ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 2, Summer 1998

# ON THE EIGENVALUES OF BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER DIFFERENCE EQUATIONS

PATRICIA J.Y. WONG AND RAVI P. AGARWAL

ABSTRACT. We consider the boundary value problem

$$\Delta^n y + \lambda Q(k, y, \Delta y, \dots, \Delta^{n-2}y) = \lambda P(k, y, \Delta y, \dots, \Delta^{n-1}y),$$
$$n \geq 2, \quad 0 \leq k \leq N,$$
$$\Delta^i y(0) = 0, \quad 0 \leq i \leq n-3,$$
$$\alpha\Delta^{n-2}y(0) - \beta\Delta^{n-1}y(0) = 0,$$
$$\gamma\Delta^{n-2}y(N+1) + \delta\Delta^{n-1}y(N+1) = 0$$

where $\lambda > 0$, $\alpha, \beta, \gamma$ and $\delta$ are constants satisfying $\alpha\gamma(N+1) + \alpha\delta + \beta\gamma > 0$, $\alpha, \gamma > 0, \beta \geq 0$ and $\delta \geq \gamma$. Upper and lower bounds for $\lambda$ are established for the existence of positive solutions of this boundary value problem.

**1. Introduction.** Let $a, b$, $b > a$, be integers. We shall denote $[a, b] = \{a, a+1, \dots, b\}$. All other interval notation will carry its standard meaning, e.g., $[0, \infty)$ denotes the set of nonnegative real numbers. Also, the symbol $\Delta^i$ denotes the $i$th forward difference operator with stepsize 1.

In this paper we shall consider the $n$th order difference equation

$$\Delta^n y + \lambda Q(k, y, \Delta y, \dots, \Delta^{n-2}y) = \lambda P(k, y, \Delta y, \dots, \Delta^{n-1}y), \quad k \in [0, N] \tag{1.1}$$

and the boundary conditions

$$\Delta^i y(0) = 0, \quad 0 \leq i \leq n-3, \tag{1.2}$$
$$\alpha\Delta^{n-2}y(0) - \beta\Delta^{n-1}y(0) = 0, \tag{1.3}$$
$$\gamma\Delta^{n-2}y(N+1) + \delta\Delta^{n-1}y(N+1) = 0 \tag{1.4}$$

Received by the editors on September 10, 1995.
*Key words and phrases.* Eigenvalues, positive solutions, difference equations.

where $n \geq 2$, $N(\geq n-1)$ is a fixed positive integer, $\lambda > 0, \alpha, \beta, \gamma$ and $\delta$ are constants so that

$$\rho = \alpha\gamma(N+1) + \alpha\delta + \beta\gamma > 0 \tag{1.5}$$

and

$$\alpha > 0, \qquad \gamma > 0, \qquad \beta \geq 0, \qquad \delta \geq \gamma. \tag{1.6}$$

Further, we assume that there exist functions $f : [0, \infty) \to (0, \infty)$ and $p, p_1, q, q_1 : [0, N] \to \Re$ such that

(i) $f$ is nondecreasing;

(ii) for $u \in [0, \infty)$,

$$q(k) \leq \frac{Q(k, u, u_1, \ldots, u_{n-2})}{f(u)} \leq q_1(k),$$
$$p(k) \leq \frac{P(k, u, u_1, \ldots, u_{n-1})}{f(u)} \leq p_1(k);$$

(iii) $q(k) - p_1(k)$ is nonnegative and is not identically zero for $k \in [0, N]$.

We shall characterize the values of $\lambda$ for which there exists a positive solution of the boundary value problem (1.1)–(1.4). By a *positive solution* $y$ of (1.1)–(1.4), we mean $y : [0, N+n] \to \Re$, $y$ satisfies (1.1) on $[0, N]$, $y$ fulfills (1.2)–(1.4), and $y$ is nonnegative on $[0, N+n]$, positive on $[n-1, N+n-2]$. If, for a particular $\lambda$, the boundary value problem (1.1)–(1.4) has a positive solution $y$, then we shall call $\lambda$ an *eigenvalue* and $y$ a corresponding *eigenfunction* of (1.1)–(1.4).

The motivation for the present work stems from many recent investigations [**1**–**15**]. In fact, for the special case $\lambda = 1$, applications of (1.1)–(1.4) and its continuous version have been made to singular boundary value problems by Agarwal and Wong [**2, 14**]. Further, assuming that $f$ is either superlinear or sublinear, existence results for positive solutions (when $\lambda = 1$) have also been established by Wong and Agarwal [**15**], as well as by Eloe, Henderson and Wong [**5**] in the continuous case. For a general $\lambda \geq 0$ we refer in particular to [**3, 4, 8**–**10**]. In all these papers, particular cases of the continuous version of (1.1)–(1.4) are considered. For example, in [**9**], Fink, Gatica and Hernandez deal with the boundary value problem

$$\tag{1.7} \begin{gathered} y'' + \lambda q(x) f(y) = 0, \quad x \in (0,1), \\ y(0) = y(1) = 0. \end{gathered}$$

Their results are extended in [**10**] to systems of second order boundary value problems. In [**3**] and [**8**] the authors tackle a different boundary value problem

$$\tag{1.8} \begin{gathered} y'' + ((N-1)/x)y' + \lambda q(x) f(y) = 0, \quad x \in (0,1), \\ y'(0) = y(1) = 0. \end{gathered}$$

Recently, Chyan and Henderson [**4**] have studied a more general problem than (1.7), namely,

$$\tag{1.9} \begin{gathered} y^{(n)} + \lambda q(x) f(y) = 0, \quad x \in (0,1), \\ y^{(i)}(0) = y^{(n-2)}(1) = 0, \quad 0 \le i \le n-2. \end{gathered}$$

Our results not only generalize and extend the known eigenvalue theorems for (1.7)–(1.9) to the discrete case, but also include several other known criteria discussed in [**1**].

Throughout, we shall let

$$E = \{\lambda > 0 \mid \text{(1.1)–(1.4) has a positive solution}\}.$$

We note that $E$ is the set of eigenvalues of (1.1)–(1.4).

The plan of this paper is as follows: In Section 2 we shall present some properties of a Green's function which will be used later. In Section 3 we define an appropriate Banach space and a cone so that the set $E$ can be characterized.

