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ON THE SUPREMUM OF A FAMILY OF SINGULAR COMPACTIFICATIONS

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ABSTRACT. Singular compactifications of locally compact Hausdorff spaces were first introduced over a decade ago. An elegant characterization of singular compactifications is the following: A compactification αX of the locally compact Hausdorff space X is singular if and only if $\alpha X \backslash X$ is a retract of αX. In this paper we provide a new representation of singular compactifications. It has been previously shown that the supremum of singular compactifications need not itself be a singular compactification. Examples of this fact are easy to find. We provide necessary and sufficient conditions which describe when the supremum of a family of singular compactifications is a singular compactification. We also show that there are compactifications which are not the supremum of a family of singular compactifications. For two singular functions f and g such that $S(f)$ is homeomorphic to $S(g)$ we describe conditions which show when $X \cup_f S(f)$ is equivalent to $X \cup_g S(g)$.

1. Introduction. All hypothesized topological spaces will be assumed to be locally compact and Hausdorff.

Two compactifications αX and γX of a space X are said to be *equivalent* if there is a homeomorphism $f : \alpha X \to \gamma X$ from αX onto γX which fixes the points of X. This defines an equivalence relation of the family of all compactifications of X. When we speak of a compactification αX of X, it will be understood that we are referring to the equivalence class of αX. The notation $\alpha X \cong \gamma X$ will mean that αX is equivalent to γX. We will say that the compactification αX is less than or equal to the compactification γX, denoted by $\alpha X \leq \gamma X$ if there is a continuous function $f : \gamma X \to \alpha X$ of γX onto αX which acts as the identity on X. This defines a partial order on the family $K(X)$ of all compactifications of X. It is well known that $K(X)$ is a complete lattice with respect to the partial order $\leq$ (see [**1**, 2.19]). If

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αX and γX are compactifications of X such that $\alpha X \leq \gamma X$, we will denote the projection map from γX onto αX which fixes the points of X by $\pi_{\gamma\alpha}$.

The family of compactifications studied here was first defined and discussed in [**4**]. We introduce the object of our study in the following definitions which appear in [**4**].

Definitions 1.1. A *singular compactification induced by the function* f is constructed as follows: Let $f : X \to K$ be a continuous function from the space X into a compact set K. Let the singular set, $S(f)$, of f be defined as the set $\{x \in \operatorname{cl}_x f[X]$: for any neighborhood U of x, $\operatorname{cl}_x f^{\leftarrow}[U]$ is not compact$\}$. If $S(f) = K$, then f is said to be a *singular map*. It is easy to verify that $S(f)$ is closed in K and that if f is a singular map then $f[X]$ is dense in $S(f)$. If f is a singular map the *singular compactification of* X *induced by* f, denoted by $X \cup_f S(f)$, is the set $X \cup S(f)$ where the basic neighborhoods of the points in X are the same as in the original space X, and the points of $S(f)$ have neighborhoods of form $U \cup (f^{\leftarrow}[U] \backslash F)$ where U is open in $S(f)$ and F is a compact subset of X. This defines a compact Hausdorff topology on $X \cup_f S(f)$ in which X is a dense subspace. We will say that a compactification αX of X is a *singular compactification* if αX is equivalent to $X \cup_f S(f)$ for some singular map f.

We begin by stating some basic properties known to be possessed by singular compactifications. The following is Theorem 4 in [**11**].

Theorem 1.2 [**11**]. *The singular compactifications of* X *are precisely those compactifications* αX *of* X *whose remainder* $\alpha X \backslash X$ *is a retract of* αX.

Since a constant map is continuous, $\omega X \backslash X$ is a retract of ωX, the one-point compactification of X. So ωX is always a singular compactification.

It is easily verified that if $r : \alpha X \to \alpha X \backslash X$ is a retraction from αX onto $\alpha X \backslash X$, then $r|_X$ is a singular map which induces the singular compactification $X \cup_{r|_X} (Sr|_X)$.

We also have the following important result from [**11**].

Theorem 1.3 [**11**, Theorem 7]. *If αX is a singular compactification and γX is any compactification of X less than αX, then γX is also a singular compactification.*

Notation 1.4. For any compactification γX of X, $C_\gamma(X)$ will denote the set $\{f|_x : f \in C(\gamma X)\}$. If f is a bounded real-valued singular function, f will be regarded as a function from X into $\mathrm{cl}_{\mathbf{R}} f[X]$, i.e., we are letting K (in our definition of singular map) be $\mathrm{cl}_{\mathbf{R}} f[X]$. The set S_γ will denote the set of all singular maps in $C_\gamma(X)$. Thus S_β denotes the collection of all singular maps in $C^*(X)$. If $G \subseteq C_\gamma(X)$, G^γ will denote the set of extensions f^γ to γX of the functions f in G. The following is a generalization of Theorem 1.1 of [**2**]. The proof appears in [**9**].

Lemma 1.5. *Let f be a continuous function from a space X to a compact Hausdorff space Z. Let $Y = \mathrm{cl}_Z f[X]$ and $K_X = \{F \subseteq X : F$ is compact$\}$. Then $S(f) = \cap\{\mathrm{cl}_Y f[X\backslash F] : F \in K_X\}$.*

Proposition 1.6 is a generalization of Lemma 1 in [**3**].

Proposition 1.6. *If αX is a compactification of X, K is a compact Hausdorff space and $f : X \to K$ is a continuous function which extends to $f^\alpha : \alpha X \to K$, then $f^\alpha[\alpha X\backslash X] = S(f)$.*

Proof. We will first show that $f^\alpha[\alpha X\backslash X]$ is contained in $\mathrm{cl}_{\mathrm{cl}_k f[X]} f[X\backslash F]$ for all $F \in K_X$ and apply the previous lemma. Let $F \in K_X$. Then $\alpha X\backslash X \subseteq \mathrm{cl}_{\alpha X}(X\backslash F)$. Hence $f^\alpha[\alpha X\backslash X] \subseteq f^\alpha[\mathrm{cl}_{\alpha X}(X\backslash F)] \subseteq \mathrm{cl}_{\mathrm{cl}_k f[X]} f[X\backslash F]$. Since this is true for all $F \in K_X$, $f^\alpha[\alpha X\backslash X] \subseteq \cap\{\mathrm{cl}_{\mathrm{cl}_k f[X]} f[X\backslash F] : F \in K_X\}$. By the previous lemma $f^\alpha[\alpha X\backslash X] \subseteq S(f)$.

Let $p \in K\backslash f^\alpha[\alpha X\backslash X]$. Let U be an open neighborhood (in K) of p such that $\mathrm{cl}_K U$ misses $f^\alpha[\alpha X\backslash X]$. Then $\mathrm{cl}_X f^\leftarrow[U] \subseteq f^\leftarrow[\mathrm{cl}_{\mathrm{cl}_K f[X]} U]$, which is a compact subset of X. This implies that p cannot belong to $S(f)$. Hence $S(f) = f^\alpha[\alpha X\backslash X]$. □

Corollary 1.7. *Let $f : X \to K$ be a continuous map into a compact Hausdorff space such that $f[X]$ is dense in K. Let $E_f(X)$ denote the set of all compactifications αX of X such that $f : X \to K$ extends to $f^\alpha : \alpha X \to K$. Then f is a singular map if and only if $f^\alpha[\alpha X \backslash X]$ contains $f[X]$ for some (equivalently for all) $\alpha X \in E_f(X)$.*

Proof. $(\Rightarrow)$. If f is a singular map, then $S(f) = K = \mathrm{cl}_K f[X]$ (by definition). By Proposition 1.6, $f^\alpha[\alpha X \backslash S] = S(f) = \mathrm{cl}_K f[X]$ for all $\alpha X \in E_f(X)$, hence $f[X]$ is contained in $f^\alpha[\alpha X \backslash X]$.

