ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 2, Summer 1998

RANDOM FOURIER SERIES AND ABSOLUTE SUMMABILITY

NASSER TOWGHI

ABSTRACT. In this article it is shown that many classical results concerning absolute summability of Fourier series can be obtained by random methods. An estimate on the modulus of continuity of random Fourier series is obtained. This estimate is then applied to obtain some new sufficient conditions for the absolute summability of the Fourier series.

1. Introduction. A classical result of Bernstein [**1**] states that if f is a periodic Lipschitz function of order greater than $1/2$, then its Fourier series converges absolutely. Later Hyslop [**3**] extended Bernstein's theorem by using absolute Cesaro summability, see Section 3 for the definition of absolute summability. Hyslop proved that if f is a Lipschitz function of order α and $0 < \alpha \leq 1/2$, then the Fourier series of f is summable $|C, \beta|$ whenever $\alpha + \beta > 1/2$. Hyslop's result was extended by McFadden [**9**] and Lal [**7**] to the case of summability for certain types of Nörlund means, see Section 3. There are a considerable number of sufficient conditions for the absolute summability and absolute convergence of Fourier series, see for example [**10**] and the references therein. Our contribution is that most of the sufficiency conditions for absolute summability are obtained by random methods using Khinchin's inequality and the modified versions of the three above-mentioned results, see for example Theorem 3.7 and Theorem 3.8. In some instances we obtain modest extensions of the classical results.

2. Random Fourier series. We consider the randomization of the Fourier series of f. Let

Received by the editors on January 19, 1996.
1991 AMS *Mathematic Subject Classification.* Primary 42A28, 42A61.
Key words and phrases. Absolute summability, modulus of continuity, p-variation, Rademacher functions, subnormal random variables, Cesaro summability of order α, absolute Nörlund summability.

$$c_0 + \sum_{n=1}^{\infty} c_n(f)\cos(nt + \phi_n) \tag{1}$$

be the Fourier series of f. We randomize the Fourier series of f, that is, let

$$F(s,t) = \sum_{n=1}^{\infty} c_n(f)\cos(nt + \phi_n)X_n(s), \tag{2}$$

where X_n is a subnormal sequence of random variables defined on some probability space (Ω, P). Henceforth, F will always represent a randomization of the function f. For the purposes of proving absolute summability results, we will often let $X_n(s) = r_n(s)$ where r_n denotes the nth Rademacher function defined on the unit interval $[0,1]$, that is, $r_n(s) = 1 - 2\varepsilon_n$ where $s = \sum_{k=1}^{\infty} \varepsilon_k/(2^k)$ is the binary expansion of $s \in [0,1]$. In other words, $c_n(F(s,\cdot)) = \pm c_n(f)$. We introduce some notation. Recall that the modulus of continuity of f is the function $\omega(f,h) = \omega(h) = \sup_{0<|x-y|\le h}\{|f(x) - f(y)|\}$, and the p-modulus of continuity of f is the function

$$\omega_p(f,h) = \omega_p(h) = \sup_{0<|x-y|\le h} \left(\int_{-\pi}^{\pi} |f(x) - f(y)|^p \right)^{1/p}.$$

For $\alpha > 0$, let $\mathrm{Lip}\,(\alpha, p, [-\pi,\pi]) = \mathrm{Lip}\,(\alpha, p) = \{f : \omega_p(f,h) \le Ch^\alpha\}$, and $C_\alpha([-\pi,\pi]) = C_\alpha = \{f : \omega(f,h) \le Ch^\alpha\}$. For $p \ge 1$, let

$$V_p(f,[-\pi,\pi]) = V_p(f) = \left[\sup_\tau \left\{ \sum_{j=1}^{n} |f(t_j) - f(t_{(j-1)})|^p \right\} \right]^{1/p},$$

where τ denotes the partition $\{-\pi = t_0 \le t_1 \le \cdots \le t_n = \pi\}$ of $[-\pi,\pi]$. Let

$$BV_p[-\pi,\pi] = BV_p = \{f : V_p(f) < \infty\}.$$

We have the following relation: $C_{1/p} \subset BV_p \subset \mathrm{Lip}\ (1/p)$.

We need the following theorem.

Theorem 2.1 [**5**, p. 86]. *Let f be a measurable function and its Fourier series be given by* (1). *For each positive integer j, let $s_j = (\sum_{n=2^{j-1}}^{2^j} |c_j|^2)^{1/2}$. Suppose that $s_j = O(2^{-\beta j} j^{-\gamma})$, where β and γ satisfy either $\beta = 0$ and $\gamma > 1$ or $\beta > 0$. Then, almost surely, as $h \downarrow 0$,*

(a) $\beta = 0$, $\gamma > 1$ *implies* $\omega(F(s, \cdot), h) = O(|\log h|^{1-\gamma})$.

(b) $0 < \beta < 1$ *implies* $\omega(F(s, \cdot), h) = O(h^\beta |\log h|^{1/2-\gamma})$.

