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ON THE PROFINITE TOPOLOGY OF THE AUTOMORPHISM GROUP OF A RESIDUALLY TORSION FREE NILPOTENT GROUP

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ABSTRACT. Let G be a finitely generated residually torsion-free nilpotent group. Then every polycyclic-by-finite subgroup H of $\operatorname{Aut} G$ is closed in the profinite topology on $\operatorname{Aut} G$.

1. It is known that, if G is a finitely generated (f.g.) residually finite ($\mathcal{RF}$) group, then its automorphism group $\operatorname{Aut} G$ is also $\mathcal{RF}$, cf. [**2**]. This gives a motive to ask if the subgroup separability of a group implies the subgroup separability of its automorphism group.

Although the automorphism group of a free group is not subgroup separable, see Proposition 1 below, we prove that polycyclic-by-finite groups of automorphisms of a residually torsion-free nilpotent group G are closed in the profinite topology on $\operatorname{Aut} G$.

2. The profinite topology on a group G is the topology in which a base for the open sets is the set of all cosets of normal subgroups of finite index in G.

A group is said to be subgroup separable if all of its f.g. subgroups are closed in the profinite topology on G or, equivalently, if any pair of distinct finitely generated subgroups of G may be mapped to distinct subgroups in some finite quotient of G. Note that G is $\mathcal{RF}$ if and only if the trivial subgroup is closed in the profinite topology on G.

Proposition 1. *Let $F = \langle a, b\rangle$ be the free group of rank 2. There exists a subgroup of* $\operatorname{Aut} F$ *which is not closed in the profinite topology on* $\operatorname{Aut} F$.

Proof. The group K with presentation $K = \langle t, x, y | t^{-1}xt =$

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$xy, t^{-1}yt = y\rangle$ is not subgroup separable, cf. [**4**]. It is not difficult to see that the group K is a subgroup of $\mathrm{Aut}\, F$ by the identification of x with τ_a, the inner automorphism induced by a, y with τ_b and t by the automorphism $\phi : (a \to ab, b \to b)$. But the subgroup separability is inherited by subgroups. Therefore, the $\mathrm{Aut}\, F$ is not subgroup separable. $\square$

Lemma 2. *Let G be a finitely generated residually torsion-free nilpotent group. Then there exists a central series $G = N_1 \geq N_2 \geq \cdots \geq N_i \geq \cdots$ with torsion-free factors, N_i, $i = 1, 2, \ldots$, characteristic in G and $\cap_i N_i = 1$.*

Proof. Let $G = G_1 \geq G_2 \geq \cdots \geq G_i \geq \cdots$ be the lower central series of G, for each i define $\mathcal{N}_i = \{R_{ij} \mid R_{ij} \triangleleft G, G_i \triangleleft R_{ij}$ and G/R_{ij} is torsion free$\}$. Let $N_i = \cap\{R_{ij} \in \mathcal{N}_i\}$. It is easy to see that, for every $i = 1, 2, \ldots$, the subgroups N_i are characteristic in G. Indeed, for $\alpha \in \mathrm{Aut}\, G$ we have that $G_i = G_i\alpha \leq N_i\alpha$ and $G/N_i\alpha$ is torsion-free. So $N_i \leq N_i\alpha$ and by symmetry $N_i = N_i\alpha$. Also, by the definition of the family $\mathcal{N}_i$, we have that each factor G/N_i, $i = 1, 2, \ldots$, is torsion-free.

The series $G = N_1 \geq N_2 \geq \cdots \geq N_i \geq \cdots$ is central. For it, we have $[G_iN_{i+1}, G] = [G_i, G][N_{i+1}, G] \leq N_{i+1}$. Therefore, $(G_iN_{i+1})/N_{i+1} \leq Z(G/N_{i+1}) = Z_i/N_{i+1}$. So $G_i \leq Z_i$. But the group $G/Z_i \approx (G/N_{i+1})/Z(G/N_{i+1})$ is torsion-free, whence by the definition of N_i we have $N_i \leq Z_i$, which means that $[N_i, N_{i+1}] \leq N_{i+1}$. So the series $G = N_1 \geq N_2 \geq \cdots$ is central. Moreover, since G is residually torsion-free nilpotent, it is clear that $\cap_i N_i = 1$. $\square$

Let G be a finitely generated residually torsion-free nilpotent group and $G = N_1 \geq N_2 \geq \cdots$ the central series defined in the previous lemma. Since the terms N_i, $i = 1, 2, \ldots$, are characteristic in G, there exist homomorphisms $\phi_i : \mathrm{Aut}\, G \to \mathrm{Aut}\,(G/N_{i+1})$, $i = 1, 2, \ldots$, and $f_i : (\mathrm{Aut}\, G)\phi_i \to \mathrm{Aut}\,(G/N_i)$ with $(\sigma\phi_i)f_i = \sigma\phi_{i-1}$ for every $\sigma \in \mathrm{Aut}\, G$. Let K_i be the kernel of ϕ_i, $i = 1, 2, \ldots$. Since $\phi_i \circ f_i = \phi_{i-1}$, it is easy to see that K_{i-1}/K_i is isomorphic to a subgroup of $\mathrm{Ker}\, f_i$, $i = 1, 2, \ldots$. Let α be an element of $\mathrm{Ker}\, f_i$. Then, for every $gN_{i+1} \in G/N_{i+1}$, we have $(gN_{i+1})\alpha = gcN_{i+1}$ for some $c = c(g) \in N_i$. Moreover, the group $\mathrm{Ker}\, f_i$ induces the identity on $G/N_2 \approx (G/N_{i+1})/(N_2/N_{i+1})$ for $i = 2, 3, \ldots$. Therefore, by Proposition 3 [**5**, page 49] $\operatorname{Ker} f_i$ induces the identity on every factor $(N_j/N_{i+1})/(N_{j+1}/N_{i+1}) \approx N_j/N_{j+1}$, $j = 1, \ldots, i$. Whence the restriction of α on N_i/N_{i+1} is the identity. Consequently, we have, for an element α of $\operatorname{Ker} f_i$, $(gG_{i+1})\alpha^n = g \cdot c^n G_{i+1}$. It gives that the torsion freeness of G_i/G_{i+1} implies the torsion freeness of K_i/K_{i+1}.