$(\Leftarrow)$ Suppose now that $f^\alpha[\alpha X \backslash X]$ contains $f[X]$ for some $\alpha X \in E_f(X)$. Since $f[X]$ is dense in K (by hypothesis) $\mathrm{cl}_K f[X] = K$. We must show that $S(f) = K$. Let $p \in K$ and U be an open neighborhood of p in $K = \mathrm{cl}_K f[X]$. Then $f^{\alpha\leftarrow}[U]$ meets $\alpha X \backslash X$, hence $\mathrm{cl}_{\alpha X} f^{\leftarrow}[U]$ meets $\alpha X \backslash X$. Since $\mathrm{cl}_X f^{\leftarrow}[U]$ is dense in $\mathrm{cl}_{\alpha X} f^{\leftarrow}[U]$, $\mathrm{cl}_X f^{\leftarrow}[U]$ cannot be compact. Hence p belongs to $S(f)$. Since $K = S(f)$, f is singular. □

Let $G \subseteq C^*(X)$. The *evaluation map* e_G induced by G is the function $e_G : X \to \Pi\{I_g : g \in G\}$ (where, for each g, I_g is a closed interval containing $g[X]$) defined by $e_G(x) = \langle g(x) \rangle_{g \in G}$. Note that the closure in $\Pi_{g \in G} I_g$ of $e_G[X]$ is a compact set.

If αX is a compactification of X and $G \subseteq C_\alpha(X)$, then G^α will denote the family of all extensions of the functions in G to αX.

By the *uniform norm topology* or *metric topology* on $C^*(X)$ we will mean the topology on $C^*(X)$ in which the closure of sets is the closure under uniform convergence. The metric on $C^*(X)$ is defined as follows: $d(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$ (see the introductory paragraph of Chapter 16 of [**12**]).

The following result is the only theorem in [**14**].

Proposition 1.8 [**14**]. *Let $G \subseteq C^*(X)$. Then there exists a smallest compactification to which all functions in G extend.*

The following notation was introduced in paragraph 2 of [**8**].

Notation 1.9. If G is contained in $C^*(X)$, the symbol $\omega_G X$ will denote the smallest compactification to which all functions in G extend. If f belongs to $C^*(X)$, $\omega_f X$ will denote the smallest compactification of X to which f extends.

Proposition 1.10. *Let αX be a compactification of X and $G \subseteq C_\alpha(X)$. Then G^α separates the points of $\alpha X \backslash X$ if and only if $e_G^\alpha(= e_{G^\alpha})$ is one-to-one on $\alpha X \backslash X$.*

Proof. Suppose G^α separates the points of $\alpha X \backslash X$. Let p and q be distinct points in $\alpha X \backslash X$. Then there exists a function f in G such that $f^\alpha(p) \neq f^\alpha(q)$. Hence $e_G^\alpha(p) \neq e_G^\alpha(q)$. It follows that e_{G^α} is one-to-one on $\alpha X \backslash X$. Conversely, if e_{G^α} is one-to-one on $\alpha X \backslash X$, then, if p and q are distinct points in $\alpha X \backslash X$, $e_G^\alpha(p) \neq e_G^\alpha(q)$; this can only happen if there exists some function f in G such that $f^\alpha(p) \neq f^\alpha(q)$. Hence G^α separates the points of $\alpha X \backslash X$. □

For further reference we formally present the following easy result which appears in the proof of Theorem 1 of [**8**].

Proposition 1.11 [**8**]. *Let $G \subseteq C^*(X)$ and αX be a compactification of X. Then $\alpha X \cong \omega_G X$ if and only if each function g in G extends to g^α in $C(\alpha X)$ and G^α separates the points of $\alpha X \backslash X$.*

Proof. ($\Rightarrow$). Suppose $\alpha X \cong \omega_G X$. Then, by definition of $\omega_G X$, every function f in G extends to a function f^α in $C(\alpha X)$. Furthermore, G^α must separate the points of $\alpha X \backslash X$ for, if not, we may collapse any two points in $\alpha X \backslash X$ which are not separated by G^α to obtain a compactification strictly smaller than $\omega_G X$ to which each member of G extends, thus obtaining a contradiction. Hence G^α separates the points of $\alpha X \backslash X$.

($\Leftarrow$). Since every function f in G extends to a function f^α in $C(\alpha X)$, then $\omega_G X$ is less than or equal to αX (by definition of $\omega_G X$). Since G^α separates the points of $\alpha X \backslash X$, then $\omega_G X$ cannot be strictly less than αX, hence $\alpha X \cong \omega_G X$. This proves the proposition. □

Compactifications of the form $\omega_G X$ are briefly discussed in [**8**]. From Propositions 1.6 and 1.11 we see that $\omega_f X\backslash X$ is homeomorphic to $S(f)$ for all $f \in C^*(X)$. (Since $\{f^{\omega_f}\}$ separates the points of $\omega_f X\backslash X$, it is one-to-one on $\omega_f X\backslash X$; hence, $f^{\omega_f}|_{\omega_f X\backslash X}$ is a homeomorphism.)

Note that if αX is a compactification and $C_\alpha(X) = \{f|_X : f \in C(\alpha X)\}$ is its associated subalgebra, then, since $C(\alpha X)$ separates points of $\alpha X\backslash X$, $\omega_{C_\alpha(X)} X \cong \alpha X$. Hence, every compactification αX can be expressed in the form $\omega_G X$ for some $G \subseteq C^*(X)$.

We will denote the one-point compactification of X by ωX; hence, $C_\omega(X) = \{g|_X : g \in C(\omega X)\}$, see Notation 1.4.

In paragraph 2 of [**8**], the author presents the following definition.

Definition 1.12. If f and g belong to $C^*(X)$, we will say that f is *equivalent* to g, denoted by $f \cong g$, if $f - g \in C_\omega(X)$. If G and F are subsets of $C^*(X)$, G is said to be equivalent to F, denoted by $G \cong F$, if every function g in G is equivalent to some function f in F and conversely.

If G is contained in $C_\alpha(X)$, $\langle G\rangle$ will denote the subalgebra generated by G and $\mathrm{cl}_{C_\alpha(X)}\langle G\rangle$ will denote its closure in the uniform norm topology on $C_\alpha(X)$.

In Corollary 1 of [**8**], we have the following useful proposition:

Proposition 1.13 [**8**]. *If $G \subseteq C^*(X)$, then $C_{\omega_G}(X) = \mathrm{cl}_{C_{\omega_G}(X)} \langle C_\omega(X) \cup G\rangle$, (the closure in the uniform norm topology of the subalgebra generated by $C_\omega(X) \cup G$) where $C_{\omega_G}(X) = \{f|_X : f \in C(\omega_G X)\}$ (as in Notation* 1.4*).*

Also, in Theorem 1 of [**3**], we have the following result:

Theorem 1.14 [**3**]. *If αX is a compactification of X and $G \subseteq S_\alpha$, then $\alpha X = \sup\{X \cup_f S(f) : f \in G\}$ if and only if G^α separates the points of $\alpha X\backslash X$.*

On page 29 of [**11**], the author describes a method of constructing a compactification of X by using the singular set of a function f even if this function is not a singular map. The construction of this compactification is very similar to the construction of singular compactifications. We describe it here. Let $f : X \to Y$ be a continuous map from a space X to a compact Hausdorff space Y. We define a topology on the set $X \cup S(f)$ as follows: The basic open neighborhoods of the points in X will be the same as in the original space X. If $p \in S(f)$ we define a basic open neighborhood of p to be any set of form $V \cup [f^{\leftarrow}[O] \backslash F]$ where O is an open neighborhood of p in Y, $V = O \cap S(f)$ and F is a compact set in X.

Notation 1.15. We will denote $X \cup S(f)$ equipped with the topology described above by $X \cup^* S(f)$.

It is shown in Theorem 9 of [**11**] that $X \cup^* S(f)$ is indeed a Hausdorff compactification of X. We note that if f is a singular map, then $X \cup_f S(f) \cong X \cup^* S(f)$.

We will also make use of the following previously established results.

Proposition 1.16 [**11**, Lemma 1]. *If $f : X \to Y$ is a singular function mapping X into a closed subspace K of the compact Hausdorff space Y and $g : Y \to Z$ is continuous so that $\mathrm{cl}_Z(g \circ f[X]) = Z$, then $g \circ f$ is a singular function.*

Proposition 1.17 [**8**, Corollary 3]. *If F and G are two equivalent subsets of $C^*(X)$, then $\omega_G X$ is equivalent to $\omega_F X$.*

(It is also shown immediately following Corollary 3 in [**8**] that the converse of the above statement fails.)