(c) $\beta = 1$, $\gamma < 1/2$ *implies* $\omega(F(s, \cdot), h) = O(h |\log h|^{1-\gamma})$.

(d) $\beta = 1$, $1/2 < \gamma \leq 1$ *implies* $\omega(F(s, \cdot), h) = O(h |\log h|^{1/2})$.

(e) $\beta = 1, 1 < \gamma$ *or* $\beta > 1$ *implies* $\omega(F(s, \cdot), h) = O(h)$.

By slightly changing the proof of Theorem 2.1, one obtains the following result.

Theorem 2.2. *If $\sum_{n=0}^{\infty} |c_n(f)|^2 n^\beta (\log n)^\gamma < \infty$, then almost surely as $h \downarrow 0$,*

(a) $\beta = 0$, $\gamma > 1$ *implies* $\omega(F(s, \cdot), h) = O(|\log h|^{(1-\gamma)/2})$.

(b) $0 < \beta < 2$ *implies* $\omega(F(s, \cdot), h) = O(h^{\beta/2} |\log h|^{(1-\gamma)/2})$.

(c) $\beta = 2$, $\gamma < 1$ *implies* $\omega(F(s, \cdot), h) = O(h |\log h|^{(1-\gamma)/2})$.

(d) $\beta = 2$, $1 < \gamma \leq 2$ *implies* $\omega(F(s, \cdot), h) = O(h |\log h|^{1/2})$.

(e) $\beta = 2, 2 < \gamma$ *or* $\beta > 1$ *implies* $\omega(F(s, \cdot), h) = O(h)$.

Lemma 2.3. *Let ϕ be a positive, nonincreasing function, $r > 0$ and $\sum_{n=1}^{\infty} |\omega_p(f, 1/n)|^r \phi(n) < \infty$.*

(A) *Suppose that one of the following three conditions is satisfied*:

(i) $p \geq 2$,

(ii) $\sum_{n=2^{j-1}}^{2^j - 1} |c_n(f)|^p \leq C$,

(iii) $\alpha p > 1$, (*here C in* (ii) *is a fixed constant depending on f but not on j*),

then $s_j = O([2^{-j}/\phi(2^j)]^{1/r})$.

(B) *If $p < 2$ and $f \in L_q$ for some $2 \leq q < \infty$, then* $s_j =$

$O([2^{-j}/\phi(2^j)]^{p(q-2)/(2r(q-p))}$. *If* $q=\infty$, *then* $s_j = O([2^{-j}/\phi(2^j)]^{p/(2r)}$.

(C) *If* $p < 2$ *and* $\sum_{n=1}^{\infty} |\omega_p(f,1/n)|^r \phi(n) < \infty$, *then* $s_j = O(2^{j(1/p-1/2)}[2^{-j}/\phi(2^j)]^{1/r})$.

Proof. Suppose $p \geq 2$. By a familiar argument, see, e.g., [**6**, p. 32], we have

$$s_j = \left(\sum_{n=2^{j-1}}^{2^j} |c_n|^2 \right)^{1/2} \leq C\omega_p(f, 2^{-j}).$$

If $\sum_{n=2^{j-1}}^{2^j-1} |c_n(f)|^p \leq C$, by Hölder's inequality,

$$s_j \leq C \left[\sum_{2^{j-1}}^{2^j-1} |c_n(f)|^{p'} \right]^{1/p'}.$$

By an argument similar to the one given in [**6**, p. 32], it can be shown that

$$\sum_{n=2^{j-1}}^{2^j} |c_n(f)|^{p'} \leq C \sum_{n=2^{j-1}}^{2^j} |c_{n,j}(f)|^{p'},$$

where $c_{n,j}(f)$ is the nth Fourier coefficient of the function $t \to f(t + 2^{-j}) - f(t)$. Since $p' > 2$, an application of the Hausdorff-Young inequality gives us

$$\begin{aligned} \left(\sum_{n=2^{j-1}}^{2^j} |c_{n,j}(f)|^{p'} \right)^{1/p'} &\leq \left[C \int_{-\pi}^{\pi} |f(x+2^{-j}) - f(x)|^p \, dx \right]^{1/p} \\ &\leq C\omega_p(f, 2^{-j}). \end{aligned}$$

Thus $s_j \leq C\omega_p(f, 2^{-j})$. If $\alpha p > 1$, then

$$\begin{aligned} \sum_{n=2^{j-1}}^{2^j-1} |c_n(f)|^p &= \sum_{n=2^{j-1}}^{2^j-1} O(\omega_1(f,1/n)^p) \\ &\leq C \sum_{n=2^{j-1}}^{2^j-1} (\omega_p(f,1/n)^p) < \infty. \end{aligned}$$

That is, condition (iii) implies (ii). Therefore, $s_j \le C\omega_p(f, 2^{-j})$. Consequently, if any one of the conditions (i) through (iii) is satisfied, we have

$$\begin{aligned}\phi(2^j)s_j^r &\le C\phi(2^j)|\omega_p(f, 2^{-j})|^r \\ &\le C2^{-j}\sum_{n=2^{j-1}}^{2^j}\left|\omega_p\left(f, \frac{1}{n}\right)\right|^r \phi(n) \\ &\le C2^{-j}\sum_{n=1}^{\infty}\omega_p\left(f, \frac{1}{n}\right)\phi(n) \le C2^{-j}.\end{aligned}$$