Lemma 3. *Let G be a finitely generated group. Then, for every soluble subgroup H of $\operatorname{Aut} G$, we have $\cap_{N \in \mathcal{N}} HN \leq \cap_i HK_i$, where $\mathcal{N} = \{N \mid N \triangleleft \operatorname{Aut} G$ with N of finite index in $\operatorname{Aut} G\}$.*

Proof. For every $i = 1, 2, \ldots$, we have that $H/H \cap K_i \approx HK_i/K_i \leq \operatorname{Aut} G/K_i$. So the group $H/(H \cap K_i)$ is a polycyclic-by-finite subgroup of $\operatorname{Aut} G/K_i \approx (\operatorname{Aut} G)\phi_i \leq \operatorname{Aut}(G/G_{i+1})$. But $\operatorname{Aut}(G/G_{i+1}) \leq GL(n, Z)$ for some $n \in Z^+$, cf. [**5**, Theorem 6, p. 96]. Therefore, HK_i/K_i is closed in the profinite topology on $\operatorname{Aut} G/K_i$, cf. [**5**, Theorem 5, p. 61]. Namely, $\cap_{N \in \mathcal{N}}(HK_i/K_i)\cdot(NK_i/K_i) = HK_i/K_i$, whence $\cap_{N \in \mathcal{N}} HN \leq HK_i$ for every $i = 1, 2, \ldots$, and finally $\cap_{N \in \mathcal{N}} HN \leq \cap_i HK_i$. □

Theorem 4. *Let G be a finitely generated residually torsion-free nilpotent group. Then every polycyclic-by-finite subgroup H of $\operatorname{Aut} G$ is closed in the profinite topology on $\operatorname{Aut} G$.*

Proof. Let H be a polycyclic-by-finite subgroup of $\operatorname{Aut} G$. By the previous lemma it is enough to prove $\cap_i HK_i \leq H$. Since H is polycyclic-by-finite, there exist a normal subgroup N of H of finite index in H and a normal series $N = N_1 \triangleright N_2 \triangleright \cdots \triangleright N_{m+1} = 1$ of N such that N_j/N_{j+1} is an infinite cyclic group for $j = 1, \ldots, m$. If $r \cdot N_2$ is a generator of N_1/N_2, then there exists a term K_i of the normal series $\operatorname{Aut} G = K_0 \triangleright K_1 \triangleright K_2 \triangleright \cdots$ such that $r \in K_i$ but $r \notin K_{i+1}$ ($\cap_i K_i = 1$ since $\cap_i G_i = 1$), cf. [**1**, Theorem 1.2]. Suppose that $i \geq 1$; then the group K_i/K_{i+1} is torsion-free because G_i/G_{i+1} is torsion-free. So $\langle r \rangle \cap K_{i+1} = 1$ and the Hirsch number $h(N_1)$ of N_1 is strictly greater than $h(N_1 \cap K_{i+1})$. Now for the group $\overline{N}_1 = N_1 \cap K_{i+1}$ we can find a term K_{r+1} of the series $K_1 \triangleright K_2 \triangleright \cdots$ such that $h(\overline{N}_1) > h(\overline{N}_1 \cap K_{r+1})$. Therefore, after many finite steps, we can find a subgroup K_n such that $N \cap K_n$ is finite. This means that $H \cap K_n$ is finite. But then, since K_n/K_{n+1} is torsion-free, we have that $H \cap K_n \leq K_{n+1}$. Whence $H \cap K_n \subseteq \cap_i K_i = 1$. If $r \in K_0$ but $r \notin K_1$, then since K_0/K_1 is torsion-free by finite, we can suppose that $\langle r^m \rangle \cap K_2 = 1$ for some $m \geq 1$. But then $\langle r \rangle \cap K_2 = 1$, since r is of infinite order. So the argument above can be adjusted.

Now let $x \in \cap_i HK_i$. Then, for each $i \in \mathbf{N}$, we have $x = h_i k_i$, $h_i \in H$, $k_i \in K_i$. For $i \geq n$, we have $h_i k_i = h_n k_n$, whence $h_n^{-1} h_i = k_n k_i^{-1} \in H \cap K_n = 1$, namely, $k_n = k_i$ for every $i \geq n$, which means that $k_n \in \cap_i K_i = 1$. So $x = h_n \in H$, and we have finished. □

Corollary 4.1. *Let F be an* f.g.*-free group. Then every polycyclic-by-finite subgroup H of* Aut G *is closed in the profinite topology on* Aut G.

Corollary 4.2. *Polycyclic-by-finite subgroups of the automorphism group of a surface group G are closed in the profinite topology on* Aut G.

Proof. It is known, cf. [**3**], that a surface group G is residually torsion-free nilpotent. □

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