Theorem 1.18 [**8**, Theorem 2]. *If $G \subseteq C^*(X)$ separates the points from the closed sets in X, then $\sup\{\omega_f X : f \in G\} = \omega_G X$.*

2. The supremum of singular compactifications. The main results of this section are given in Theorems 2.6, 2.11 and 2.12. In Theorem 2.6 we show that if αX is a singular compactification, then αX can be expressed in the form $\omega_{S_\alpha} X$. We also show that the converse of this theorem fails. In Theorem 2.12 we describe when a compactification of form $\omega_G X$ where $G \subseteq S_\beta$ is a singular compactification. This will simultaneously describe when the supremum of a family of singular compactifications is a singular compactification. In Theorem 2.11 we show that a singular compactification αX can always be expressed in the form $X \cup e_F S(e_F) \cong \omega_F X$ where F is a subalgebra of $C_\alpha(X)$, $F \subseteq S_\alpha$, e_F is a singular map and e_F^α separates points of $\alpha X \backslash X$.

Lemma 2.1. *Let $f : X \to Y$ be a continuous function from the space X into a compact Hausdorff space Y. If αX is a compactification of X and f extends to $f^\alpha : \alpha X \to Y$ so that f^α separates the points of $\alpha X \backslash X$, then αX is equivalent (as a compactification of X) to $X \cup^* S(f)$.*

Proof. By Proposition 1.6, $f^\alpha[\alpha X \backslash X] = S(f)$. We define a function $j : \alpha X \to X \cup^* S(f)$ as follows: $j(x) = f^\alpha(x)$ if x belongs to $\alpha X \backslash X$ and $j(x) = x$ if x belongs to X. Clearly j is one-to-one. We now verify that j is continuous. It is sufficient to verify that j pulls back open neighborhoods of points in $S(f)$ to open sets in αX. Recall that the open neighborhoods of points in $S(f)$ are of the form $V \cup (f^{\leftarrow}[O] \backslash F)$ where O is an open set in Y, $V = O \cap S(f)$ and F is a compact set in X. Note that

$$\begin{aligned} j^{\leftarrow}[V \cup f^{\leftarrow}[O]] &= j^{\leftarrow}[V] \cup f^{\leftarrow}[O] \\ &= (f^{\alpha\leftarrow}[V] \cap \alpha X \backslash X) \cup f^{\leftarrow}[O] \\ &= f\alpha^{\leftarrow}[O], \end{aligned}$$

which is an open subset of αX.

It follows that $j^{\leftarrow}[V \cup f^{\leftarrow}[O] \backslash F]$ is open in αX, hence j is continuous. The lemma follows. □

The following corollary is an easy consequence of the lemma.

Corollary 2.2. *If αX is a compactification of X, then αX can be expressed in the form of $X \cup^* S(f)$, i.e., αX is equivalent to $X \cup^* S(e_{C_\alpha(X)})$.*

In Theorem 2.1 of [**15**], the author proves the following statement: "If X is locally compact and K is a Hausdorff space, then there exists a compactification αX of X such that $\alpha X \backslash X$ is homeomorphic to K if and only if K is a continuous image of $\beta X \backslash X$." In Corollary 2.3 we give a similar result referring specifically to *singular* compactifications.

Corollary 2.3. *Let X be a locally compact Hausdorff space and Y a compact Hausdorff space. Then there exists a topology on the disjoint union $X \cup Y$ of X and Y such that the resulting topological space is a singular compactification of X if and only if Y is homeomorphic to the singular set of some evaluation map e_G induced by a subset G of $C^*(X)$.*

Proof. ($\Rightarrow$). This direction follows from Corollary 2.2.

($\Leftarrow$). This direction follows from the definition of $X \cup^* S(f)$. □

Theorem 2.4. a) *Let $f \in C^*(X)$. Then $\omega_f X$ is equivalent to $X \cup^* S(f)$. In particular, if f is a singular map, then $\omega_f X$ is a singular compactification and $\omega_f X$ is equivalent to $X \cup_f S(f)$.*

b) *If $G \subseteq C^*(X)$ and $\omega_G X$ is a singular compactification, then $t = e_G^{\omega_G} \circ r|_X$ is a singular map (where $r : \omega_G X \to \omega_G X \backslash X$ is a retraction map) and $\omega_G X$ is equivalent to $X \cup_t S(t)$.*

Proof. The proof of part a) follows from the fact that f^ω separates the points of $\omega_f X \backslash X$ (see Proposition 1.11, Proposition 1.6 and Lemma 2.1). If f is a singular map, then $X \cup^* S(f)$ is equivalent to $X \cup_f S(f)$ (Notation 1.15).

We now prove part b). Let $G \subseteq C^*(X)$ and suppose that $\omega_G X$ is a singular compactification. Let $r : \omega_G X \to \omega_G X \backslash X$ be a retraction. Recall that $r|_X$ is a singular map. Since the composition of a continuous function with a singular function is singular (Proposition 1.16), then $t = e_G^{\omega_G} \circ r|_X$ is a singular map. Since t extends continuously to $e_G^{\omega_G} \circ r$ and $e_G^{\omega_G} \circ r$ separates points of $\omega_G X \backslash X$, it follows from Lemma 2.1 that $\omega_G X$ is equivalent to $X \cup_t S(t)$. □

In what follows, we will show that it is possible to express any singular compactification in a form that involves only real-valued singular maps.

We require the following lemma.

Lemma 2.5. *If αX is a singular compactification, then every $f \in C_\alpha(X)$ is equivalent to some function $h \in S_\alpha$ (see Definition* 1.12*).*

Proof. Let αX be a singular compactification, and let $f \in C_\alpha(X)$. Then there exists a retraction map r mapping αX onto $\alpha X \backslash X$. We have already see that $r|_X$ is a singular map. Then $f^\alpha \circ r|_X$ is a singular map and belongs to S_α, (Proposition 1.16). Let $G = f - f^\alpha \circ r|_X$. If $x \in \alpha X \backslash X$, $g^\alpha(x) = f^\alpha(x) - f^\alpha \circ r(x) = f^\alpha(x) - f^\alpha(x) = 0$. Therefore $g^\alpha|_{\alpha X \backslash X}$ is the $\mathbf{0}$-function on $\alpha X \backslash X$. Hence $g \in C_\omega(X)$. Thus f is equivalent to $h = f^\alpha \circ r|_X$. □

In Theorem 2.6 we show that if αX is a singular compactification then αX can be expressed in the form $\omega_{S_\alpha} X$.

Theorem 2.6. *If αX is a singular compactification, then αX is equivalent to $\omega_{S_\alpha} X$. Hence every singular compactification αX of X is the supremum of the family $\{X \cup_f S(f) : f \in S_\alpha\}$ of singular compactifications.*

Proof. Let $\gamma X = \omega_{S_\alpha} X$. By Proposition 1.13, $C_\gamma(X) = \mathrm{cl}_{C_\gamma(X)} \langle C_\omega(X) \cup S_\alpha \rangle$. Since $C_\omega(X) \cup S_\alpha \subseteq C_\alpha(X)$, then $\omega_{S_\alpha} X \leq \alpha X$.