Therefore, $s_j = O([2^{-j}/\phi(2^j)]^{1/r})$. This proves (A). Suppose $p < 2$, $2 \le q < \infty$ and $f \in L_q$. Let $\theta = p(q-2)/(q-p)$. Since $p < 2$, $\theta < p$. Applying Hölder's inequality with exponents p/θ and $p/(p-\theta)$, we obtain

$$\begin{aligned}&\left[\int_{-\pi}^{\pi}|f(x+2^{-j}) - f(x)|^2\,dx\right]^{1/2} \\ &\quad = \left[\int_{-\pi}^{\pi}|f(x+2^{-j}) - f(x)|^{\theta}|f(x+2^{-j}) - f(x)|^{2-\theta}\,dx\right]^{1/2} \\ &\quad \le C\left[\int_{-\pi}^{\pi}|f(x+2^{-j}) - f(x)|^p\,dx\right]^{\theta/(2p)}\|f\|_q^{q(\theta-p)/(2p)} \\ &\quad \le C|\omega_p(f, 2^{-j})|^{\theta/2} \\ &\quad = C|\omega_p(f, 2^{-j})|^{p(q-2)/(2(q-p))}.\end{aligned}$$

Arguing as in part (A), we obtain $s_j = O([2^{-j}/\phi(2^j)]^{p(q-2)/(2r(q-p))}$. The case $q = \infty$ is proved in a similar manner. To prove (C), apply Hölder's inequality, with exponents $p'/2$ and $p'/p\prime - 2$ to obtain

$$\begin{aligned}s_j &\le C2^{j(1/2-1/p')}\left[\sum_{n=2^{j-1}}^{2^j}|c_n(f)|^{p'}\right]^{1/p'} \\ &= C2^{j(1/p-1/2)}\left[\sum_{n=2^{j-1}}^{2j}|c_n(f)|^{p'}\right]^{1/p'}.\end{aligned}$$

By an argument similar to the one given in [**6**, p. 32], it can be shown that

$$\sum_{n=2^{j-1}}^{2^j}|c_n(f)|^{p'} \le C\sum_{n=2^{j-1}}^{2^j}|c_{n,j}(f)|^{p'},$$

where $c_{n,j}(f)$ is the nth Fourier coefficient of the function $t \to f(t + 2^{-j}) - f(t)$. Since $p' > 2$, an application of the Hausdorff-Young inequality gives us $s_j \le C2^{j(1/p-1/2)}\omega_p(f, 2^{-j})$. Arguing as we did for the case $p \ge 2$, we obtain that $s_j \le C2^{j(1/p-1/2)}[2^{-j}/\phi(2^j)]^{1/r}$. This completes the proof. □

Combining Lemma 2.3 and Theorem 2.1, we obtain

Corollary 2.4. *Suppose* $\sum_{n=1}^{\infty}(\omega_p(f,1/n))^r/(n^\beta(\log n)^\gamma) < \infty$.

(A) *If one of the following three conditions is satisfied*:

(i) $p \ge 2$,

(ii) $\sum_{n=2^{j-1}}^{2^j-1}, |c_n(f)|^p \le C$,

(iii) $\alpha p > 1$,

then almost surely as $h \downarrow 0$,

(a) $(1-\beta)/r = 0$, $\gamma/r > 1$ *implies* $\omega(F(s,\cdot),h) = O(|\log h|^{1-\gamma/r})$.

(b) $0 < (1-\beta)/r < 1$ *implies* $\omega(F(s,\cdot),h) = O(h^{(1-\beta)/r}|\log h|^{1/2-\gamma/r})$.

(c) $(1-\beta)/r = 1$, $\gamma/r < 1/2$ *implies* $\omega(F(s,\cdot),h) = O(h|\log h|^{1-\gamma/r})$.

(d) $(1-\beta)/r = 1$, $1/2 < \gamma/r \le 1$ *implies* $\omega(F(s,\cdot),h) = O(h|\log h|^{1/2})$.

(e) $(1-\beta)/r = 1$, $1 < \gamma/r$ *or* $\beta > 1$ *implies* $\omega(F(s,\cdot),h) = O(h)$.

(B) *If* $p < 2$ *and* $f \in L_q$ *such that* $2 \le q < \infty$, *then* (a) *through* (e) *of part* (A) *holds with* $(1-\beta)/r$ *replaced by* $\delta(1-\beta)/r$ *everywhere, where* $\delta = p(q-2)/(2(q-p))$. *If* $q = \infty$, *then* (a) *through* (e) *of part* (A) *holds with* $(1-\beta)/r$ *replaced by* $p(1-\beta)/(2r)$ *everywhere.*

(C) *If* $p < 2$, *then* (a) *through* (e) *of part* (A) *holds with* $(1-\beta)/r$ *replaced by* $1/2 - 1/p + (1-\beta)/r$ *everywhere.*

Proof of the following corollary is similar to the proof of the last result. We omit the proof.

