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A NOTE ON ASCOLI'S THEOREM

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ABSTRACT. In this note we study several properties of the space $(G, \mathcal{L}_u)$ where G is a vectorial subspace of continuous functions from a topological space T into a Hausdorff topological vector space F and $\mathcal{L}_u$ is the topology of uniform convergence on the members of a cover $\mathcal{L}$ of T directed by inclusion, when $(G, \mathcal{L}_u)$ satisfies Ascoli's theorem. We give a sufficient condition for the members $\mathcal{L}$ to be functionally bounded, in T, and we apply this result in two ways. First, we prove that, in this case, $(G, \mathcal{L}_u)$ is a topological vector space and, second, if T is a Tychonoff space and F is complete, we prove that the only spaces $(G, \mathcal{L}_u)$ which satisfy Ascoli's theorem are, up to topological isomorphisms, those spaces such that every member of $\mathcal{L}$ is compact. We also obtain an application of this result when the topological vector space F is the usual topological vector space $\mathbf{R}$ of real numbers.

1. Introduction. Let T be a topological space and let $(Y, \mathcal{U})$ be a uniform space. If G is a subset of Y^T, the set of all functions from T to Y, we say that G is pointwise bounded if $\{f(x) : f \in G\}$ is relatively compact in $(Y, \mathcal{U})$ for every $x \in T$ and we say that G is equicontinuous if, for every $x \in T$ and any $V \in \mathcal{U}$, there exists a neighborhood W of the point x such that $(f(x), f(x')) \in V$ whenever $f \in G$ and $x' \in W$. Given a topology $\mathcal{T}$ on a subset $\mathcal{H}$ of Y^T, $\mathcal{T}$ is said to satisfy Ascoli's theorem if a subset K of $\mathcal{H}$ is $\mathcal{T}$-compact if and only if K is $\mathcal{T}$-closed, pointwise bounded and equicontinuous.

Compactness criteria in function spaces, in particular those of Ascoli type, have applications in various fields of mathematics, for instance in functional analysis. The prototype of Ascoli's theorem was proved by Ascoli in [**3**] and independently by Arzelà, who acknowledged Ascoli's priority in [**2**]. This classical theorem of Ascoli-Arzelà was the starting-point for investigations on compactness in function spaces, in particular for spaces of continuous functions.

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In this note we study several properties of the space $(G, \mathcal{L}_u)$ where G is a vectorial subspace of continuous functions from a topological space T into a Hausdorff topological vector space F and $\mathcal{L}_u$ is the topology of uniform convergence on the members of a cover $\mathcal{L}$ of T directed by inclusion, when $(G, \mathcal{L}_u)$ satisfies Ascoli's theorem. We give a sufficient condition for the members of cover $\mathcal{L}$ to be functionally bounded, in T, and we apply this result in two ways. First, we prove that, in this case, $(G, \mathcal{L}_u)$ is a topological vector space and, second, if T is a Tychonoff space and F is complete, we prove that the only spaces $(G, \mathcal{L}_u)$ which satisfy Ascoli's theorem are, up to topological isomorphisms, those spaces such that every member of $\mathcal{L}$ is compact. We also obtain an application of this result when the topological vector space F is the usual topological vector space $\mathbf{R}$ of real numbers.

Notation. Let T be a topological space and $\mathcal{L}$ a cover of T directed by inclusion, i.e.,

$$A, B \in \mathcal{L} \quad \Longrightarrow \quad \exists\, C \in \mathcal{L} \;\mid\; A \cup B \subset C$$

If F is a topological vector space over the field $\mathbf{K}$, where $\mathbf{K}$ is the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers, $\mathcal{B}$ is a basis at 0 in F, and G is a subgroup of Y^T, the family

$$M(S, V) = \{f \in G : f(S) \subset V\}, \quad S \in \mathcal{L},\ V \in \mathcal{B}$$

is a basis at 0 in G for a topology, translation invariant. This topology is called the $\mathcal{L}$-topology or the $\mathcal{L}_u$-topology of uniform convergence on the members of $\mathcal{L}$ [**15**].

A subset B of a topological space T is said to be functionally bounded, in T, if every real-valued continuous function on T is bounded on B. As usual, $C(T)$ will denote the ring of all continuous real-valued functions on the topological space T. The set of points of T where a member f of $C(T)$ is equal to zero is called the zero-set of f and will be denoted by $Z(f)$. All topological vector spaces are assumed to be Hausdorff.

2. The results. We begin with a lemma. A family $\{V_i\}_{i \in I}$ of subsets of T is called locally finite if, for every $x \in T$, there exists a neighborhood U of x such that the set $\{i \in I : V_i \cap U \neq \varnothing\}$ is finite.