Let $f \in C_\alpha(X)$. By Lemma 2.5, $f \cong g$ for some $g \in S_\alpha$. It is shown in Proposition 1.17 that, if $f \cong g$ then $\omega_f X$ is equivalent to $\omega_g X$. Now $C_{\omega_f}(X) = \mathrm{cl}_{C_{\omega_f} X} \langle C_\omega(X) \cup \{f\} \rangle$. Then $\mathrm{cl}_{C_{\omega_f}(X)} \langle C_\omega(X) \cup \{f\} \rangle = \mathrm{cl}_{C_{\omega_g}(X)} \langle C_\omega(X) \cup \{g\} \rangle \subseteq C_\gamma(X)$. It follows that $C_\alpha(X) \subseteq C_\gamma(X)$; consequently, $\alpha X \leq \omega_{S_\alpha} X$. Since $\omega_{S_\alpha} X \leq \alpha X$ and $\alpha X \leq \omega_{S_\alpha} X$, αX is equivalent to $\omega_{S_\alpha} X$. By Proposition 1.11, S_α^α separates the points of $\alpha X \backslash X$. By Theorem 1.14, $\omega_{S_\alpha} X$ is equivalent to $\sup\{X \cup_f S(f) : f \in S_\alpha\}$. Hence αX is the supremum of the family of singular compactifications $\{X \cup_f S(f) : f \in S_\alpha\}$ (where $X \cup_f S(f) \cong \omega_f X$ for each f in S_α, by Theorem 2.4 a)). This is the assertion of the theorem. □

The converse of the above theorem fails as we shall now see. Recall that S_β is the set of all real-valued singular functions. On page 20 of [**11**], it is shown that $\beta \mathbf{N} = \sup\{\mathbf{N} \cup_f S(f) : f \in S_\beta\}$. Since

$S_\beta^{\omega_{S_\beta}}$ separates points of $(\omega_{S_\beta}\mathbf{N})\backslash\mathbf{N}$ (Proposition 1.11), then $\omega_{\S_\beta}\mathbf{N} = \sup\{\mathbf{N}\cup_f S(f) : f \in S_\beta\}$ (by Theorem 1.14). Hence $\beta\mathbf{N}$ is equivalent to $\omega_{S_\beta}\mathbf{N}$, the smallest compactification to which all real-valued singular maps extend. Since $\beta\mathbf{N}\backslash\mathbf{N}$ is not separable, $\beta\mathbf{N}\backslash\mathbf{N}$ cannot be the continuous image of a separable space, hence $\beta\mathbf{N}$ cannot be a singular compactification. It follows that not every compactification αX of form $\omega_{S_\alpha}X$ is singular.

We know that there are compactifications which are singular and that there are compactifications which are not singular but which are the supremum of a family of singular compactifications ($\beta\mathbf{N}$ for example). In [**11**], the author asks:

> Can every compactification of X be expressed as the supremum of singular compactifications?

We give the following example which provides a negative answer to the question.

Example 2.7. Consider the two-point compactification of $\mathbf{R}$, $\alpha\mathbf{R} = \mathbf{R} \cup \{p_1, p_2\}$. We claim that $\alpha\mathbf{R}$ cannot be the supremum of singular compactifications, i.e., $\alpha\mathbf{R}$ cannot be expressed in the form $\omega_{S_\alpha}\mathbf{R}$.

Proof. Suppose that $\alpha\mathbf{R} = \sup\{\gamma\mathbf{R} : \gamma\mathbf{R}$ is a singular compactification, $\gamma\mathbf{R} \leq \alpha\mathbf{R}\}$. Since every singular compactification γX can be expressed in the form $\gamma X = \omega_{S_\gamma}X = \sup\{X \cup_f S(f) : f \in S_\gamma\} = \{\omega_f X : f \in S_\gamma\}$, see Theorems 2.6 and 2.4 a), then $\alpha\mathbf{R} = \sup\{\sup \omega_f\mathbf{R} : f \in S_\gamma\} : \gamma\mathbf{R} \leq \alpha\mathbf{R}\} = \sup\{\omega_f\mathbf{R} : f \in S_\alpha\}$, where S_α is the set of all singular real-valued maps which extend to $\alpha\mathbf{R}$. By Theorem 1.14, S_α^α separates the points of $\alpha\mathbf{R}\backslash\mathbf{R}$. Let $f \in S_\alpha$ be such that $f^\alpha(p_1) \neq f^\alpha(p_2)$. By Proposition 1.6, $f^\alpha[\alpha\mathbf{R}\backslash\mathbf{R}] = S(f) = \{f^\alpha(p_1), f^\alpha(p_2)\}$. But since f is a singular map, f maps $\mathbf{R}$ onto $S(f) = \{f^\alpha(p_1), f^\alpha(p_2)\}$. Since $\mathbf{R}$ is connected this is clearly a contradiction. Consequently, $\gamma\mathbf{R}$ is not the supremum of singular compactifications. We have thus shown that not every compactification γX can be expressed in the form $\omega_{S_\gamma}X$. □

In Theorem 2.6 and Example 2.7 we have shown that the two-point compactification of $\mathbf{R}$ is not a singular compactification. (This is also obvious from the fact that $\mathbf{R}$ is connected and $\alpha\mathbf{R}\backslash\mathbf{R}$ is not.) Note that the fact that the two-point compactification of $\mathbf{R}$ is not singular does not imply that the largest compactification of $\mathbf{R}$ which is singular must be the one-point compactification of $\mathbf{R}$. It simply implies that no compactification $\gamma\mathbf{R}$ of $\mathbf{R}$ larger than the two-point compactification of $\mathbf{R}$ is singular (by Theorem 1.3). Note that $\sin(x)$ is a singular map and that $S(\sin(x)) = [-1, 1]$. Then $\mathbf{R} \cup_{\text{sine}} S(\text{sine})$ is a singular compactification of $\mathbf{R}$ which is not comparable with the two-point compactification of $\mathbf{R}$. We are however guaranteed that $\beta\mathbf{R}$ is not singular by the existence of a nonsingular compactification of $\mathbf{R}$ and by Theorem 1.3. (Again, since $\beta\mathbf{R}$ is connected and $\beta\mathbf{R}\backslash\mathbf{R}$ isn't, we have an even simpler reason why $\beta\mathbf{R}$ is not a singular compactification.)

We now consider the following question:

> When is the supremum of a collection of singular compactifications a singular compactification?

We begin with a brief discussion of the question. Given a family $A = \{\alpha_i X : i \in A\}$ of singular compactifications of a space X, we seek ways of recognizing when the supremum, say αX, of A is itself a singular compactification. There are many possible approaches to this problem: one could look for a property possessed by the family A which will guarantee that αX is a singular compactification. But αX may be the supremum of many families of singular compactifications. Each one of these families (including the family of all singular compactifications less than or equal to αX) would have to possess this particular property. That αX is the supremum of the collection A tells us that αX is not a compactification such as the two-point compactification of $\mathbf{R}$ (which is not the supremum of any collection of singular compactifications (see Example 2.7)). After some reflection, we have chosen to study αX as the supremum of "some" family of singular compactifications rather than αX "the supremum of the collection A of singular compactifications." This approach has turned out to be the most fruitful. Proposition 2.8 will show us that suprema of singular compactifications are precisely compactifications of the form $\omega_G X$, where G is contained in S_β. Given this result, to answer our question, *it will only be necessary to characterize those compactifications of form $\omega_G X$, where G is contained in S_β.*

We begin with the following proposition:

Proposition 2.8. *Let X be a topological space. The compactification*

αX of X is a supremum of a collection of singular compactifications if and only if αX is equivalent to $\omega_G X$ for some G contained in S_α.

Proof. ($\Rightarrow$). Suppose $A = \{\alpha_i X : i \in A\}$ is a collection of singular compactifications such that αX is $\sup\{\alpha_i X : i \in A\}$. Then

$$\begin{aligned} \alpha X &= \sup\{\omega_{S_{\alpha_i}} X : i \in A\} \quad \text{(Theorem 2.6)} \\ &= \sup\{\sup\{X \cup_f S(f) : f \in S_{\alpha_i}\} : i \in A\} \\ &\quad \text{(by Proposition 1.11 and Theorem 1.14)} \\ &= \sup\{X \cup_f S(f) : f \in \cup\{S_{\alpha_i} : i \in A\}\}. \end{aligned}$$

Hence, by Theorem 1.14, $(\cup S_{\alpha_i})^\alpha$ separates the points of $\alpha X \backslash X$. Thus, αX is equivalent to $\omega_{\cup S\alpha_i} X$. Now $\cup S_{\alpha_i}$ is contained in S_α. Hence we have shown that the supremum of a collection of singular compactifications is of the form $\omega_G X$, where G is a subset of S_α.

($\Leftarrow$). Suppose that αX is equivalent to $\omega_G X$, where G is contained in S_α. Then, by Proposition 1.11 and Theorem 1.14, αX is the supremum of the collection $\{X \cup_f S(f) : f \in G\}$ of singular compactifications. $\square$

Our question can then be reformulated as follows:

If $G \subseteq S_\beta$, when is $\omega_G X$ a singular compactification?

One may conjecture that $[X \cup_f S(f)] \vee [X \cup_g S(g)]$ is singular if and only if the evaluation map $h = f \times g$ is a singular map (as stated in Theorem 8 of [**11**]). We provide the following counterexample.