Corollary 2.5. *Let* $f \in \mathrm{Lip}\,(\alpha, p)$.

(A) *Suppose that one of the following three conditions is satisfied*:

(i) $p \geq 2$,

(ii) $\sum_{n=2^{j-1}}^{2^j-1} |c_n(f)|^p \leq C$,

(iii) $\alpha p > 1$,

then for almost all s,

$$\omega(F(s,\cdot),h) = O(h^{\alpha}[|\log h|]^{1/2}). \tag{3}$$

(B) *If* $p < 2$ *and* $f \in L_q$ *such that* $2 \leq q < \infty$, *then* (3) *holds almost surely with* α *replaced by* $\alpha\delta$, *where* $\delta = p(q-2)/(2(q-p))$. *If* $q = \infty$, *then* (3) *holds almost surely with* α *replaced by* $p\alpha/2$.

(C) *If* $f \in \mathrm{Lip}\,(\alpha, p)$, $p < 2$ *and* $1/p < \alpha + 1/2$, *then, for almost all* s,

$$\omega(F(s,\cdot),h) = O([h^{\alpha+1/2-1/p}|\log h|]^{1/2}). \tag{4}$$

By slightly modifying the proof of Theorem 2.1, we obtain

Theorem 2.6. *If* $f \in \mathrm{Lip}\,(\alpha, p)$ *and* $p < 2$, *then almost surely*

$$\omega(F(s,\cdot),h) = O([h^{\alpha p}\omega(f,h)^{(2-p)}|\log h|]^{1/2}). \tag{5}$$

Since $BV_p \subset \mathrm{Lip}\,(1/p,p) \cap L_\infty$, we obtain

Corollary 2.7. *Let* $f \in BV_p$. *If* $p \leq 2$, *then almost surely* $\omega(F(s,\cdot),h) = O(\sqrt{h|\log h|})$. *If* $p \geq 2$, *then almost surely* $\omega(F(s,\cdot),h) = O([h^{1/p}|\log h|^{1/2})$.

Corollary 2.8. *If* $f \in BV_p \cap C_\alpha$ *and* $p < 2$, *then, for almost all* s,

$$\omega(F(s,\cdot),h) = O([h^{1+\alpha(2-p)}|\log h|]^{1/2}). \tag{6}$$

In particular, almost surely, $F(s,\cdot) \in C_\beta$ *where* $\beta < (1/2+\alpha(2-p)/2) = \alpha + (1-\alpha p)/2$.

3. Absolute summability of Fourier series. Let $\sum a_n$ be a given infinite series with a sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of constants. Let

$$P_n = \sum_{k=0}^{n} p_k; \quad P_{-k} = p_{-k} = 0 \quad \text{for } k \geq 1.$$

The Nörlund transform of the sequence S_n generated by the sequence $\{p_n\}$ is the sequence $\{t_n\}$ given by

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k = \frac{1}{P_n} \sum_{k=0}^{n} P_k a_{n-k}, \quad P_n \neq 0.$$

If the series $\sum_{n=1}^{\infty} |t_n - t_{n-1}|$ converges, then $\sum a_n$ is said to be absolutely Nörlund summable or summable $|N, p_n|$. When $p_n = A_n^{\alpha-1}$ are Cesaro numbers of order $\alpha - 1$, where A_n^{α} is given by the following formulae

$$\sum_{n=0}^{\infty} A_n^{\alpha} x^n = (1-x)^{-\alpha-1},$$

then the summability $|N, p_n|$ is the same as the summability $|C, \alpha|$ (absolute Cesaro summability of order α). Extending Bernstein's theorem on absolute convergence of the Fourier series, Hyslop [**3**] proved the following theorem.

Theorem 3.1 [**3**]. *Let $0 < \alpha \leq 1/2$, $\beta > 0$ and $\alpha + \beta > 1/2$. If $\omega(f, h) = O(h^{\alpha} |\log h|^{\gamma})$, then the Fourier series is of summable $|C, \beta|$ everywhere.*

Actually Hyslop proved the above theorem under the condition that $\omega(f, h) = O(h^{\alpha})$; however, his argument also proves Theorem 3.1. In fact, Hyslop's proof with minor modifications gives us the following result.

Theorem 3.2. *If $f \in L_p$, $1 < p \leq 2$, $\beta < 1/p$ and $\sum \infty_{n=1} (\omega_p(f, (1/n))/n^{\beta+(1/p')} < \infty$, then the Fourier series of f is almost everywhere $|C, \beta|$ summable.*

Corollary 3.3. *$\omega_p(f, 1/n) = O(n^{-\alpha}(\log n)^{\gamma})$ and $\alpha + \beta > 1/2$, then the Fourier series of f is almost everywhere $|C, \beta|$ summable.*

Extending Hyslop's result, McFadden [**9**] proved the following result.