**2. Preliminaries.** To obtain a solution of (1.1)–(1.4), we need a mapping whose kernel $g(i,j)$ is the Green's function of the boundary value problem

$$\begin{gathered} -\Delta^n y = 0, \qquad \Delta^i y(0) = 0, \quad 0 \le i \le n-3, \\ \alpha \Delta^{n-2} y(0) - \beta \Delta^{n-1} y(0) = 0, \\ \gamma \Delta^{n-2} y(N+1) + \delta \Delta^{n-1} y(N+1) = 0. \end{gathered}$$

It can be verified that

$$G(i,j) = \Delta^{n-2} g(i,j), \quad \text{w.r.t. } i$$

is the Green's function of the boundary value problem

$$-\Delta^2 w = 0, \qquad \alpha w(0) - \beta \Delta w(0) = 0,$$
$$\gamma w(N+1) + \delta \Delta w(N+1) = 0.$$

Further, we have [**14**]

$$(2.1) \quad G(i,j) = \frac{1}{\rho} \begin{cases} [\beta + \alpha(j+1)][\delta + \gamma(N+1-i)] & j \in [0, i-1], \\ (\beta + \alpha i)[\delta + \gamma(N-j)] & j \in [i, N]. \end{cases}$$

We observe that the conditions (1.5) and (1.6) imply that $G(i,j)$ is nonnegative on $[0, N+2] \times [0, N]$ and positive on $[1, N+1] \times [0, N]$.

**Lemma 2.1** [**15**]. *For* $(i,j) \in [1, N] \times [0, N]$, *we have*

$$(2.2) \qquad G(i,j) \geq K G(j,j)$$

*where* $0 < K < 1$ *is given by*

$$(2.3) \qquad K = \frac{(\beta + \alpha)(\delta + \gamma)}{(\beta + \alpha N)(\delta + \gamma N)}.$$

**Lemma 2.2** [**15**]. *For* $(i,j) \in [0, N+2] \times [0, N]$, *we have*

$$(2.4) \qquad G(i,j) \leq L G(j,j)$$

*where* $L > 1$ *is given by*

$$(2.5) \qquad L = \begin{cases} (\beta + \alpha)/\beta & \beta > 0, \\ 2 & \beta = 0. \end{cases}$$

We shall need the following notations in Section 3. For a nonnegative $y$ which is not identically zero on $[0, N]$, we denote

$$\theta = \sum_{l=0}^{N} G(l,l)[q_1(l) - p(l)] f(y(l))$$

and

$$\Gamma = \sum_{l=0}^{N} G(l,l)[q(l) - p_1(l)]f(y(l)).$$

In view of (i)–(iii), it is clear that $\theta \geq \Gamma > 0$. Further, we define the constant

$$\xi = \frac{K\Gamma}{L\theta}.$$

It is noted that $0 < \xi < 1$.

**3. Main results.** Let $B$ be the Banach space defined by

$$B = \{y : [0, N+n] \to \Re \mid \Delta^i y(0) = 0, 0 \leq i \leq n-3\}$$

with the norm $\|y\| = \max_{k \in [0,N+2]} |\Delta^{n-2} y(k)|$, and let

$$C = \Big\{ y \in B \;\Big|\; \Delta^{n-2} y(k) \text{ is nonnegative and is not identically zero on } [0, N+2];\ \min_{k \in [1,N]} \Delta^{n-2} y(k) \geq \xi \|y\| \Big\}$$

be a cone in $B$. Further, we let

$$C_M = \{y \in C \mid \|y\| \leq M\}.$$

**Lemma 3.1** [**15**]. *Let $y \in B$. For $0 \leq i \leq n-3$, we have*

(3.1) $$|\Delta^i y(k)| \leq \frac{k^{(n-2-i)}}{(n-2-i)!}\|y\|, \quad k \in [0, N+n-i].$$

*In particular,*

(3.2) $$|y(k)| \leq \frac{(N+n)^{(n-2)}}{(n-2)!}\|y\|, \quad k \in [0, N+n].$$

**Lemma 3.2** [**15**]. *Let $y \in C$. For $0 \leq i \leq n-3$, we have*

(3.3) $$\Delta^i y(k) \geq 0, \quad k \in [0, N+n-i]$$

*and*

$$\Delta^i y(k) \ge \frac{(k-1)^{(n-2-i)}}{(n-2-i)!}\xi\|y\|, \quad k \in [1, N+n-2-i]. \tag{3.4}$$

*In particular,*

$$y(k) \ge \xi\|y\|, \quad k \in [n-1, N+n-2]. \tag{3.5}$$

*Remark* 3.1. If $y \in C$ is a solution of (1.1)–(1.4), then (3.3) and (3.5) imply that $y$ is a positive solution of (1.1)–(1.4).

To obtain a positive solution of (1.1)–(1.4), we shall seek a fixed point of the operator $\lambda S$ in the cone $C$, where $S : C \to B$ is defined by

$$\begin{aligned} Sy(k) = \sum_{l=0}^{N} g(k,l)[Q(l, y, \Delta y, \dots, \Delta^{n-2}y) \\ - P(l, y, \Delta y, \dots, \Delta^{n-1}y)], \\ k \in [0, N+n]. \end{aligned} \tag{3.6}$$

It follows that

$$\begin{aligned} \Delta^{n-2}Sy(k) = \sum_{l=0}^{N} G(k,l)[Q(l, y, \Delta y, \dots, \Delta^{n-2}y) \\ - P(l, y, \Delta y, \dots, \Delta^{n-1}y)], \\ k \in [0, N+2], \end{aligned}$$

and in view of condition (ii) we get for $k \in [0, N+2]$,

$$\begin{aligned} \sum_{l=0}^{N} G(k,l)[q(l) - p_1(l)]f(y(l)) \le \Delta^{n-2}Sy(k) \\ \le \sum_{l=0}^{N} G(k,l)[q_1(l) - p(l)]f(y(l)). \end{aligned} \tag{3.7}$$

**Theorem 3.1.** *There exists a $c > 0$ such that the interval $(0, c] \subseteq E$.*

*Proof.* Let $M > 0$ be given. Define

$$c = M\left\{\frac{L}{\rho} f\left(\frac{(N+n)^{(n-2)}}{(n-2)!}M\right) \cdot \sum_{l=0}^{N}(\beta+\alpha l)[\delta+\gamma(N-l)][q_1(l)-p(l)]\right\}^{-1}. \tag{3.8}$$