Consider the natural numbers $\mathbf{N}$. Let A denote the even natural numbers, B denote the odd natural numbers and $C = \{3\}$. Define the maps $f : \mathbf{N} \to \{0, 1\}$ and $g : \mathbf{N} \to \{0, 1\}$ as follows: $f[A \cup C] = \{0\}$ and $f[B \backslash C] = \{1\}$, $g[A] = \{1\}$ and $g[B] = \{0\}$. Clearly, both f and g are singular maps and $X \cup_f S(f)$ and $X \cup_g S(g)$ each describe a two-point compactification. We verify that $X \cup_f S(f)$ is equivalent to $X \cup_g S(g)$, hence their supremum will be their common value, a singular compactification.

Let $k : X \cup_f S(f) \to S \cup_f S(g)$ be defined as follows: $k(x) = x$ if $x \in X$ and $k(0) = 1$ and $k(1) = 0$. Now $\{1\} \cup g \leftarrow (1)$ is a basic open neighborhood of 1 in $X \cup_g S(g)$. Now $k^{\leftarrow}[\{1\} \cup g^{\leftarrow}[1]] = \{0\} \cup g^{\leftarrow}(1) =$

$\{0\} \cup A = \{0\} \cup f^{\leftarrow}(0) \backslash \{3\}$ a basic open neighborhood of 0 in $X \cup_f S(f)$. Consider now the set $\{0\} \cup g^{\leftarrow}(0)$ a basic open neighborhood of 0 in $X \cup_g S(g)$. Now $k^{\leftarrow}[\{0\} \cup g^{\leftarrow}(0)] = \{1\} \cup B = \{1\} \cup (f^{\leftarrow}[1] \cup \{3\}) = (\{1\} \cup f^{\leftarrow}[1]) \cup \{3\}$ an open neighborhood of 1 in $X \cup_f S(f)$. Thus, k is continuous; hence, $X \cup_f S(f)$ is equivalent to $X \cup_g S(g)$. Note that the evaluation map $h = f \times g$ maps A to $\{(0,1)\}$, $B \backslash C$ to $\{(1,0)\}$ and C to $\{(0,0)\}$. But $h^{\leftarrow}[\{(0,0)\}] = \{3\}$. Since h pulls back an open set, $\{(0,0)\}$ to an open set, $\{3\}$, whose closure in $\mathbf{N}$ is compact, then h cannot be singular. Consequently the fact that $[X \cup_f S(f)] \vee [X \cup_g S(g)]$ is singular is not sufficient to imply that $h = f \times g$ is a singular map.

Remark 2.9. For further reference, we would like to emphasize an important point illustrated in the above example. In this example, $\{f, g\}$ is contained in S_β, hence, by Theorem 1.14, $\omega_{\{f,g\}}X$ is equivalent to $X \cup_f S(f) \vee X \cup_g S(g)$. We have shown that the evaluation map $e_{\{f,g\}} = f \times g$ is not a singular map even though $\omega_{\{f,g\}}X$ was proven to be a singular compactification. Hence, if G is an arbitrary subset of S_β, it is not sufficient that $\omega_G X$ be a singular compactification for e_G to be a singular map, i.e., "*$\omega_G X$ being singular does not imply that e_G is singular.*"

Before we pursue our goal of characterizing those spaces of form $\omega_G X$ which are singular we need a little more preparation. In Theorem 2.6 we have shown that, if αX is a singular compactification, then αX is equivalent to $\omega_{S_\alpha} X$. Given the original definition of a singular compactification, one would naturally like to find a singular map which induces $\omega_{S_\alpha} X$, i.e., a singular map f such that $\omega_{S_\alpha} X$ is equivalent (as a compactification of X) to $X \cup_f S(f)$). In Remark 2.9, we have shown that this singular map need not be e_{S_α}. We now describe some properties possessed by singular maps which induce a singular compactification of form $\omega_G X$.

Theorem 2.10. *Let $G \subseteq S_\beta$. Then the following are equivalent:*

1) *$\omega_G X$ is a singular compactification.*

2) *There is a singular function $k : X \to K$ mapping X densely into some compact Hausdorff space K which extends to $k^{\omega_G} : \omega_G X \to K$ such that k^{ω_G} is one-to-one on $\omega_G X \backslash X$ (hence $\omega_G X$ is equivalent to $X \cup_k S(k)$).*

Proof. 1) $\Rightarrow$ 2). Since $\omega_G X$ is a singular compactification then there exists a retraction map $r : \omega_G X \to \omega_G X \setminus X$ (by Theorem 1.2). We claim that the map $t = (e_G^{\omega_G})|_{\omega_G X\setminus X} \circ r|_X$ is the required function. Clearly $\omega_G X \cong X \cup_t S(t)$ (by Theorem 2.4 b)). Since $r|_X$ is a singular map then, by Proposition 1.16, t is a singular map. Observe that $t^{\omega_G}|_{\omega_G X\setminus X} = e_G \omega_G|_{\omega_G X\setminus X}$. By Proposition 1.11, G^{ω_G} separates the points of $\omega_G X\setminus X$, hence by Proposition 1.10 $t^{\omega_G} = (e_G^{\omega_G})|_{\omega_G X\setminus X} \circ r$ is one-to-one on $\omega_G X \setminus X$.

2)$\Rightarrow$ 1). Let k be a singular map such that k^{ω_G} separates the points of $\omega_G X\setminus X$. By Lemma 2.1, $\omega_G X \cong X \cup^* S(k) = X \cup_k S(k)$ (Notation 1.15). Hence, $\omega_G K$ is singular. □

The next theorem describes more specifically a singular map which induces a singular compactification $\omega_G X$ ($G \subseteq S_\beta$).

Theorem 2.11. *Let αX be a singular compactification of X. Let $r : \alpha X \to \alpha X\setminus X$ be a retraction map, and define F to be $\{f \circ r|_X : f \in C(\alpha X)\}$. Then $F \subseteq S_\alpha$, F is a subalgebra of $C_\alpha(X)$, e_F is a singular map, e_F^α separates points of $\alpha X\setminus X$, and $\alpha X \cong X \cup e_F S(e_F) \cong \omega_F X$.*

Proof. Let αX, the mapping r and the family of functions F be as described in the statement of the theorem. By Proposition 1.16, $F \subseteq S_\alpha$. It is easily verified that F is a subalgebra of $C_\alpha(X)$. We will now show that e_F is a singular map. Let $J = \prod_{g\in F} S(g)$ and $t \in \mathrm{cl}_J e_F[X]$ and U be a basic open neighborhood of t in J of the form $J \cap [\cap\{U_{f_k} \circ r|_X : k = 1 \text{ to } n\}]$ where $\{f_1, \dots , f_n\} \subseteq C(\alpha X)$. Then $e_F^{\leftarrow}(t) \subseteq e_F^{\leftarrow}[U] = \cap\{(f_k \circ r|_X)^{\leftarrow}[U_{f_k} \circ r|_X] : k = 1 \text{ to } n\} = \cap\{r|_X^{\leftarrow} \circ f_k^{\leftarrow}[U_{f_k\circ r|_X}] : k = 1 \text{ to } n\}$. Suppose $\mathrm{cl}_X \cap \{r|_X^{\leftarrow} \circ f_k^{\leftarrow}[U_{f_k\circ r|_X}] : k = 1 \text{ to } n\}$ is compact. Note that $\cap\{r|_X^{\leftarrow} \circ f_k^{\leftarrow}[U_{f_k\circ r|_X}] : k = 1 \text{ to } n\} = r|_X^{\leftarrow}[\cap\{f_k^{\leftarrow}[U_{f_k\circ r|_X}] : k = 1 \text{ to } n\}]$. Hence $\mathrm{cl}_X r|_X^{\leftarrow}[\cap\{f_k^{\leftarrow}[U_{f_k\circ r|_X}] : k = 1 \text{ to } n\}]$ is compact. But this contradicts the fact that $r|_X$ is a singular map. Hence e_F is a singular map.