Theorem 3.4 [**9**]. *Let p_n be nonnegative and nonincreasing, $\lim_{n\to\infty} p_n = 0$, $|\Delta p_n| = |p_{n+1} - p_n|$ nonincreasing and satisfying the conditions*

$$\sum_{k=1}^{\infty} P_k^2 k^{-2} \le A, \qquad \sum_{k=1}^{\infty} P_k^{-1} k^{-\alpha-1/2} \le A.$$

If $\omega(f, h) = O(h^{\alpha}|\log h|^{\gamma})$, then the Fourier series is of summable $|N, p_n|$ everywhere.

The final result in this direction is the following theorem of M. Izumi and S. Izumi [**4**] and Lal [**7**].

Theorem 3.5. *Let $f \in L_p$, $1 < p \le 2$, p_n be nonnegative and nonincreasing, $\lim_{n\to\infty} p_n = 0$, $|\Delta p_n| = |p_{n+1} - p_n|$ nonincreasing and satisfying the conditions*

$$\sum_{k=1}^{\infty} P_k^2 k^{-2} \le A, \qquad \sum_{k=1}^{\infty} \omega_p\left(f, \frac{1}{k}\right) P_k^{-1} k^{-1/p'} \le A.$$

Then the Fourier series is of summable $|N, p_n|$ almost everywhere.

We introduce some notation. Let

$$\Phi_x(t) = f(x+t) + f(x-t) - 2f(x).$$

If the Fourier series of $f(x)$ is $\sum_{j=0}^{\infty} c_j(f)\cos(jt + \phi_j)$, then the Fourier series of Φ_x is

$$\begin{aligned} &\sum_{j=0}^{\infty} c_j(f)[\cos(j(t+x)+\phi_j) \\ &\qquad + \cos(j(t-x)+\phi_j) - 2\cos(jx+\phi_j)] \\ &\qquad\qquad = \sum_{j=0}^{\infty} A_j(x)\cos(jt+\phi_j) \end{aligned} \tag{7}$$

where $A_j(x) = c_j(f)\cos(jx + \phi_j)$. Let $Q(n,k) = p_k P_n - p_n P_k$ and $\tau_n(f,x) = t_n(f,x) - t_{n-1}(f,x)$, where $\{t_n(f,x)\}$ is the Nörlund transform of the Fourier series of f at x. Then

$$\begin{aligned}
P_n P_{n-1}\tau_n(f,x) &= \int_0^\pi \Phi_x(t)\sum_{k=0}^{n-1} Q(n,k)\cos((n-k)t + \phi_{n-k})\,dt \\
&= \int_0^\pi \sum_{j=0}^{\infty} c_j(f)[\cos(j(t+x)+\phi_j) + \cos(j(t-x)+\phi_j \\
&\qquad - 2\cos(jx+\phi_j)\sum_{k=0}^{n-1} Q(n,k)\cos((n-k)t+\phi_{n-k})\,dt \\
&= \int_0^\pi \sum_{k=0}^{n-1} Q(n,k)c_{(n-k)}(f)\cos((n-k)x+\phi_{n-k}) \\
&\qquad\qquad \cdot \cos^2((n-k)t+\phi_{n-k}))\,dt \\
&= C\sum_{k=0}^{n-1} Q(n,k)c_{(n-k)}(f)\cos((n-k)x+\phi_{n-k}).
\end{aligned}$$

Let $F(s,t) = \sum_{j=0}^{\infty} r_j(s)c_j(f)\cos(jt+\phi_j)$, where $r_0(s) = 1$ and r_j is the jth Rademacher function. Then

$$\begin{aligned}
\tau_n(F,x) &= \tau_n(F(s,\cdot),x) \\
&= \frac{C}{P_n P_{n-1}}\sum_{k=0}^{n-1} Q(n,k)c_{(n-k)}(f)r_k(s)\cos((n-k)x+\phi_{n-k}).
\end{aligned} \tag{8}$$

We need the following lemma.

Lemma 3.6. *The measure* (*probability*) *of the set* $\{s : \int_{-\pi}^{\pi} |\tau(F(s,\cdot), x)|\,dx \geq C\int_{-\pi}^{\pi} |\tau(f,x)|\,dx\}$ *is positive, where* C *is some fixed universal constant.*

Proof.

$$\int_0^1 \int_{-\pi}^{\pi} |\tau_n(F(s,\cdot),x|\,dx\,ds = \int_{-\pi}^{\pi}\int_0^1 |\tau_n(F(s,\cdot),x)|\,ds\,dx$$

(Khinchin's inequality $\Rightarrow$)

$$\begin{aligned}
&\geq \int_{-\pi}^{\pi} C_K \left[\int_0^1 |\tau_n(F(s,\cdot),x)|^2\, ds \right]^{1/2} dx \\
&= \int_{-\pi}^{\pi} \frac{C_K}{P_n P_{n-1}} \left[\sum_{k=0}^{n-1} |Q(n,k) c_{(n-k)}(f) \right. \\
&\qquad \left. \cdot \cos((n-k)x + \phi_{n-k})|^2 \right]^{1/2} dx
\end{aligned}$$