Let $y \in C_M$ and $0 < \lambda \le c$. We shall prove that $\lambda Sy \in C_M$. For this, first we shall show that $\lambda Sy \in C$. From (3.7) and (iii) we find

$$\Delta^{n-2}\lambda Sy(k) \ge \lambda \sum_{l=0}^{N} G(k,l)[q(l)-p_1(l)]f(y(l)) \ge 0, \quad k \in [0, N+2]. \tag{3.9}$$

Further, it follows from (3.7) and Lemma 2.2 that

$$\begin{aligned}\Delta^{n-2}Sy(k) &\le \sum_{l=0}^{N} G(k,l)[q_1(l)-p(l)]f(y(l))\\ &\le L\sum_{l=0}^{N} G(l,l)[q(l)-p(l)]f(y(l)),\\ &\qquad k \in [0, N+2]\end{aligned}$$

Therefore,

$$\|Sy\| \le L\sum_{l=0}^{N} G(l,l)[q_1(l)-p(l)]f(y(l)) = L\theta. \tag{3.10}$$

Now, on using (3.7), Lemma 2.1 and (3.10) we find for $k \in [1, N]$,

$$\begin{aligned}\Delta^{n-2}\lambda Sy(k) &\ge \lambda\sum_{l=0}^{N} G(k,l)[q(l)-p_1(l)]f(y(l))\\ &\ge \lambda K\sum_{l=0}^{N} G(l,l)[q(l)-p_1(l)]f(y(l))\\ &= \lambda K\Gamma \ge \lambda\xi\|Sy\| = \xi\|\lambda Sy\|.\end{aligned}$$

Hence,

$$\min_{k\in[1,N]} \Delta^{n-2}\lambda Sy(k) \geq \xi\|\lambda Sy\|. \tag{3.11}$$

It follows from (3.9) and (3.11) that $\lambda Sy \in C$.

Next, on using (3.7), Lemma 2.2, (3.2), (2.1) and (3.8) successively, we get

$$\begin{aligned}\Delta^{n-2}(\lambda Sy)(k) &\leq \lambda\sum_{l=0}^{N} G(k,l)[q_1(l)-p(l)]f(y(l))\\ &\leq L\lambda\sum_{l=0}^{N} G(l,l)[q_1(l)-p(l)]f(y(l))\\ &\leq L\lambda\sum_{l=0}^{N} G(l,l)[q_1(l)-p(l)]f\left(\frac{(N+n)^{(n-2)}}{(n-2)!}M\right)\\ &= \frac{L\lambda}{\rho}\sum_{l=0}^{N}(\beta+\alpha l)[\delta+\gamma(N-l)]\\ &\qquad\qquad \cdot[q_1(l)-p(l)]f\left(\frac{(N+n)^{(n-2)}}{(n-2)!}M\right)\\ &\leq M, \qquad k\in[0,N+2],\end{aligned}$$

which implies

$$\|\lambda Sy\| \leq M.$$

Hence, $(\lambda S)(C_M) \subseteq C_M$. Also, the standard arguments yield that $\lambda S$ is completely continuous. By the Schauder fixed point theorem, $\lambda S$ has a fixed point in $C_M$. Clearly, this fixed point is a positive solution of (1.1)–(1.4) and therefore $\lambda$ is an eigenvalue of (1.1)–(1.4). Since $0<\lambda\leq c$ is arbitrary, it follows immediately that $(0,c]\subseteq E$. □

**Theorem 3.2.** *Suppose that* $\lambda_0 \in E$*. Then, for each* $0<\lambda<\lambda_0$*,* $\lambda\in E$*.*

*Proof.* The proof requires the monotonicity and the compactness of the operator $S$ on the cone, $C$, and is similar to that of Theorem 3.2 in [**9**]. □

The following corollary is immediate from Theorem 3.2.

**Corollary 3.1.** *$E$ is an interval.*

Next we shall establish conditions under which $E$ is a bounded or an unbounded interval. For this, we need the following results.

**Theorem 3.3.** *Let $\lambda$ be an eigenvalue of (1.1)–(1.4) and $y \in C$ be a corresponding eigenfunction.*

(a) *Suppose that $\delta = \gamma = 1$ and $\beta = 0$. If*

$$\Delta^{n-1} y(0) = \nu \tag{3.12}$$

*for some $\nu > 0$, then $\lambda$ satisfies*

$$a\nu(N+2)\left[f\left(\frac{(N+n)^{(n-1)}\nu}{(n-1)!}\right)\right]^{-1} \le \lambda \le a_1\nu(N+2)[f(0)]^{-1} \tag{3.13}$$

*where*

$$a = \left\{\sum_{l=0}^{N+1}(N+1-l)[q_1(l)-p(l)]\right\}^{-1} \tag{3.14}$$

*and*

$$a_1 = \left\{\sum_{l=0}^{N+1}(N+1-l)[q(l)-p_1(l)]\right\}^{-1}. \tag{3.15}$$

(b) *Suppose that $\delta > \gamma$ and $\beta = 0$. If (3.12) holds for some $\nu > 0$, then $\lambda$ satisfies*

$$b\nu[\gamma(N+1)+\delta]\left[f\left(\frac{(N+n)^{(n-1)}\nu}{(n-1)!}\right)\right]^{-1} \le \lambda \le b_1\nu[\gamma(N+1)+\delta][f(0)]^{-1} \tag{3.16}$$

*where*

$$b = \left\{\sum_{l=0}^{N}[\gamma(N-l)+\delta][q_1(l)-p(l)]\right\}^{-1} \tag{3.17}$$

*and*

$$b_1 = \left\{ \sum_{l=0}^{N} [\gamma(N-l)+\delta][q(l)-p_1(l)] \right\}^{-1}. \tag{3.18}$$