We now show that $\alpha X \cong X \cup_{e_F} S(e_F)$. Recall that $C_{\omega_F}(X) = \mathrm{cl}_{C_\alpha(X)}\langle C_\omega(X) \cup F\rangle$ (Theorem 1.13). Since $F \subseteq S_\alpha \subseteq C_\alpha(X)$, then $\omega_F X \leq \alpha X$. Thus every function $f^\alpha \circ r|_X$ in F extends to the function $f^\alpha \circ r$ on αX. Let x and y be distinct points in $\alpha X\setminus X$. Then $r(x) = x \neq y = r(y)$. Since $C(\alpha X)$ separates the points of $\alpha X\setminus X$ then there is a function f in $C(\alpha X)$ such that $f(x) \neq f(y)$. This means that F^α separates the points of $\alpha X \backslash X$. Hence, $\omega_F X$ is equivalent to $\sup\{\omega_f X : f \in F\} \cong \alpha X$ (by Theorem 1.14). Thus,

$$\begin{aligned} \alpha X &\cong X \cup^* S(e_F) \quad \text{(Lemma 2.1)} \\ &\cong X \cup_{e_F} S(e_F) \quad \text{(by Notation 1.15).} \quad \square \end{aligned}$$

We now proceed to answer our question: When is the supremum of singular compactifications a singular compactification? (Equivalently, when is a compactification of form $\omega_G X$ (where G is contained in S_β) singular?

In what follows, we will require the following concepts. If B is a collection of functions in $C^*(X)$, a *maximal stationary set of* B is a subset of X maximal with respect to the property that every f in B is constant on it.

The maximal stationary sets of a subalgebra are briefly discussed in [**12**, 16.31].

Let $G \subseteq C^*(X)$, x a point in X and $G^+ = \{f - \mathbf{r} : f \in G, r \in \mathbf{R}\}$. The symbol ${}_X K_G$ will denote the set $\cap\{Z(f) : f \in G^+, x \in Z(f)\}$. Thus $y \in {}_x K_G$ if and only if $f(y) = f(x)$ for each $f \in G$. Suppose αX is a compactification of X such that G (hence $G+$) is a subset of $C_\alpha(X)$. For $x \in \alpha X$, let ${}_x K_{G^\alpha} = \cap\{Z(f^\alpha) : f \in G^+, x \in Z(f^\alpha)\}$. It is clear that the subset ${}_x K_G$ (${}_x K_{G^\alpha}$) is a maximal stationary set of G (G^α) which contains the point x. It is easily observed that, given $G \subseteq C^*(X)$, the collection $\{{}_X K_G : x \in X\}$ forms a partition of X.

Theorem 2.12. *Let αX be a compactification of X. Let G be a subset of S_α such that the evaluation map $e_{G^\alpha} : \alpha X \to \Pi_{f \in G} S(f)$ separates the points of $\alpha X \backslash X$. Then αX is equivalent to $\omega_G X$. Furthermore, the following are equivalent*:

1) *e_G is a singular map and $\omega_G X (\cong \alpha X)$ is equivalent to the singular compactification $X \cup_{e_G} S(e_G)$.*

2) $e_G[X] \subseteq e_G^{\omega_G}[\omega_G X \backslash X]$.

3) *e_G is a singular map.*

4) *e_F is a singular map for every finite subset F of G.*

5) ${}_xK_G^{\omega_G} \cap (\omega_G X \backslash X)$ *is a singleton set for every* $x \in X$.

Proof. That αX is equivalent to $\omega_G X$ follows from Proposition 1.11.

1) $\Rightarrow$ 3). Obvious.

3) $\Rightarrow$ 1). By Lemma 2.1, $\omega_G X$ is equivalent to $X \cup^* S(e_G)$ (since $e_G^{\omega_G}$ separates the points of $\omega_G X \backslash X$). Since e_G is singular, $X \cup^* S(e_G)$ is equivalent to $X \cup_{e_G} S(e_G)$ (Notation 1.15).

2) $\Leftrightarrow$ 3). This is a special case of Corollary 1.7.

3) $\Rightarrow$ 4). Let $P = \Pi_{f \in G} f[X]$. Suppose the function $e_G : X \to \mathrm{cl}_P e_G[X]$ is a singular map. Define $M_{G,F} : \Pi_{f\in G} f[X] \to \Pi_{f \in F} f[X]$ by $M_{G,F}(\langle f(x)\rangle_{f\in G}) = \langle f(x)\rangle_{f \in F}$. Then $e_F = M_{G,F} \circ e_G$ so e_F is singular (by Proposition 1.16).

4) $\Rightarrow$ 2). Let $G \subseteq S_\beta$, and suppose that, for every finite subset F of G, e_F is a singular map. Then, since 3) $\Rightarrow$ 1), $X \cup_{e_F} S(e_F)$ is equivalent to $\omega_F X$. We will show that $e_G[X] \subseteq e_{G^{\omega_G}}[\omega_G X \backslash X]$ by showing that $e_{G^{\omega_G}}[\omega_G X\backslash X] \cap e_G[X]$ is densely contained in $e_G[X]$. Let $p \in e_G[X]$. Then $p = e_G(x) = \langle f(x) \rangle_{f \in G}$ for some $x \in X$. Let U be a basic open neighborhood of p in $\Pi_{f \in G} S(f)$. We will show that $U \cap e_G^{\omega_G}[\omega_G X \backslash X]$ is nonempty. Let $F = \{f \in G : \pi_f[U] \neq S(f)\}$ where π_f is the fth projection map with domain $\Pi_{f\in G} S(f)$. We will denote the elements of F as $\{f_1, f_2, \dots, f_n\}$ (where the indices correspond to the nontrivial components of U). Since $\omega_F X$ is a singular compactification, $e_F[X] \subseteq e_F^{\omega_F}[\omega_F X \backslash X]$ (by 1) $\Rightarrow$ 3) $\Rightarrow$ 2)). Consequently, there exists a point y in $\omega_F X \backslash X$ such that $e_F^{\omega_F}(y) = (f_1^{\omega_F}(y), f_{2^{\omega_F}}(y), \dots, f_n^{\omega_F}(y)) = (f_1(x), \dots, f_n(x)) = e_F(x)$. Now $\omega_F X$ is equivalent to $\omega_G X$, hence there is a function $\pi : \omega_G X \to \omega_F X$ which maps $\omega_G X$ onto $\omega_F X$, fixing the points of X. Let $u \in \pi^{\leftarrow}_{\omega_G \omega_F}(y) \subseteq \omega_G X \backslash X$. Then $f_k^{\omega_G}(u) = f_k^{\omega_F \circ} \pi_{\omega_G \omega_F}(u) = f_{k_F^\omega}(y) = f_k(x)$ for $k = 1$ to n. Then, for each $k = 1$ to n, $f_k^{\omega_G}(u) = f_k(x) \in U_k$. Therefore, $f_{k_G^\omega}(u) \in U$; hence $U \cap e_F^{\omega_G}[\omega_G X \backslash X]$ is nonempty. Recall that U was an arbitrary basic open neighborhood of p in $e_G[X]$. Hence we have shown that $e_F^{\omega_G}[\omega_G X\backslash X] \cap e_G[X]$ is dense in $e_G[X]$. Since $e_{F_G^\omega}[\omega_G X \backslash X]$ is compact, then $e_G[X] \subseteq e_G^{\omega_G}[\omega_G X \backslash X]$.

2) $\Rightarrow$ 5). Suppose $G \subseteq S_\alpha$ and $e_G[X] \subseteq e_G^{\omega_G}[\omega_G X \backslash X]$. We first note that, if f is a real-valued singular map, then, for any $r \in \mathbf{R}$, $f - \mathbf{r}$ is a singular map (Proposition 1.16). Let $x \in X$ and $K =$

${}_xK_G^{\omega_G} = \cap\{Z(f^{\omega_G}) : f \in G^+, x \in Z(f)\}$. By hypothesis, there exists a $y \in \omega_G X\backslash X$ such that $e_G(x) = e_G^{\omega_G}(y)$. Then $f^{\omega_G}(y) = f(x)$ for all $f \in G$. It follows that y belongs to $K \cap (\omega_G X\backslash X)$. We now verify that $(\omega_G X\backslash X) \cap K = \{y\}$. Suppose z belongs to $K \cap (\omega_G X\backslash X)$ and $z \neq y$. Then, since G^{ω_G} separates the points of $\omega_G X\backslash X$, there is an $f \in G$ such that $f^{\omega_G}(z) \neq f^{\omega_G}(y)$. If $f(x) = \varepsilon$, then $(f - \varepsilon)(x) = 0$. Since $f - \varepsilon \in G^+$ and since y and z belong to K, then $(f^{\omega_G} - \varepsilon)(y) = 0$ and $(f^{\omega_G} - \varepsilon)(z) = 0$. Consequently, $f^{\omega_G}(y) = f^{\omega_G}(z) = \varepsilon$, a contradiction. It follows that $K \cap (\omega_G X\backslash X) = \{y\}$.