(Minkowski's inequality $\Rightarrow$)

$$\begin{aligned}
&\geq \frac{C_K}{P_n P_{n-1}} \left[\sum_{k=0}^{n-1} \left(\int_{-\pi}^{\pi} |Q(n,k) c_{(n-k)}(f) \right.\right. \\
&\qquad \left.\left. \cdot \cos((n-k)x + \phi_{n-k})|\, dx \right)^2 \right]^{1/2}
\end{aligned}$$

($|\cos(\cdot)| \geq |\cos(\cdot)^2 \Rightarrow$)

$$\begin{aligned}
&\geq \frac{C_K}{P_n P_{n-1}} \left[\sum_{k=1}^{n-1} \left(\int_{-\pi}^{\pi} |Q(n,k) c_{(n-k)}(f) \right.\right. \\
&\qquad \left.\left. \cdot \cos((n-k)x + \phi_{n-k})|^2\, dx \right)^2 \right]^{1/2}
\end{aligned}$$

($C = C_K(\int_{-\pi}^{\pi} |\cos((n-k)x + \phi_{n-k})|^2\, dx)^2 \Rightarrow$)

$$\begin{aligned}
&= \frac{C}{P_n P_{n-1}} \left[\sum_{k=0}^{n-1} |Q(n,k) c_{(n-k)}(f)|^2 \right]^{1/2} \\
&= \frac{C}{P_n P_{n-1}} \left[\int_{-\pi}^{\pi} \left| \sum_{k=0}^{n-1} Q(n,k) c_{(n-k)}(f) \right.\right.
\end{aligned}$$

$$\cdot \cos((n-k)x+\phi_{n-k})\bigg|^2\,dx\bigg]^{1/2}$$
$$= C\bigg[\int_{-\pi}^{\pi} |\tau_n(f,x)|^2\,dx\bigg]^{1/2}$$
$$\geq C\int_{-\pi}^{\pi} |\tau_n(f,x)|\,dx.$$

This proves the lemma. □

In view of Theorems (3.1), (3.4) and Lemma (3.6), we obtain the following theorems.

Theorem 3.7. *Let p_n be a sequence satisfying the hypothesis of Theorem* 3.4. *Let*

$$F_{\Phi_x}(s,t) = \sum_{n=0}^{\infty} r_n(s)A_n(x)\cos(nt+\phi_n), \tag{9}$$

where $A_n(x)$ is defined in Equation (7). *If almost surely $\omega(F_{\Phi_x}(s,\cdot),h) = O(h^\alpha|\log h|^\gamma)$ as $h \downarrow 0$, then Fourier series of f at x is $|N,p_n|$ summable.* (α *appears in the hypothesis of Theorem* 3.4).

Theorem 3.8. *Let $0 < \alpha \leq 2$, $\beta > 0$ and $\alpha + \beta > 1/2$. If almost surely $\omega(F_{\Phi_x}(s,\cdot),h) = O(h^\alpha|\log h|^\gamma)$ as $h \downarrow 0$, then Fourier series of f at x is $|C,\beta|$ summable.*

We now show that number of classical results concerning summability of Fourier series can be obtained as easy consequences of Theorem 3.2 and/or the above two results. In some cases we obtain some extension of the classical results. We cannot deduce Lal's result, Theorem 3.5, from Theorem 3.7. However, we do obtain some results which cannot be obtained as consequences of Theorem 3.5, see Theorems 3.10 through 3.15. Combining Corollary 2.4, Corollary 2.5, Theorems 3.7 and 3.8, we obtain the following results.

Theorem 3.9. *Let* $f \in \mathrm{Lip}\,(\alpha, p)$.

(A) *Suppose one of the following three conditions is satisfied*:

(i) $p \geq 2$,

(ii) $\sum_{n=2^{j-1}}^{2^j-1} |c_n(f)|^p \leq C$,

(iii) $\alpha p > 1$.

Then the Fourier series of f *at each* x *is* $|C, \beta|$ *summable whenever* $\alpha + \beta > 1/2$.

(B) *If* $p < 2$ *and* $f \in L_q$, *such that* $2 \leq q < \infty$, *then the Fourier series of* f *at* x *is* $|C, \beta|$ *summable whenever* $\alpha\delta + \beta > 1/2$ *where* $\delta = p(q-2)/(2(q-p))$. *If* $q = \infty$, *then the Fourier series of* f *at* x *is* $|C, \beta|$ *summable whenever* $p\alpha/2 + \beta > 1/2$.

(C) *If* $p < 2$ *and* $1/p < \alpha + 1/2$, *then the Fourier series of* f *at* x *is* $|C, \beta|$ *summable whenever* $\alpha + \beta > 1/p$.