(c) *Suppose that* $\delta = \gamma = 1$ *and* $\beta > 0$. *If*

$$\Delta^{n-2} y(0) = \mu, \qquad \Delta^{n-1} y(0) = \nu \tag{3.19}$$

*for some* $\mu, \nu > 0$ *such that* $\alpha\mu = \beta\nu$, *then* $\lambda$ *satisfies*

$$a[\mu+\nu(N+2)]\left[ f\left( \frac{(N+n)^{(n-2)}\mu}{(n-2)!} + \frac{(N+n)^{(n-1)}\nu}{(n-1)!} \right) \right]^{-1} \le \lambda \le a_1[\mu+\nu(N+2)][f(0)]^{-1} \tag{3.20}$$

*where* $a, a_1$ *are defined in* (3.14) *and* (3.15), *respectively.*

(d) *Suppose that* $\delta > \gamma$ *and* $\beta > 0$. *If* (3.19) *holds for some* $\mu, \nu > 0$ *such that* $\alpha\mu = \beta\nu$, *then* $\lambda$ *satisfies*

$$b\{\gamma[\mu+\nu(N+1)]+\delta\nu\}\left[ f\left( \frac{(N+n)^{(n-2)}\mu}{(n-2)!} + \frac{(N+n)^{(n-1)}\nu}{(n-1)!} \right) \right]^{-1} \le \lambda \le b_1\{\gamma[\mu+\nu(N+1)]+\delta\nu\}[f(0)]^{-1} \tag{3.21}$$

*where* $b, b_1$ *are defined in* (3.17) *and* (3.18), *respectively.*

*Proof.* (a) In this case the boundary conditions (1.2)–(1.4) reduce to

$$\begin{aligned} \Delta^i y(0) &= 0, \quad 0 \le i \le n-2, \\ \Delta^{n-2} y(N+2) &= 0. \end{aligned} \tag{3.22}$$

Clearly, the eigenfunction $y$ that satisfies (3.12) is the unique solution of the initial value problem

$$\Delta^n y + \lambda Q(k, y, \Delta y, \dots, \Delta^{n-2} y) = \lambda P(k, y, \Delta y, \dots, \Delta^{n-1} y), \quad k \in [0, N], \tag{3.23}$$

$$\Delta^i y(0) = 0, \quad 0 \le i \le n-2, \tag{3.24}$$
$$\Delta^{n-1} y(0) = \nu.$$

Since

$$\begin{aligned}\Delta^n y(k) &= \lambda[P(k, y, \Delta y, \dots, \Delta^{n-1} y) - Q(k, y, \Delta y, \dots, \Delta^{n-2} y)]\\ &\le \lambda[p_1(k) - q(k)] f(y(k)) \le 0,\end{aligned}$$

we have $\Delta^{n-1} y$ is nonincreasing and hence

$$\Delta^{n-1} y(k) \le \Delta^{n-1} y(0) = \nu, \quad k \in [0, N+1]. \tag{3.25}$$

Using the initial conditions (3.24) and (3.25), we find for $k \in [0, N+2]$,

$$\Delta^{n-2} y(k) = \sum_{l=0}^{k-1} \Delta^{n-1} y(l) \le \sum_{l=0}^{k-1} \nu = \nu k.$$

This in turn leads to

$$\Delta^{n-3} y(k) = \sum_{l=0}^{k-1} \Delta^{n-2} y(l) \le \sum_{l=0}^{k-1} \nu l = \nu \frac{k^{(2)}}{2!},$$
$$k \in [0, N+3].$$

Continuing the process we obtain for $k \in [0, N+n]$,

$$y(k) \le \nu \frac{k^{(n-1)}}{(n-1)!} \le \nu \frac{(N+n)^{(n-1)}}{(n-1)!}. \tag{3.26}$$

Now, in view of (ii), (i) and (3.26), we get for $k \in [0, N]$,

$$\begin{aligned}\lambda[q(k) - p_1(k)] f(0) &\le -\Delta^n y(k)\\ &\le \lambda[q_1(k) - p(k)] f\left(\nu \frac{(N+n)^{(n-1)}}{(n-1)!}\right).\end{aligned} \tag{3.27}$$

Summing (3.27) from 0 to $(k-1)$ provides

$$\phi_1(k) \le \Delta^{n-1} y(k) \le \phi_2(k), \quad k \in [0, N+1] \tag{3.28}$$

where

$$\phi_1(k) = \nu - \lambda f\left(\nu \frac{(N+n)^{(n-1)}}{(n-1)!}\right) \sum_{l=0}^{k-1} [q_1(l) - p(l)]$$

and

$$\phi_2(k) = \nu - \lambda f(0) \sum_{l=0}^{k-1} [q(l) - p_1(l)].$$

Again, we sum (3.28) from 0 to $(k-1)$, and subsequently change the order of summation to obtain

$$\phi_3(k) \leq \Delta^{n-2} y(k) \leq \phi_4(k), \quad k \in [0, N+2] \tag{3.29}$$

where

$$\phi_3(k) = \nu k - \lambda f\left(\nu \frac{(N+n)^{(n-1)}}{(n-1)!}\right) \sum_{l=0}^{k-1} (k-1-l)[q_1(l) - p(l)]$$

and

$$\phi_4(k) = \nu k - \lambda f(0) \sum_{l=0}^{k-1} (k-1-l)[q(l) - p_1(l)].$$

Since the solution $y$ of (3.23) and (3.24) is an eigenfunction corresponding to $\lambda$, it satisfies the boundary condition $\Delta^{n-2} y(N+2) = 0$, see (3.22). Therefore, in inequality (3.29) we must have

$$\phi_3(N+2) \leq 0 \quad \text{and} \quad \phi_4(N+2) \geq 0,$$

or equivalently

$$\lambda \geq a\nu(N+2)\left[f\left(\frac{(N+n)^{(n-1)}\nu}{(n-1)!}\right)\right]^{-1} \tag{3.30}$$

and

$$\lambda \leq a_1 \nu(N+2)[f(0)]^{-1}. \tag{3.31}$$

The inequality (3.13) follows immediately.

