive solution of equation (3) for $t \geq t_0 \geq 0$. Then we also have $x(t-h_i) > 0$, $i \in \mathbf{N}_m$ and $f(x(t-h_1),\dots,x(t-h_m)) > 0$ for $t \geq t_0 + \bar{h}$.

Multiply (3) by $e^{\int_T^t a(s)\,ds}$, $t > T \geq t_0 + \bar{h}$, set $z(t) = x(t)e^{\int_T^t a(s)\,ds}$, substitute into (3) and obtain (24). Integrate (24) from τ_k to $\tau_k + h$ and deduce

$$(27)\quad z(\tau_k+h) - z(\tau_k+0) - \sum_{s=k}^{k+l-1} b_s z(\tau_s) + L\int_{\tau_k}^{\tau_k+h} p(s)e^{\int_{s-h}^{s} a(u)\,du}\prod_{i=1}^{m} z^{\alpha_i}(s-h_i)\,ds \leq 0.$$

For each $s \in [\tau_k, \tau_k + h]$, let the interval $[s - h_i, s - h]$ contain $l_i^{(k)}$ points of jump, $l_i^{(k)} \in \mathbf{N}$, $l_i^{(k)} \leq l$. Then

$$z(s - h_i) \geq \frac{z(s-h)}{(1+M)^{l_i^{(k)}}} \geq \frac{z(s-h)}{(1+M)^l}$$

and

$$\prod_{i=1}^{m} z^{\alpha_i}(s - h_i) \geq \prod_{i=1}^{m} \frac{z^{\alpha_i}(s-h)}{(1+M)^{\alpha_i l}} = \frac{z(s-h)}{(1+M)^l}. \tag{28}$$

From (27) and (28) we obtain that

$$\frac{1}{(1+M)^l} \int_{\tau_k}^{\tau_k + h} p(s) e^{\int_{s-h}^{s} a(u)\,du} z(s-h)\,ds \leq (1 + b_k) z(\tau_k) + \sum_{s=k+1}^{k+l-1} b_s z(\tau_s),$$

$$\frac{1}{(1+M)^l} z(\tau_k) \int_{\tau_k}^{\tau_k + h} p(s) e^{\int_{s-h}^{s} a(u)\,du}\,ds \leq (1+M) z(\tau_k) + M \sum_{s=k+1}^{k+l-1} z(\tau_s). \tag{29}$$

But

$$\begin{aligned} z(\tau_{k+1}) &\leq z(\tau_k + 0) = (1 + b_k) z(\tau_k) \leq (1+M) z(\tau_k), \\ z(\tau_{k+2}) &\leq z(\tau_{k+1} + 0) \leq \cdots \leq (1+M)^2 z(\tau_k), \\ &\ldots\ldots\ldots\ldots\ldots \\ z(\tau_{k+l-1}) &\leq \cdots \leq (1+M)^{l-1} z(\tau_k). \end{aligned}$$

Then

$$\begin{aligned} \sum_{s=k+1}^{k+l-1} z(\tau_s) &< z(\tau_k) \sum_{i=1}^{l-1} (1+M)^i \\ &= z(\tau_k)(1+M) \frac{[(1+M)^{l-1} - 1]}{M}. \end{aligned} \tag{30}$$

From (29) and (30) it follows that

$$\frac{L}{(1+M)^l} z(\tau_k) \int_{\tau_k}^{\tau_k+h} p(s) e^{\int_{s-h}^{s} a(u)\,du}\, ds \le z(\tau_k)(1+M)^l,$$

$$\int_{\tau_k}^{\tau_k+h} p(s) e^{\int_{s-h}^{s} a(u)\,du}\, ds \le \frac{(1+M)^{2l}}{L}.$$

The last inequality contradicts condition 2 of Theorem 3.

The proofs of Assertions 2 and 3 of the theorem do not principally differ from the proof of Assertion 1. □

Corollary 4. *Let the following conditions hold*:

1. *Conditions* H1–H3 *and* H7 *are met.*
2. $b_k > -1$, $k \in \mathbf{N}$.
3. $$\liminf_{k\to\infty} \int_{\tau_k - h}^{\tau_k} a(s)\, ds \ge s > 0, \quad s = \text{const}.$$
4. $$\limsup_{k\to\infty} \int_{\tau_k}^{\tau_k+h} p(s)\, ds > \frac{(1+M)^{2l}}{Le^s}.$$

Then the assertions of Theorem 3 *are valid.*

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