Theorem 3.10. *Let* p_n *be nonnegative and nonincreasing,* $\lim_{n\to\infty} p_n = 0$, $|\Delta p_n| = |p_{n+1} - p_n|$ *nonincreasing and satisfying the conditions*

$$\sum_{k=1}^{\infty} P_k^2 k^{-2} \leq A, \qquad \sum_{k=1}^{\infty} P_k^{-1} k^{-\beta-1/2} \leq A.$$

Let $f \in \mathrm{Lip}\,(\alpha, p)$.

(A) *Suppose one of the following three conditions is satisfied*:

(i) $p \geq 2$,

(ii) $\sum_{n=2^{j-1}}^{2^j-1} |c_n(f)|^p \leq C$,

(iii) $\alpha p > 1$.

Then the Fourier series of f *at each* x *is* $|N, p_n|$ *summable whenever* $\alpha \geq \beta$.

(B) *If* $p < 2$ *and* $f \in L_q$ *such that* $2 \leq q < \infty$, *then the Fourier series of* f *at* x *is* $|N, p_n|$ *summable whenever* $\alpha\delta \geq \beta$, *where* $\delta = p(q-2)/(2(q-p))$. *If* $q = \infty$, *then the Fourier series of* f *at* x *is* $|N, p_n|$ *summable whenever* $p\alpha/2 \geq \beta$.

(C) *If* $p < 2$ *and* $1/p < \alpha + 1/2$, *then the Fourier series of* f *at* x *is* $|N, p_n|$ *summable whenever* $\alpha + 1/2 \geq \beta + 1/p$.

Theorem 3.11. *Suppose* $\sum_{n=1}^{\infty}(\omega_p(f,1/n))^r/(n^{\beta}(\log n)^{\gamma}) < \infty$.

(A) *If* $0 < (1-\beta)/r$ *and any one of the following three conditions is satisfied*:

(i) $p \geq 2$,

(ii) $\sum_{n=2^{j-1}}^{2^j-1} |c_n(f)|^p \leq C$,

(iii) $\alpha p > 1$,

then the Fourier series of f *is* $|C,\theta|$ *summable whenever* $0 < (1-\beta)/r + \theta > 1/2$.

(B) *If* $p < 2$ *and* $f \in L_q$ *such that* $2 \leq q < \infty$ *and* $\delta(1-\beta)/r > 0$ *where* $\delta = p(q-2)/(2(q-p))$, *then the Fourier series of* f *is* $|C,\theta|$ *summable whenever* $\delta(1-\beta)/r + \theta > 1/2$. *If* $q = \infty$, *then the Fourier series of* f *is* $|C,\theta|$ *summable whenever* $p(1-\beta)/(2r) + \theta > 1/2$.

(C) *If* $p < 2$, *then the Fourier series of* f *is* $|C,\theta|$ *summable whenever* $1/2 - 1/p + (1-\beta)/r + \theta > 1/2$.

Theorem 3.12. *Let* p_n *be nonnegative and nonincreasing,* $\lim_{n\to\infty} p_n = 0$, $|\Delta p_n| = |p_{n+1} - p_n|$ *nonincreasing and satisfying the conditions*

$$\sum_{k=1}^{\infty} P_k^2 k^{-2} \leq A, \qquad \sum_{k=1}^{\infty} P_k^{-1} k^{-\alpha-1/2} \leq A.$$

Suppose $\sum_{n=1}^{\infty}(\omega_p(f,1/n))^r/(n^{\beta}(\log n)^{\gamma}) < \infty$.

(A) *If* $0 < (1-\beta)/r$ *and any one of the following three conditions is satisfied*:

(i) $p \geq 2$,

(ii) $\sum_{n=2^{j-1}}^{2^j-1} |c_n(f)|^p \leq C$,

(iii) $\alpha p > 1$,

then the Fourier series of f *is* $|N,p_n|$ *summable whenever* $0\langle(1-\beta)/r \geq \alpha\theta\rangle$.

(B) *If* $p < 2$ *and* $f \in L_q$ *such that* $2 \leq q < \infty$ *and* $\delta(1-\beta)/r > 0$ *where* $\delta = p(q-2)/(2(q-p))$, *then the Fourier series of* f *is* $|N,p_n|$ *summable whenever* $\delta(1-\beta)/r \geq \alpha$. *If* $q = \infty$, *then the Fourier series of* f *is* $|N,p_n|$ *summable whenever* $p(1-\beta)/(2r) \geq \alpha$.

(C) *If $p < 2$, then the Fourier series of f is $|N, P_n|$ summable whenever $1/2 - 1/p + (1-\beta)/r \geq \alpha$.*

Combining Theorems 2.2 and 3.2, we obtain

Theorem 3.13. *Suppose $\sum_{n=0}^{\infty} |c_n(f)|^2 n^{\beta} (\log n)^{\gamma} < \infty$.*

(A) *If $0 < \beta$, then the Fourier series of f is $|C, \alpha|$ summable whenever $\beta > 0$, $\beta/2 + \alpha > 1/2$.*

(B) *If $\beta = 0$ and $\gamma > 1$, then the Fourier series of f is $|C, \alpha|$ summable whenever $\alpha > 1/2$.*

(C) *If $\beta \geq 0$ and $\beta/2 + \alpha = 1/2$ and $\gamma > 3$, then the Fourier series of f is $|C, \alpha|$ summable.*

We should point out that F.T. Wang [**11**] has proved the following stronger version of part (C) of Theorem 3.13. Wang's proof is direct, and we can't seem to obtain his result by random methods.