(b) Here the boundary conditions (1.2)–(1.4) reduce to

$$\begin{aligned}\Delta^i y(0) &= 0, \quad 0 \le i \le n-2,\\ \gamma\Delta^{n-2}y(N+1) + \delta\Delta^{n-1}y(N+1) &= 0.\end{aligned} \tag{3.32}$$

It is obvious that the eigenfunction $y$ that satisfies (3.12) is the unique solution of the initial value problem (3.23), (3.24). As in case (a) we get the inequalities (3.28) and (3.29). It follows that

$$\begin{aligned}\gamma\phi_3(k) + \delta\phi_1(k) &\le \gamma\Delta^{n-2}y(k) + \delta\Delta^{n-1}y(k)\\ &\le \gamma\phi_4(k) + \delta\phi_2(k).\end{aligned} \tag{3.33}$$

Since $y$ satisfies $\gamma\Delta^{n-2}y(N+1) + \delta\Delta^{n-1}y(N+1) = 0$ (from (3.32)), in inequality (3.33) it is necessary that

$$\gamma\phi_3(N+1) + \delta\phi_1(N+1) \le 0$$

and

$$\gamma\phi_4(N+1) + \delta\phi_2(N+1) \ge 0$$

which respectively lead to

$$\lambda \ge b\nu[\gamma(N+1) + \delta]\left[f\left(\frac{(N+n)^{(n-1)}\nu}{(n-1)!}\right)\right]^{-1} \tag{3.34}$$

and

$$\lambda \le b_1\nu[\gamma(N+1) + \delta][f(0)]^{-1}. \tag{3.35}$$

Coupling (3.34) and (3.35), we get (3.16).

(c) In this case the boundary conditions (1.2)–(1.4) become

$$\begin{aligned}\Delta^i y(0) &= 0, \quad 0 \le i \le n-3,\\ \Delta^{n-2}y(N+2) &= 0,\\ \alpha\Delta^{n-2}y(0) - \beta\Delta^{n-1}y(0) &= 0.\end{aligned} \tag{3.36}$$

Clearly, the eigenfunction $y$ that satisfies (3.19) is the unique solution of the difference equation (3.23), together with the initial conditions

$$\begin{aligned} \Delta^i y(0) &= 0, \quad 0 \le i \le n-3, \\ \Delta^{n-2} y(0) &= \mu, \qquad \Delta^{n-1} y(0) = \nu. \end{aligned} \tag{3.37}$$

As in case (a), we see that $\Delta^{n-1} y$ is nonincreasing and hence (3.25) holds. In view of the initial conditions (3.37) and (3.25), we find

$$\begin{aligned} \Delta^{n-2} y(k) &= \mu + \sum_{l=0}^{k-1} \Delta^{n-1} y(l) \\ &\le \mu + \sum_{l=0}^{k-1} \nu = \mu + \nu k, \\ & k \in [0, N+2]. \end{aligned}$$

It follows that, for $k \in [0, N+3]$,

$$\Delta^{n-3} y(k) = \sum_{l=0}^{k-1} \Delta^{n-2} y(l) \le \sum_{l=0}^{k-1} (\mu + \nu l) = \mu k + \nu \frac{k^{(2)}}{2!}.$$

Continuing the process, we obtain for $k \in [0, N+n]$,

$$\begin{aligned} y(k) &\le \mu \frac{k^{(n-2)}}{(n-2)!} + \nu \frac{k^{(n-1)}}{(n-1)!} \\ &\le \mu \frac{(N+n)^{(n-2)}}{(n-2)!} + \nu \frac{(N+n)^{(n-1)}}{(n-1)!}. \end{aligned} \tag{3.38}$$

Now it follows from (ii), (i) and (3.38) that, for $k \in [0, N]$,

$$\begin{aligned} \lambda[q(k) - p_1(k)] f(0) &\le -\Delta^n y(k) \\ &\le \lambda[q_1(k) - p(k)] \\ &\quad \cdot f\left(\mu \frac{(N+n)^{(n-2)}}{(n-2)!} + \nu \frac{(N+n)^{(n-1)}}{(n-1)!}\right). \end{aligned} \tag{3.39}$$

Summing (3.39) from 0 to $(k-1)$ gives

$$\phi_5(k) \le \Delta^{n-1} y(k) \le \phi_6(k), \quad k \in [0, N+1] \tag{3.40}$$

where

$$\phi_5(k) = \nu - \lambda f\left(\mu \frac{(N+n)^{(n-2)}}{(n-2)!} + \nu \frac{(N+n)^{(n-1)}}{(n-1)!}\right) \cdot \sum_{l=0}^{k-1} [q_1(l) - p(l)]$$

and

$$\phi_6(k) = \nu - \lambda f(0) \sum_{l=0}^{k-1} [q(l) - p_1(l)].$$

Once again, we sum (3.40) from 0 to $(k-1)$ to get

$$\phi_7(k) \le \Delta^{n-2} y(k) \le \phi_8(k), \quad k \in [0, N+2] \tag{3.41}$$

where

$$\phi_7(k) = \mu + \nu k - \lambda f\left(\mu \frac{(N+n)^{(n-2)}}{(n-2)!} + \nu \frac{(N+n)^{(n-1)}}{(n-1)!}\right) \cdot \sum_{l=0}^{k-1} (k-1-l)[q_1(l) - p(l)]$$

and

$$\phi_8(k) = \mu + \nu k - \lambda f(0) \sum_{l=0}^{k-1} (k-1-l)[q(l) - p_1(l)].$$

Since $y$ satisfies the boundary condition $\Delta^{n-2} y(N+2) = 0$ (see (3.36)), in inequality (3.41) we must have

$$\phi_7(N+2) \le 0 \quad \text{and} \quad \phi_8(N+2) \ge 0$$

or equivalently

$$\lambda \ge a[\mu + \nu(N+2)] \left[ f\left( \frac{(N+n)^{(n-2)}\mu}{(n-2)!} + \frac{(N+n)^{(n-1)}\nu}{(n-1)!} \right) \right]^{-1} \tag{3.42}$$

and

$$\lambda \le a_1[\mu + \nu(N+2)][f(0)]^{-1}. \tag{3.43}$$

The inequality (3.20) follows immediately.