Lemma 1. *Let T be a topological space and B a subset of T not functionally bounded. Then there exist a sequence $\{x_n\}_{n=1}^{\infty}$ of points of B and a sequence $\{f_n\}_{n=1}^{\infty}$ of elements of $C(T)$ such that $f_n(x_n) = 1$ for $n = 1, 2, ...$ and $[T \setminus Z(f_n)] \cap [T \setminus Z(f_m)] = \varnothing$ if $n \neq m$. Moreover, the sequence $\{T \setminus Z(f_n)\}_{n=1}^{\infty}$ is locally finite.*

Proof. Let g be a real-valued continuous function on T such that $g \geq 0$ and such that its restriction to B is not bounded. We can take a sequence $\{x_n\}_{n=1}^{\infty}$ of points of B and a sequence $\{r_n\}_{n=1}^{\infty}$ of real numbers such that

$$g(x_n) = r_n, \quad 1 < r_1, \quad r_n + 1 < r_{n+1}, \quad n = 1, 2, \dots .$$

Now we consider the sequence $\{W_n\}_{n=1}^{\infty}$ of open subsets of $\mathbf{R}$, where $W_n =]r_n - 1/2, r_n + 1/2[$, $n = 1, 2, \dots$. Since $\mathbf{R}$ is a Tychonoff space, there exists a sequence $\{h_n\}_{n=1}^{\infty}$ in $C(\mathbf{R})$ satisfying

$$0 \leq h_n \leq 1, \quad h_n(\mathbf{R} \setminus W_n) = 0, \quad h_n(r_n) = 1, \quad n = 1, 2, \dots .$$

We define $f_n = h_n \circ g$, $n = 1, 2, \dots$. One can easily verify that the sequences $\{f_n\}_{n=1}^{\infty}$ and $\{T \setminus Z(f_n)\}_{n=1}^{\infty}$ have the required properties. □

Let F be a topological vector space and $z \in F$; we use $[0, z]$ to denote the closed segment of extremes 0 and z and $C(T, F)$, respectively $C^*(T, F)$, to denote the vectorial space of continuous functions, respectively continuous bounded functions, from a topological space T into F. If $\mathcal{L}$ is a cover of T directed by inclusion, a key result is the following.

Theorem 2. *Let G be a subgroup of F^T and $z \in F \setminus \{0\}$. If G contains the set*

$$(*) \qquad \{f \in C(T, F) : f(T) \subset [0, z]\}$$

and $(G, \mathcal{L}_u)$ satisfies Ascoli's theorem, then every $S \in \mathcal{L}$ is functionally bounded, in T.

Proof. Let B be an element of $\mathcal{L}$ not functionally bounded. It follows from the lemma above that there exist a sequence $\{f_n\}_{n=1}^{\infty}$ in $C(T)$ and a sequence $\{x_n\}_{n=1}^{\infty}$ of points of B satisfying:

$$0 \leq f_n \leq 1, \quad f_n(x_n) = 1, \quad n = 1, 2, \ldots$$

and $\{T \setminus Z(f_n)\}_{n=1}^{\infty}$ is a locally finite family which consists of pairwise disjoint open sets.

We define $h_n(x) = \sup\{f_p(x) : p = 1, 2, \ldots n\}$, $n = 1, 2, \ldots$, and we take the sequence $\{m_n\}_{n=1}^{\infty}$ in $C(T, F)$, where

$$m_n(x) = h_n(x)z \quad \text{for every} \quad x \in T, \quad n = 1, 2, \ldots .$$

The family $\{T \setminus Z(f_n)\}_{n=1}^{\infty}$ being locally finite every function in $\{h_n\}_{n=1}^{\infty}$ is continuous. One can easily show that the sequence $\{m_n\}_{n=1}^{\infty}$ is in G, because G satisfies the property $(*)$, and that it is pointwise bounded, because $\{T \setminus Z(f_n)\}_{n=1}^{\infty}$ is locally finite. Now we are going to prove that $\{m_n\}_{n=1}^{\infty}$ is equicontinuous. Consider a point $x \in T$ and a symmetric neighborhood V at 0 in F. Since the sequence $\{T \setminus Z(f_n)\}_{n=1}^{\infty}$ is locally finite, there exist a finite set L of natural numbers and a neighborhood W of x such that

$$W \cap [T \setminus Z(f_n)] = \varnothing$$

whenever $n \notin L$. The functions of the sequence $\{m_n\}_{n=1}^{\infty}$ being continuous, there exists an open set M, with $x \in M$, such that the condition $y \in M$ implies

$$m_p(y) - m_p(x) \in V \quad \text{whenever} \quad p \leq \max\{n : n \in L\}.$$

Now one can easily verify that $m_p(y) - m_p(x) \in V$, $n = 1, 2, \ldots$ whenever $y \in W \cap M$; therefore, $\{m_n\}_{n=1}^{\infty}$ is equicontinuous.