Theorem 3.14 [**11**]. *Suppose $\sum_{n=0}^{\infty} |c_n(f)|^2 n^{\beta} (\log n)^{\gamma} < \infty$.*

(C$'$) *If $\beta = 0$ and $\gamma > 2$, then the Fourier series of f is $|C, 1/2|$ summable almost everywhere.*

(C$''$) *If $\beta > 0$, $\alpha < 1/2$, $\beta/2 + \alpha = 1/2$ and $\gamma > 2$, then the Fourier series of f is $|C, \alpha|$ summable almost everywhere.*

Theorem 3.15. *Let p_n be nonnegative and nonincreasing, $\lim_{n\to\infty} p_n = 0$, $|\Delta p_n| = |p_{n+1} - p_n|$ nonincreasing and satisfying the conditions*

$$\sum_{k=1}^{\infty} P_k^2 k^{-2} \leq A, \qquad \sum_{k=1}^{\infty} P_k^{-1} k^{-\alpha - 1/2} \leq A.$$

If $\sum_{n=0}^{\infty} |c_n(f)|^2 n^{\beta} (\log n)^{\gamma} < \infty$ and $0 < \beta$, then the Fourier series of f is $|N, \alpha|$ summable whenever $\beta/2 \geq \alpha > 1/2$.

As a corollary of Theorem 3.10, we obtain

Corollary 3.16. *Suppose $f \in BV_p$. If $p < 2$, then the Fourier series* *of f is $|C,\beta|$ summable whenever $\beta > 0$. If $f \in BV_p$, then the Fourier series of f at x is $|C,\beta|$ summable whenever $\beta + 1/p > 1/2$.*

As corollaries of (3.10) and (3.15), we obtain the following results of N. Matsuyama and L.S. Bosanquet.

Corollary 3.17 [**8**, Theorem 1]. *If $f \in \text{Lip}\ (\alpha, p)$, $0 < \alpha \le 1$, $1 \le p \le 2$, $\alpha p \le 1$, then the Fourier series of f is $|C, 1/p - \alpha + \varepsilon|$ summable almost everywhere, ε being any positive number.*

Corollary 3.18 [**8**, Theorem 2]. *If $f \in \text{Lip}\ (\alpha, p)$, $0 < \alpha \le 1$, $1 \le p \le 2$, $\alpha p > 1$, then the Fourier series of f is $|C, 1/p' + \varepsilon|$ summable almost everywhere, ε being any positive number.*

Corollary 3.19 [**2**]. *If $f \in BV_1$, then the Fourier series of f is $|C, \varepsilon|$ summable almost everywhere, ε being any positive number.*

The next result of F.T. Wang is a corollary of Theorem 3.13.

Corollary 3.20 [**11**]. *If $\sum_{n=0}^{\infty} |c_n(f)|^2 (\log n)^{1+\varepsilon} < \infty$, then the Fourier series of f is $|C, \alpha|$ summable almost everywhere whenever $\alpha > 1/2$.*

REFERENCES

1. S. Bernstein, *Sur la convergence absolue des séries trigonométriques*, Comptes Rendus **158** (1914), 1661–1664.

2. L.S. Bosanquet, *The absolute Cesaro summability of Fourier series*, Proc. London Math. Soc. (3) **41** (1936), 517–528.

3. J.M. Hyslop, *On the absolute summability of Fourier series*, Proc. London Math. Soc. (3) **43** (1937), 475–483.

4. M. Izumi and S. Izumi, *Nörlund summability of Fourier series*, Pacific J. Math. **26** (1968), 289–301.

5. J.P. Kahane, *Some random series of functions*, Cambridge University Press, London, 1985.

6. Y. Katznelson, *An introduction to harmonic analysis*, Dover Publications, Inc., New York, 1968.

7. S.N. Lal, *On a theorem of M. Izumi and S. Izumi*, Pacific J. Math. **35** (1970), 661–667.

8. N. Matsuyama, *Note on Fourier series* (XXV); *On the* $|C|$*-summability of Fourier series*, Tohoku Math. J. **2** (1950), 51–56.

9. L. McFadden, *Absolute Nörlund summability*, Duke Math. J. **9** (1942), 168–207.

10. Y. Okuyama, *Absolute summability of Fourier series and orthogonal series*, Lecture Notes in Math. **1067** (1984), 1–23.

11. F.T. Wang, *The absolute Cesaro summability of trigonometric series*, Duke Math. J. **9** (1942), 567–572.

Department of Mathematics, University of Arizona, Tucson, Arizona 85721

Current address: Booth Research Center, 233 Glenbrook Road, U-31, University of Connecticut, Storrs CT 06269-4031
E-mail address: nasser@optic-pc2.brc.uconn.edu