(d) It is obvious that the eigenfunction $y$ that satisfies (3.19) is the unique solution of the initial value problem (3.23), (3.37). As in case (c) we get the inequalities (3.40) and (3.41) which lead to

$$\begin{aligned}\gamma\phi_7(k)+\delta\phi_5(k) &\le \gamma\Delta^{n-2}y(k)+\delta\Delta^{n-1}y(k)\\ &\le \gamma\phi_8(k)+\delta\phi_6(k).\end{aligned} \tag{3.44}$$

Since $y$ satisfies the boundary condition $\gamma\Delta^{n-2}y(N+1)+\delta\Delta^{n-1}y(N+1)=0$, in inequality (3.44) it is necessary that

$$\gamma\phi_7(N+1)+\delta\phi_5(N+1)\le 0$$

and

$$\gamma\phi_8(N+1)+\delta\phi_6(N+1)\ge 0$$

which reduce to

$$\begin{aligned}\lambda \ge b\{\gamma[\mu+\nu(N+1)]+\delta\nu\}\\ \cdot\left[f\left(\frac{(N+n)^{(n-2)}\mu}{(n-2)!}+\frac{(N+n)^{(n-1)}\nu}{(n-1)!}\right)\right]^{-1}\end{aligned} \tag{3.45}$$

and

$$\lambda\le b_1\{\gamma[\mu+\nu(N+1)]+\delta\nu\}[f(0)]^{-1}. \tag{3.46}$$

Combining (3.45) and (3.46), we get (3.21). □

**Theorem 3.4.** *Let $\lambda$ be an eigenvalue of* (1.1)–(1.4) *and $y\in C$ be a corresponding eigenfunction. Further, let $\eta=\|y\|$. Then*

$$\begin{aligned}\lambda \ge \frac{\eta\rho}{L}\Bigg\{f\left(\frac{(N+n)^{(n-2)}\eta}{(n-2)!}\right)\\ \cdot\sum_{l=0}^{N}(\beta+\alpha l)[\delta+\gamma(N-l)][q_1(l)-p(l)]\Bigg\}^{-1}.\end{aligned} \tag{3.47}$$

*Also, there exists a* $c > 0$ *such that*

$$\lambda \leq \frac{\eta\rho}{f(c\eta)} \left\{ \sum_{l \in J} (\beta + \alpha l)[\delta + \gamma(N - l)][q(l) - p_1(l)] \right\}^{-1} \tag{3.48}$$

*where*

$$J = \begin{cases} [1, [(N+1)/2]] & n = 2, \\ [n-1, N] & n \geq 3. \end{cases} \tag{3.49}$$

*Proof.* We observe that $\Delta^n y$ is nonpositive and hence $\Delta^{n-2} y$ is concave on $[0, N+2]$. This, together with the fact that $\Delta^{n-2} y$ is nonnegative, implies the existence of a unique $k_0 \in [1, N+1]$ such that

$$\eta = \|y\| = \Delta^{n-2} y(k_0).$$

To prove that (3.47) holds, we use (3.7), Lemma 2.2, (3.2) and (2.1) successively to get

$$\begin{aligned} \eta = \Delta^{n-2} y(k_0) &= \Delta^{n-2} \lambda S y(k_0) \\ &\leq \lambda \sum_{l=0}^{N} G(k_0, l)[q_1(l) - p(l)] f(y(l)) \\ &\leq \lambda L \sum_{l=0}^{N} G(l, l)[q_1(l) - p(l)] f(y(l)) \\ &\leq \lambda L \sum_{l=0}^{N} G(l, l)[q_1(l) - p(l)] f\left( \frac{(N+n)^{(n-2)} \eta}{(n-2)!} \right) \\ &= \frac{\lambda L}{\rho} f\left( \frac{(N+n)^{(n-2)} \eta}{(n-2)!} \right) \\ &\quad \cdot \sum_{l=0}^{N} (\beta + \alpha l)[\delta + \gamma(N - l)][q_1(l) - p(l)]. \end{aligned}$$

The inequality (3.47) follows immediately.

Next, to prove (3.48) we shall consider four cases.

*Case* 1. $\delta=\gamma=1$, $\beta=0$. Here $\Delta^{n-2}y(0)=\Delta^{n-2}y(N+2)=0$. By the concavity of $\Delta^{n-2}y$, we find

$$\tag{3.50} \begin{aligned} \Delta^{n-2}y(k) &\ge \begin{cases} (\eta/k_0)k & k\in[0,k_0],\\ \eta/(N+2-k_0)(N+2-k) & k\in[k_0,N+2] \end{cases}\\ &\ge \frac{\eta}{(N+2)^2}k(N+2-k), \quad k\in[0,N+2]. \end{aligned}$$

Thus, on using (1.2) and (3.50) we get for $k\in[0,N+3]$,

$$\begin{aligned} \Delta^{n-3}y(k) &= \sum_{l=0}^{k-1}\Delta^{n-2}y(k)\\ &\ge \sum_{l=0}^{k-1}\frac{\eta}{(N+2)^2}l(N+2-l)\\ &= \frac{\eta}{(N+2)^2}\left[(N+1)\frac{k^{(2)}}{2}-\frac{k^{(3)}}{3}\right]. \end{aligned}$$

Continuing the summation process, we obtain

$$\tag{3.51} y(k)\ge\frac{\eta}{(N+2)^2}\psi(k), \quad k\in[0,N+n],$$

where

$$\psi(k)=(N+1)\frac{k^{(n-1)}}{(n-1)!}-2\frac{k^{(n)}}{n!}.$$

We note that

$$\Delta\psi(k)=\frac{k^{(n-2)}}{(n-2)!}\left[N+1-\frac{2(k-n+2)}{n-1}\right]$$

is nonnegative for $k\in I$ where

$$I=\begin{cases}[0,[(N+1)/2]] & n=2,\\ [0,N+2] & n\ge 3.\end{cases}$$

Hence, in particular, $\psi(k)$ is nondecreasing for $k\in J\subset I$, see (3.49). Consequently, for $k\in J$,

$$\tag{3.52} \psi(k)\ge\left.\begin{cases}\psi(1) & n=2\\ \psi(n-1) & n\ge 3\end{cases}\right\}=N+1.$$

It follows from (3.51) and (3.52) that

$$y(k) \geq c\eta, \quad k \in J \tag{3.53}$$

where

$$c = \frac{N+1}{(N+2)^2} > 0. \tag{3.54}$$

Now, in view of (3.7), (3.53) and (2.1), we find

$$\begin{aligned}\eta &\geq \Delta^{n-2}y(n-1) = \Delta^{n-2}\lambda S y(n-1)\\ &\geq \lambda \sum_{l=0}^{N} G(n-1,l)[q(l)-p_1(l)]f(y(l))\\ &\geq \lambda \sum_{l\in J} G(n-1,l)[q(l)-p_1(l)]f(y(l))\\ &\geq \lambda \sum_{l\in J} G(n-1,l)[q(l)-p_1(l)]f(c\eta)\\ &= \frac{\lambda}{\rho} f(c\eta) \sum_{l\in J} (\beta+\alpha l)[\delta+\gamma(N-l)][q(l)-p_1(l)]\end{aligned}$$

from which (3.48) follows immediately.