Next we show that $\{m_n\}_{n=1}^{\infty}$ is not relatively compact. Let V be a symmetric neighborhood at 0 in F such that $z \notin V$. Since $h_p(x_j) = 0$ and $h_k(x_j) = 1$ whenever $p < j < k$, we have that

$$m_k(x_j) - m_p(x_j) \notin V$$

if $p < j < k$; by consequence $\{m_n\}_{n=1}^{\infty}$ is not relatively compact in $(G, \mathcal{L}_u)$.

The $\mathcal{L}_u$-topology being stronger than the pointwise convergence topology, the closure S in $(G, \mathcal{L}_u)$ of the equicontinuous family $\{m_n\}_{n=1}^{\infty}$ is also equicontinuous [**9**, Theorem 14]. Since $[0, z]$ is compact, S is pointwise bounded, by consequence $(G, \mathcal{L}_u)$ does not satisfy Ascoli's theorem. $\square$

The following proposition is a rather straightforward consequence of the definitions. It points out that the condition $(*)$ of the above theorem is not very restrictive.

Proposition 3. *Let G be a subgroup of F^T. If G satisfies one the following conditions, then G satisfies condition $(*)$:*

(a) *$F = \mathbf{K}$ and $\{f \in C(T, \mathbf{R}) : 0 \leq f \leq 1\} \subset G$.*

(b) *$C^*(T, \mathbf{K}) \subset G \subset C(T, \mathbf{K})$.*

(c) *T is a topological vector space, $G \neq \{0\}$ and every function in G is linear.*

It is well known that, if f is a continuous function from a space T into a space Y and B is a functionally bounded subset (in T), then $f(B)$ is functionally bounded (in Y). The functionally bounded subsets of a topological vector space being totally bounded [**17**, Lemma 2.25], one can apply several results of the theory of topological vector spaces, see [**15**, Theorems 3.1, 3.2] in order to obtain the following theorem.

Theorem 4. *Let G be a vectorial subspace of $C(T, F)$ which satisfies the property $(*)$, and let $\mathcal{L}$ be a cover of T directed by inclusion. If $(G, \mathcal{L}_u)$ satisfies Ascoli's theorem, then $(G, \mathcal{L}_u)$ is a topological vector space. Moreover, if F is locally convex, $(G, \mathcal{L}_u)$ is also locally convex.*

One the most important cases in functional analysis is when T is a Tychonoff space and F is complete, for instance, F a Fréchet space, a Banach space, a Hilbert space or F the field $\mathbf{K}$ ($\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$). In this situation, we prove that it suffices to study Ascoli's theorem in spaces $(G, \mathcal{H}_u)$ such that the members of $\mathcal{H}$ are compact. We will denote by γT the topological completion of a Tychonoff space T. It is known that γT is a subspace of the real compactification of Hewitt vT

of T and that $\gamma T = \upsilon T$ if and only if the cardinal of every closed discrete subspace of T is nonmeasurable [**8**]. If F is a topological complete vector space and $f \in C(T, F)$, we use f^γ to denote the continuous extension of f to γT. Let $\mathcal{K}(T)$ be the set of equicontinuous and pointwise bounded subsets of $C(T)$. On each subset $H \in \mathcal{K}(T)$ we place the pointwise convergence topology. The space $M(T)$ of linear forms on $C(T)$ that are continuous on each $H \in \mathcal{K}(T)$, is a locally convex, separated, complete vector space, with the topology of $\mathcal{K}$-convergence defined by the semi-norms:

$$\|u\| = \sup\{u(h) : h \in H\}, \quad H \in \mathcal{K}(T).$$

The following construction of the topological completion of a Tychonoff space is needed in the sequel.

Proposition 5 [**6**, Theorems 1, 2]. *Let T be a Tychonoff space. The following assertions hold*:

(i) *T is a subspace of $M(T)$.*

(ii) *The topological completion γT of T is the closure of T in $M(T)$, i.e., the characters, except zero, of $C(T)$ that are members of $M(T)$.*

It is known that a uniform structure on a Tychonoff space can be described by a suitable nonempty family $\mathcal{D}$ of pseudometrics, see [**8**] for details. In the following we take this point of view.