5) $\Rightarrow$ 2). Let $G \subseteq S_\beta$, $G^+ = \{f - \mathbf{r} : f \in G, r \in \mathbf{R}\}$ and $K = \cap\{Z(f^{\omega_G}) : f \in G^+,\ x \in Z(f)\}$ for each $x \in X$. Suppose $K \cap (\omega_G X\backslash X)$ is a singleton for each $x \in X$. Let $x_0 \in X$ and suppose $K \cap (\omega_G X\backslash X) = \{y_0\}$. We wish to show that $e_G(x_0) \in e_G^{\omega_G}[\omega_G X\backslash X]$. Suppose that for some $f \in G$, $f(x) = \varepsilon$. Then $f - \varepsilon \in G^+$ and $(f - \varepsilon)(x_0) = 0$. By definition of K, $(f^{\omega_G} - \varepsilon)(y_0) = 0$. Hence, $f^{\omega_G}(y_0) = \varepsilon = f(x_0)$. Consequently, since f was arbitrarily chosen in G, $f(x_0) = f^{\omega_G}(y_0)$ for all $f \in G$. Hence, $e_G(x_0) = \langle f(x_0)\rangle_{f\in G} = \langle f^{\omega_G}(y_0)\rangle_{f\in G} = e_G^{\omega_G}(y_0) \in e_G^{\omega_G}[\omega_G X\backslash X]$. All the parts of the theorem have thus been established. □

We have just characterized compactifications of a space X which are the suprema of singular compactifications.

Earlier, we provided an example of a compactification αX, namely the two-point compactification of $\mathbf{R}$, which could not be expressed in the form $\omega_{S_\alpha} X$. We now present a condition which guarantees that a compactification αX is equivalent to $\omega_{S_\alpha} X$. (Recall however that this does not imply that αX is a singular compactification; see Remark 2.9.)

Proposition 2.13. *If $\alpha X\backslash X$ is not totally disconnected, then αX is equivalent to $\omega_{S_\alpha} X$.*

Proof. If $\alpha X\backslash X$ is not totally disconnected, then $\alpha X\backslash X$ has a connected component K which is not a singleton. Let p and q be distinct elements of K. Let r and s be any two distinct elements of $\alpha X\backslash X$. Then there exists an $f \in C(\alpha, X)$ such that $\mathbf{0} \leq f \leq \mathbf{1}$, $f[\{p, r\}] = \{0\}$ and $f[\{q, s\}] = \{1\}$. Since K is connected, f maps

$\alpha X \backslash X$ onto $[0,1]$. Since $f[\alpha X \backslash X] = S(f)$ (see Proposition 1.6), $f|_X$ maps X into $S(f)$. Hence $f|_X$ is a singular map. Therefore, we have shown that S^α_α separates points of $\alpha X \backslash X$. It follows that

$$\begin{aligned} \alpha X &= \sup\{\omega_f X : f \in S_\alpha\} \quad \text{(by Theorem 1.14)} \\ &= \omega_{S_\alpha} X \quad \text{(by Theorem 1.18).} \qquad \square \end{aligned}$$

The converse of Proposition 2.13 fails (even for connected spaces X). A space X which is almost compact, noncompact (so that $\beta X \backslash X$ is simultaneously connected and totally disconnected) witnesses the failure of the converse of Proposition 2.13.

Proposition 2.14. *Let X be a strongly zero-dimensional not almost compact space. then βX is the supremum of the family of the two-point singular compactifications of X. Hence $\beta X = \omega_{S_\beta} X$.*

Proof. Since X is strongly zero-dimensional, then βX is zero-dimensional (see 3.34 of [**21**]). Let p and q be distinct points in βX, and let U be a clopen set of βX which contains p but not q. Since the singular characteristic function $f = \chi_{X \backslash (U \cap X)}$ has an extension to βX which separates p and q, then the family H of singular characteristic functions of X extends to H^β to separate the points of βX. By Theorem 1.14, $\beta X = \sup\{\omega_f X : f \in H\}$. Since $\omega_f X$ is a singular two-point compactification of X, we are done. $\square$

We now provide various results which follow quickly from some of the properties of singular compactifications described above.

Proposition 2.15. *Let $K(X)$ denote the family of all compactifications of X. Let $K = \{\alpha X \in K(X) : \alpha X \backslash X$ is homeomorphic to a closed interval of $\mathbf{r}\}$. Then $K \subseteq \{\omega_f X : f \in S_\beta\}$, and, if X is connected, then $K = \{\omega_f X : f \in S_\beta\}$. (We will consider the singleton set $\{a\}$ in $\mathbf{r}$ as the closed interval $[a,a]$ with empty interior.)*

Proof. We will first show that K is contained in $\{\omega_f X : f \in S_\beta\}$. Let $\alpha X \in K$. Then $\alpha X \backslash X$ is homeomorphic to a closed interval of

$\mathbf{R}$. Since $\alpha X\backslash X$ is an absolute retract, then $\alpha X\backslash X$ is a retract of αX (see 15D4 of [**22**]). Hence, αX is a singular compactification. Let $r : \alpha X \to \alpha X\backslash X$ denote a retraction from αX onto $\alpha X\backslash X$ and $h : \alpha X\backslash X \to \mathbf{R}$ denote a homeomorphism from $\alpha X\backslash X$ to a closed interval of $\mathbf{R}$. Since $r|_X$ is singular, then $h \circ r|_X$ is singular (by Proposition 1.16). That αX is equivalent to $X \cup_{h\circ r|_X} S(h\circ r|_X)$ follows from Lemma 2.1.

We now show that if X is connected then $\{\omega_f X : f \in S_\beta\}$ is contained in K. Let $f \in S_\beta$. Recall that $\omega_f X$ is equivalent to $X \cup_f S(f)$ (by Theorem 2.4). Since $\omega_f X$ is a singular compactification, $\omega_f X\backslash X$ is the closure of the continuous image of the connected space X. This implies that $\omega_f X\backslash X$ is a connected compact subset of $\mathbf{r}$. It follows that $\omega_f X$ belongs to K. □

We have shown that K is always contained in $\{\omega_f X : f \in S_\beta\}$. However, if X is not connected, then it may happen that K is a proper subset of $\{\omega_f X : f \in S_\beta\}$ as witnessed by the following example. Let $f \in C^*(\mathbf{N})$ be defined as follows: f maps the even numbers to $\{3\}$ and the odd numbers to $\{4\}$. Clearly f is singular. Since f maps $\omega_f\mathbf{N}\backslash\mathbf{N}$ homeomorphically onto $S(f) = \{3, 4\}$ then $\omega_f\mathbf{N}\backslash\mathbf{N}$ is not connected. Consequently $\omega_f\mathbf{N}\backslash\mathbf{N}$ is not a closed interval of $\mathbf{R}$.

Given a singular compactification αX of X, we know by definition that αX is equivalent to $X \cup_f S(f)$, where $f : X \to K$ is some singular map from X into some compact Hausdorff space K. But there may be many such maps f for which this is true. It is important to know how these maps are related to each other.

On page 35 of [**11**], the following question is asked:

> Suppose $X \cup_f S(f)$ and $X \cup_g S(g)$ are two singular compactifications and f and g both map X densely into the same space K so that $S(f) = S(g)$. When are $X \cup_f S(f)$ and $X \cup_g S(g)$ equivalent compactifications of X?

To provide further motivation of the study of this problem, consider the two functions sine and cosine on the real numbers. Note that both are singular maps and that $[-1, 1] = S(\text{sine}) = S(\text{cosine})$ is their common singular set. One would surely wonder whether the compactifications $\mathbf{R}\cup_{\text{sine}} S(\text{sine})$ and $\mathbf{R}\cup_{\text{cosine}} S(\text{cosine})$ are equivalent. The following theorem will quickly help resolve this question.