*Case* 2. $\delta > \gamma$, $\beta = 0$. In this case $\Delta^{n-2}y(0) = 0$, $\Delta^{n-2}y(N+2) \neq 0$. Hence, for $k \in [0, N+2]$,

$$\begin{aligned}\Delta^{n-2}y(k) &\geq \frac{\Delta^{n-2}y(N+2)}{N+2}k\\ &\geq \frac{\Delta^{n-2}y(N+2)}{(N+2)^2}k(N+2-k).\end{aligned} \tag{3.55}$$

Using a similar technique as in Case 1, it follows from (3.55) and successive summations that

$$y(k) \geq \frac{\Delta^{n-2}y(N+2)}{(N+2)^2}\psi(k), \quad k \in [0, N+n]. \tag{3.56}$$

From (3.56) and (3.52) we get

$$y(k) \geq \frac{\Delta^{n-2} y(N+2)}{(N+2)^2}(N+1) = c\eta, \quad k \in J, \tag{3.57}$$

where

$$c = \frac{\Delta^{n-2} y(N+2)}{\eta(N+2)^2}(N+1) > 0. \tag{3.58}$$

The rest of the proof is similar to that of Case 1.

*Case* 3. $\delta = \gamma = 1$, $\beta > 0$. In this case $\Delta^{n-2}y(0) \neq 0$, $\Delta^{n-2}y(N+2) = 0$. Thus, for $k \in [0, N+2]$,

$$\begin{aligned} \Delta^{n-2} y(k) &\geq \frac{\Delta^{n-2} y(0)}{N+2}(N+2-k) \\ &\geq \frac{\Delta^{n-2} y(0)}{(N+2)^2} k(N+2-k). \end{aligned} \tag{3.59}$$

Again, as in Case 1 it follows from (3.59) and successive summations that

$$y(k) \geq \frac{\Delta^{n-2} y(0)}{(N+2)^2}\psi(k), \quad k \in [0, N+n]. \tag{3.60}$$

From (3.60) and (3.52) we find

$$y(k) \geq \frac{\Delta^{n-2} y(0)}{(N+2)^2}(N+1) = c\eta, \quad k \in J \tag{3.61}$$

where

$$c = \frac{\Delta^{n-2} y(0)}{\eta(N+2)^2}(N+1) > 0. \tag{3.62}$$

The rest of the proof is similar to that of Case 1.

*Case* 4. $\delta > \gamma$, $\beta > 0$. Here $\Delta^{n-2}y(0) \neq 0$, $\Delta^{n-2}y(N+2) \neq 0$. Let

$$m = \min\{\Delta^{n-2} y(0), \Delta^{n-2} y(N+2)\}.$$

Then

$$\begin{aligned} \Delta^{n-2}y(k) &\geq m \\ &\geq \frac{m}{(N+2)^2}k(N+2-k), \\ &k \in [0, N+2]. \end{aligned} \tag{3.63}$$

Once again it follows from (3.63) and successive summations that

$$y(k) \geq \frac{m}{(N+2)^2}\psi(k), \quad k \in [0, N+n]. \tag{3.64}$$

From (3.64) and (3.52) we have

$$y(k) \geq \frac{m}{(N+2)^2}(N+1) = c\eta, \quad k \in J \tag{3.65}$$

where

$$c = \frac{m}{\eta(N+2)^2}(N+1) > 0. \tag{3.66}$$

The rest of the proof is similar to that of Case 1.

This completes the proof of the theorem. □

**Theorem 3.5.** *Let*

$$F_B = \left\{ f \;\middle|\; \frac{u}{f(u)} \text{ is bounded for } u \in [0, \infty) \right\},$$
$$F_0 = \left\{ f \;\middle|\; \lim_{u\to\infty} \frac{u}{f(u)} = 0 \right\},$$
$$F_\infty = \left\{ f \;\middle|\; \lim_{u\to\infty} \frac{u}{f(u)} = \infty \right\}.$$

(a) *If $f \in F_B$, then $E = (0, c)$ or $(0, c]$ for some $c \in (0, \infty)$.*

(b) *If $f \in F_0$, then $E = (0, c]$ for some $c \in (0, \infty)$.*

(c) *If $f \in F_\infty$, then $E = (0, \infty)$.*

*Proof.* (a) This is immediate from (3.48).

(b) Since $F_0 \subseteq F_B$, it follows from case (a) that $E = (0, c)$ or $(0, c]$ for some $c \in (0, \infty)$. In particular,

$$c = \sup E. \tag{3.67}$$

Let $\{\lambda_l\}_{l=1}^{\infty}$ be a monotonically increasing sequence in $E$ which converges to $c$, and let $\{y_l\}_{l=1}^{\infty}$ in $C$ be a corresponding sequence of eigenfunctions. Further, let $\eta_l = \|y_l\|$. Then (3.48) implies that no subsequence of $\{\eta_l\}_{l=1}^{\infty}$ can diverge to infinity. Thus, there exists $M > 0$ such that $\eta_l \le M$ for all $l$. In view of (3.2), we find that $y_l$ is uniformly bounded. Hence, there is a subsequence of $\{y_l\}$, relabelled as the original sequence, which converges uniformly to some $y \in C$.

Noting that $\lambda_l S y_l = y_l$, we have

$$cSy_l = \frac{c}{\lambda_l} y_l. \tag{3.68}$$

Since $\{cSy_l\}_{l=1}^{\infty}$ is relatively compact, $y_l$ converges to $y$ and $\lambda_l$ converges to $c$, it follows from (3.68) that

$$cSy = y,$$

i.e., $c \in E$. This completes the proof for case (b).