Lemma 6. *Let $\{f_i\}_{i\in I}$ be an equicontinuous family of functions from a Tychonoff space T into a topological complete vector space F. If $\mathcal{D}$ is a uniform structure on F and $d \in \mathcal{D}$, the family*

$$h_i(t) = d(f_i(t), f_i^\gamma(u_0)), \quad t \in T, \ i \in I$$

where u_0 is a fix point of γT, is also equicontinuous.

Proof. Consider $\varepsilon > 0$ and $t_1 \in T$. The family $\{f_i\}_{i\in I}$ being equicontinuous, there exists a neighborhood V of t_1 such that $d(f_i(t), f_i(t_1)) < \varepsilon$ whenever $t \in V$ and $i \in I$.

The result follows from inequality [**8**, 15G.1]

$$|d(f_i(t), f_i^\gamma(u_o)) - d(f_i(t_1), f_i^\gamma(u_o))| \le d(f_i(t), f_i(t_1)). \qquad \square$$

Theorem 7. *Let $\{f_i\}_{i\in I}$ be an equicontinuous pointwise bounded family of functions from a Tychonoff space T into a topological complete vector space F. Then the family $\{f_i^\gamma\}_{i\in I}$ is also equicontinuous and pointwise bounded.*

Proof. Let u_0 be a point of γT. Let $\mathcal{D}$ be a uniform structure on F and $d \in \mathcal{D}$. Consider the family

$$g_i(t) = \inf\{1, h_i(t)\}, \quad t \in T,\ i \in I,$$

where $\{h_i\}_{i\in I}$ is as the above lemma. By Lemma 6 $\{g_i\}_{i\in I}$ is equicontinuous; obviously it is pointwise bounded. By Proposition 5 we can take a neighborhood W of u_0 in γT defined by $\{g_i\}_{i\in I}$ and $0 < \varepsilon < 1$, i.e., $u \in W$ if and only if $|u(g_i) - u_0(g_i)| < \varepsilon$ for every $i \in I$. Therefore, by [**16**, Theorem II.4.4], we have for every $i \in I$,

$$|g_i^\gamma(u) - g_i^\gamma(u_0)| < \varepsilon, \quad u \in W. \tag{1}$$

By continuity $h_i^\gamma(u) = d(f_i^\gamma(u), f_i^\gamma(u_0))$ and $g_i^\gamma(u) = \inf\{1, h_i^\gamma(u)\}$ with $u \in \gamma T$; consequently $g_i^\gamma(u_0) = 0$. It follows now from (1) that

$$g_i^\gamma(u) = h_i^\gamma(u) = d(f_i^\gamma(u), f_i^\gamma(u_0)) < \varepsilon$$

whenever $u \in W$ and $i \in I$. Therefore $\{f_i^\gamma\}_{i\in I}$ is equicontinuous.

Now we shall show that $\{f_i^\gamma\}_{i\in I}$ is pointwise bounded. Let u_0 be a point of γT and we take $\varepsilon > 0$. The family $\{f_i^\gamma\}_{i\in I}$ being equicontinuous, there exists a neighborhood W of u_0 such that

$$|f_i^\gamma(u) - f_i^\gamma(u_0)| < \frac{\varepsilon}{2}$$

whenever $u \in W$ and $i \in I$. Let t_1 be such that $t_1 \in T \cap W$. Since $\{f_i\}_{i\in I}$ is pointwise bounded, $\{f_i(t_1)\}_{i\in I}$ is totally bounded for the uniform structure $\mathcal{D}$ [**7**, Corollary 8.3.17]. Therefore, if $d \in \mathcal{D}$, there exists a finite number of points of F, $z_1, z_2, \dots, z_n$, such that

$$\{f_i(t_1)\}_{i\in I} \subset \bigcup\{B_{\varepsilon/2}(z_j) : j = 1, \dots, n\}$$

where $B_{\varepsilon/2}(z_j) = \{y \in F : d(y, z_j) < \varepsilon/2\}$.

Pick an $i \in I$. There exists $j \in \{1, \dots, n\}$ such that $f_i(t_1) \in B_{\varepsilon/2}(z_j)$. We clearly have

$$|f_i^\gamma(u_0) - z_j| \le |f_i^\gamma(u_0) - f_i(t_1)| + |f_i(t_1) - z_j| < \varepsilon.$$

Consequently, $\{f_i^\gamma(u_0)\}_{i\in I}$ is totally bounded for the uniform structure $\mathcal{D}$. The space F being complete, $\{f_i^\gamma\}_{i\in I}$ is pointwise bounded [**7**, Corollary 8.3.17]. □

Now, we can state the following theorem.