Note that, in the following theorem, the functions f and g are not assumed to be real-valued maps. The conjecture that $X \cup_f S(f) \cong X \cup_g S(g)$ if and only if f and g agree except on a compact set, has been shown (in [**11**]) to be false.

We answer the question in the following theorem. For convenience we recall the following notation introduced in Chapter 1. If αX and γX are compactifications of X such that $\alpha X \leq \gamma X$, we will denote the projection map from γX onto αX which fixes the points of X by $\pi_{\gamma\alpha}$. If X and Y are two topological spaces "$X \cong Y$" will mean that X is homeomorphic to Y.

Proposition 2.16. *Let $f : X \to K_f$ and $g : X \to K_g$ be two singular maps from the space X into the compact spaces K_f and K_g, respectively, such that $S(f) \cong S(g)$. Then the following are equivalent*:

1) *$X \cup_f S(f)$ is equivalent to $X \cup_g S(g)$.*

2) *The function $f : X \to S(f)$ extends continuously to a function $f^* : X \cup_g S(g) \to S(f)(\cong S(g))$ in such a way that $f*$ separates the points of $S(g)$.*

Proof. $(1 \Rightarrow 2)$. Let $\alpha X = X \cup_f S(f)$ and $\gamma X = X \cup_g S(g)$, where $S(f) \cong S(g)$. Suppose that αX is equivalent to γX. Let $r : \alpha X \to \alpha X \backslash X$ be defined as follows: $r(x) = x$ if x belongs to $\alpha X \backslash X$ and $r|_X = f$. Since f is singular and $\alpha X \backslash X = S(f)$, it is easily verified that r is continuous and hence is a retraction of αX onto $\alpha X \backslash X$. Let $\pi_{\gamma\alpha}$ denote the projection map from γX onto αX, i.e., $\pi_{\gamma\alpha}$ fixes the points of X and maps $\gamma X \backslash X$ homeomorphically onto $\alpha X \backslash X$. Let $f^* : \gamma X \to S(f)$ be defined as $f^* = r \circ \pi_{\gamma\alpha}$. Note that $f^*|_{S(g)} = r \circ \pi_{\gamma\alpha}|_{S(g)} = \pi_{\gamma\alpha}|_{S(g)}$. Clearly $f^*|_X = f$ and f^* is continuous, being the composition of two continuous functions. Hence $f : X \to S(f)$ extends continuously to a function $f^* : X \cup_g S(g) \to S(f)$. Furthermore, since $\pi_{\gamma\alpha}$ maps $\gamma X \backslash X$ homeomorphically onto $\alpha X \backslash X$ and r is the identity function on $\alpha X \backslash X$, f^* separates the points of $\gamma X \backslash X$. This proves that the given condition is necessary.

$(2 \Rightarrow 1)$. Suppose now that the function f extends continuously to a function $f^* : X \cup_g S(g) \to S(f)$ in such a way that f^* separates the points of $S(g)$. It must then follow from Lemma 2.1 that $X \cup_g S(g)$ is equivalent to $X \cup^* S(f)$. But since f is singular, $X \cup^* S(f)$ is equivalent to $X \cup_f S(f)$ (see Notation 1.15). Hence $X \cup_g S(g)$ is equivalent to $X \cup_f S(f)$. This proves that the condition is sufficient. $\square$

Example 2.17. The singular compactifications $\mathbf{R} \cup_{\text{sine}} S(\text{sine})$ and $\mathbf{R} \cup_{\text{cosine}} S(\text{cosine})$ are not equivalent.

Proof. Suppose the singular compactifications $\mathbf{R} \cup_{\text{sine}} S(\text{sine})$ and $\mathbf{R} \cup_{\text{cosine}} S(\text{cosine})$ are equivalent. Then, by Proposition 2.16, 1) $\Rightarrow$ 2) the function sine : $\mathbf{R} \to [-1,1]$ extends to a function sine^* : $\mathbf{R} \cup_{\text{cosine}} S(\text{cosine}) \to [-1,1]$ such that $\text{sine}^*|_{S(\text{cosine})}$ is one-to-one on $S(\text{cosine}) = [-1,1]$. Then $\text{sine}^*|_{S(\text{cosine})}$ is a homeomorphism from $[-1,1]$ onto $[-1,1]$, hence is monotone (increasing or decreasing) and maps endpoints to endpoints. Suppose without loss of generality that $\text{sine}^*|_{S(\text{cosine})}$ maps -1 to -1 and 1 to 1. Let U be an open interval containing 1 such that $U \cap [-1,1] \subseteq (1/2,1]$ and $\text{sine}^*|^{\leftarrow}_{S(\text{cosine})}[U] = V \subseteq (1/2,1]$. Observe that $\sin^{\leftarrow}[U] \cap \cos^{\leftarrow}[\sin^{\leftarrow}[U]] = \sin^{\leftarrow}[U] \cap \cos^{\leftarrow}[V]$ is empty. Since sine^* is continuous on $\mathbf{R} \cup_{\text{cosine}} S(\text{cosine})$, $\text{sine}^{*\leftarrow}[U] = V \cup \sin^{\leftarrow}[U]$ is open in $\mathbf{R} \cup_{\text{cosine}} S(\text{cosine})$. Since $V \cup \cos^{\leftarrow}[V]$ is also open in $\mathbf{R} \cup_{\text{cosine}} S(\text{cosine})$, then $(V \cup \cos^{\leftarrow}[V]) \cap \sin^{*\,\leftarrow}[U] = V$ is open in $\mathbf{R} \cup_{\text{cosine}} S(\text{cosine})$. Since $V \subseteq S(\text{cosine})$, we have a contradiction. Hence $\mathbf{R} \cup_{\text{sine}} S(\text{sine})$ and $\mathbf{R} \cup_{\text{cosine}} S(\text{cosine})$ are not equivalent. $\square$

The following corollary offers an easy method of recognizing many pairs of singular compactifications which are equivalent.

We introduce the following definition.

Definition 2.18. We will say that two functions $f : X \to K$ and $g : X \to K$ from a space X to a space K are *homeomorphically related* if there exists a homeomorphism $h : \text{cl}_K f[X] \to \text{cl}_K g[X]$ such that $h(f(x)) = g(x)$ for all x in X.

It is clear that families of homeomorphically related functions from a space X to a space K form equivalence classes on the collection of all functions from X to K.

Corollary 2.19. *Let $f : X \to K$ and $g : X \to K$ be two singular maps on X such that $S(f) = S(g) = K$. If f and g are homeomorphically related, then $X \cup_f S(f)$ is equivalent to $X \cup_g S(g)$.*

Proof. Let $h : S(g) \to S(f)$ be a homeomorphism from $S(g)$ onto $S(f)$ such that $h(g(x)) = f(x)$. Note that $g : X \to S(g)$ extends continuously to the function $g^* : X \cup_g S(g) \to S(g)$ where g^* acts as the identity function on $S(g)$, see Theorem 1.2. Hence $h \circ g$ extends to $h \circ g^* : X \cup_g S(g) \to S(g)$ where $(h \circ g^*)|_{S(g)} = h$, a homeomorphism from $S(g)$ onto $S(f)$. Since $f = h \circ g$ on X, f extends to $f^* : X \cup_g S(g) \to S(g)$ such that f^* separates the points of $S(g)$. By Proposition 2.16, 2) $\Rightarrow$ 1), $X \cup_f S(f)$ is equivalent to $X \cup_g S(g)$. $\square$

Example 2.20. The singular compactifications $\mathbf{R} \cup_{\sin^2} S(\sin^2)$ and $\mathbf{R} \cup_{\cos^2} S(\cos^2)$ are equivalent.

Proof. It is easily seen that both $\sin^2$ and $\cos^2$ are singular maps on $\mathbf{R}$. Observe that if $h : [0,1] \to [0,1]$ is the homeomorphism defined by $h(x) = 1 - x$, then $h \circ \sin^2 = \cos^2$. Hence, by Corollary 2.19, $\mathbf{R} \cup_{\sin^2} S(\sin^2)$ is equivalent to $\mathbf{R} \cup_{\cos^2} S(\cos^2)$. $\square$

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