(c) This follows from Corollary 3.1 and (3.47). □

**Example 3.1.** Consider the boundary value problem

$$\begin{gathered}
\Delta^2 y + \lambda \left\{ \phi(k, y) + \frac{2}{[k(13-k)+3]^r} \right\} (y+2)^r \\
= \lambda \phi(k, y)(y+2)^r, \quad k \in [0, 11], \\
12y(0) - \Delta y(0) = 0, \\
12y(12) + 13\Delta y(12) = 0
\end{gathered}$$

where $\lambda > 0$, $r \ge 0$ and $\phi(k, y)$ is any function of $k$ and $y$.

Taking $f(y) = (y+2)^r$, we find

$$\frac{Q(k, y)}{f(y)} = \phi(k, y) + \frac{2}{[k(13-k)+3]^r}$$

and

$$\frac{P(k,y,\Delta y)}{f(y)} = \phi(k,y).$$

Hence, we may choose

$$q(k) = \phi(k,y) + \frac{1}{[k(13-k)+3]^r},$$
$$q_1(k) = \phi(k,y) + \frac{2}{[k(13-k)+3]^r}$$

and

$$p(k) = p_1(k) = \phi(k,y).$$

*Case* 1. $0 \le r < 1$. Since $f \in F_\infty$, by Theorem 3.5(c) the set $E = (0,\infty)$. For example, when $\lambda = 1$, the boundary value problem has a positive solution given by $y(k) = k(13-k)+1$.

*Case* 2. $r = 1$. Since $f \in F_B$, by Theorem 3.5(a) the set $E$ is an open or half-closed interval. Further, we note from Case 1 and Theorem 3.2 that $E$ contains the interval $(0,1]$.

*Case* 3. $r > 1$. Since $f \in F_0$, by Theorem 3.5(b) the set $E$ is a half-closed interval. Again, it is noted that $(0,1] \subseteq E$.

**Example 3.2.** Consider the boundary value problem

$$\Delta^3 y + \lambda\left\{\phi(k,y,\Delta y) + \frac{24k}{[k(5000-(k-1)(k-6)(k+1))+1]^r}\right\}(y+1)^r$$
$$= \lambda\phi(k,y,\Delta y)(y+1)^r, \quad k \in [0,10],$$
$$y(0) = 0,$$
$$3\Delta y(0) - 625\Delta^2 y(0) = 0,$$
$$162\Delta y(11) + 163\Delta^2 y(11) = 0,$$

where $\lambda > 0$, $r \ge 0$ and $\phi(k,y,\Delta y)$ is any function of $k$, $y$ and $\Delta y$.

Taking $f(y) = (y+1)^r$, we find

$$\frac{Q(k,y,\Delta y)}{f(y)} = \phi(k,y,\Delta y) + \frac{24k}{[k(5000-(k-1)(k-6)(k+1))+1]^r}$$

and

$$\frac{P(k, y, \Delta y, \Delta^2 y)}{f(y)} = \phi(k, y, \Delta y).$$

Hence, we may take

$$q(k) = \phi(k, y, \Delta y) + \frac{k}{[k(5000 - (k-1)(k-6)(k+1)) + 1]^r},$$
$$q_1(k) = \phi(k, y, \Delta y) + \frac{24k}{[k(5000 - (k-1)(k-6)(k+1)) + 1]^r}$$

and

$$p(k) = p_1(k) = \phi(k, y, \Delta y).$$

We note that when $\lambda = 1$ the boundary value problem has a positive solution given by $y(k) = k[5000 - (k-1)(k-6)(k+1)]$. The three cases considered in Example 3.1 also apply to this problem.

## REFERENCES

**1.** R.P. Agarwal, *Difference equations and inequalities*, Marcel Dekker, New York, 1992.

**2.** R.P. Agarwal and P.J.Y. Wong, *Existence of solutions for singular boundary value problems for higher order differential equations*, Rend. Sem. Mat. Fis. Milano **55** (1995), 249–264.

**3.** N.P. Cac, A.M. Fink and J.A. Gatica, *Nonnegative solutions of quasilinear elliptic boundary value problems with nonnegative coefficients*, preprint.

**4.** C.J. Chyan and J. Henderson, *Positive solutions for singular higher order nonlinear equations*, Differential Equations Dyn. Sys. **2** (1994), 153–160.

**5.** P.W. Eloe, J. Henderson and P.J.Y. Wong, *Positive solutions for two-point boundary value problems*, in *Dynamic systems and applications*, Vol. 2 ( G.S. Ladde and M. Sambandham, eds.), Dynamic Publisher, Atlanta, 1996, 135–144.

**6.** J. Henderson, *Singular boundary value problems for difference equations*, Dynamic Systems Appl. **1** (1992), 271–282.

**7.** ———, *Singular boundary value problems for higher order difference equations*, in *Proceedings of the first world congress on nonlinear analysts* (V. Lakshmikantham, ed.), Walter de Gruyter and Co., Berlin, 1996, 1139–1150.

**8.** A.M. Fink, *The radial Laplacian Gel'fand problem, delay and differential equations*, World Scientific Publishing, River Edge, NJ, 1992.

**9.** A.M. Fink, J.A. Gatica and G.E. Hernandez, *Eigenvalues of generalized Gel'fand models*, Nonlinear Anal. **20** (1993), 1453–1468.

**10.** A.M. Fink and J.A. Gatica, *Positive solutions of second order systems of boundary value problems*, J. Math. Anal. Appl. **180** (1993), 93–108.

**11.** D.J. Joseph and E.M. Spanow, *Nonlinear diffusion induced by nonlinear sources*, Quart. Appl. Math. **2** (1970–71), 327–342.

**12.** P.L. Lions, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Rev. **24** (1982), 441–467.

**13.** H. Wang, *On the existence of positive solutions for semilinear elliptic equations in the annulus*, J. Differential Equations **109** (1994), 1–7.

**14.** P.J.Y. Wong and R.P. Agarwal, *On the existence of solutions of singular boundary value problems for higher order difference equations*, Nonlinear Anal. **28** (1997), 277–287.

**15.** ———, *On the existence of positive solutions of higher order difference equations*, Topological Methods in Nonlinear Analysis, to appear.

Division of Mathematics, Nanyang Technological University, 469, Bukit Timah Road, Singapore 259756

Department of Mathematics, National University of Singapore, 10, Kent Ridge Crescent, Singapore 119260