Theorem 8. *Let T be a Tychonoff space and F a topological complete vector space. Let G be a vectorial subspace of $C(T,F)$ satisfying condition $(*)$ and $\mathcal{L}$ a cover of T directed by inclusion whose members are functionally bounded. Then there exist a space Y, $T \subset Y \subset \gamma T$, a cover $\mathcal{M}$ of Y directed by inclusion whose members are compact and a vectorial subspace G^* of $C(Y,F)$ such that*

(a) *The spaces $(G, \mathcal{L}_u)$ and $(G^*, \mathcal{M}_u)$ are topologically isomorphic.*

(b) *$(G, \mathcal{L}_u)$ satisfies Ascoli's theorem if and only if $(G^*, \mathcal{M}_u)$ satisfies Ascoli's theorem.*

Proof. We define

$$Y = \bigcup\{\mathrm{cl}_{\gamma T} A : A \in \mathcal{L}\}, \quad \mathcal{M} = \{\mathrm{cl}_{\gamma T} A : A \in \mathcal{L}\}$$

and we take $G^* = \{h_f : h_f = f^\gamma_{|Y}, f \in G\}$.

Since γT is topologically complete, then every member of $\mathcal{M}$ is compact [**5**, 5.4.2]. One can easily check that G^* is a vectorial subspace of $C(Y,F)$ and $\mathcal{M}$ is directed by inclusion. Now, we consider the function $\Phi : (G, \mathcal{L}_u) \to (G^*, \mathcal{M}_u)$ such that $\Phi(f) = h_f$. Obviously Φ is a topological isomorphism. This proves part (a). Now, since continuous images of compact sets are also compact, part (b) follows from Theorem 7. □

Finally we give an example of an application of the previous theorem. Let $\mathcal{L}$ be a cover of a Tychonoff space T. We say that T is an $\mathcal{L}_f$-space if, for every real-valued discontinuous function f on T, there exists an $A \in \mathcal{L}$ such that $f_{|A}$ does not admit a continuous extension to the whole T. These spaces were introduced by Blanchard-Jourlin in [**4**] and by Arkhangels'kiĭ in [**1**]. The $\mathcal{L}_f$-spaces arise in different contexts, for instance:

(a) completion of spaces $(C(T), \mathcal{L}_u)$ [**4**],

(b) z-closed projections [**11**],

(c) study of equalities $\gamma(X \times Y) = \gamma X \times \gamma Y$, $v(X \times Y) = vX \times vY$ [**13, 14**],

(d) compactness in $C_p(T)$ [**1**].

This class of spaces includes locally pseudocompact spaces and k_r-spaces, spaces X in which a real-valued function is continuous if its restriction to every compact subset of X is continuous.

An easy consequence of the definition is the following lemma.

Lemma 9. *Let T be a Tychonoff space and $\mathcal{L}$ a cover directed by inclusion of functionally bounded subset of T. Let T be an $\mathcal{L}_f$-space and we take Y and $\mathcal{M}$ as in the above theorem. A function from Y into $\mathbf{R}$ is continuous if and only if its restriction to every member of $\mathcal{M}$ is continuous.*

Proof. Let f be a real-valued function on Y such that $f_{|\mathrm{cl}_{\gamma T} A}$ is continuous for every $A \in \mathcal{L}$. Given $A \in \mathcal{L}$, since $\mathrm{cl}_{\gamma T} A$ is compact, there exists a continuous function $g \in C(Y)$ such that $g_{|A} = f_{|A}$ [**8**, 3.11(c)]. Therefore, the function $f_{|A}$ admits a continuous extension to the whole T. The space T being an $\mathcal{L}_f$-space, the function $f_{|T}$ is continuous. By density, the restriction of $f_{|T}^{\gamma}$ to each element of $\mathcal{M}$ agrees with f. Thus, f is also continuous. □

It follows from Lemma 9 and [**12**, Theorem 4.2] that, if T is an $\mathcal{L}_f$-space and the members of $\mathcal{L}$ are functionally bounded, then the space $(G^*, \mathcal{M}_u)$ of Theorem 8 satisfies Ascoli's theorem (one can also obtain this result from Lemma 9 and a straightforward adaptation of Lemma 3.2.5 in [**10**]). This lets us apply the theorem above in order to obtain the following result:

Theorem 10. *Let T be a Tychonoff space and $\mathcal{L}$ a cover directed by inclusion of functionally bounded subsets of T. If T is an $\mathcal{L}_f$-space, the functional space $(C(T), \mathcal{L}_u)$ satisfies Ascoli's theorem.*

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