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INCLUSION THEOREMS FOR CONVOLUTION PRODUCT OF SECOND ORDER POLYLOGARITHMS AND FUNCTIONS WITH THE DERIVATIVE IN A HALFPLANE

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ABSTRACT. For $\beta < 1$ and real η, let $\mathcal{R}_\eta(\beta)$ denote the family of normalized analytic functions f defined in the unit disc Δ such that $\operatorname{Re}[e^{i\eta}(f'(z)-\beta)] > 0$ for $z \in \Delta$. Given a generalized second order polylogarithm function

$$G(a,b;z) = \sum_{n=1}^{\infty} \frac{(a+1)(b+1)}{(n+a)(n+b)} z^n,$$
$$a, b \in \mathbf{C}\backslash\{-1,-2,-3,\cdots\},$$

we place conditions on the parameters a, b and β to guarantee that the Hadamard product of the power series $G(a,b;z)*f(z)$ will be univalent, starlike or convex. We also give conditions on a and b to describe the geometric nature of the function $G(a,b;z)$. By taking f in the class of convex functions, we also find a sufficient condition for $G(a,b;z)*f(z)$ to belong to the class $\mathcal{R}_0(\beta)$. Several open problems have been raised at the end.

1. Introduction and main results. Let $\mathbf{C}$ denote the complex plane, and let $\Delta = \{z \in \mathbf{C} : |z| < 1\}$. Denote by $\mathcal{H}$ the linear space of all functions f analytic in Δ, endowed with the usual topology of uniform convergence on compact subsets and by $\mathcal{A}$ the subset of $\mathcal{H}$ with the normalization $f(0) = 0 = f'(0) - 1$. We say that the function $f \in \mathcal{A}$ is *convex* (denoted by $f \in \mathcal{K}$) if f maps Δ onto a convex domain. The function $f \in \mathcal{A}$ is said to be *starlike* (denoted by $f \in \mathcal{S}^*$) if f maps Δ onto a domain which is starlike with respect to the origin. Denote by $\mathcal{S}$, $\mathcal{C}(\beta)$, $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$, the subsets consisting of functions in $\mathcal{A}$, which are, respectively, univalent, *close-to-convex of order* β, *starlike* (with respect to the origin) *of order* β and *convex of order* β, where

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$\beta < 1$. It is well known that $f(z) \in \mathcal{K}(\beta)$ if and only if $zf'(z) \in \mathcal{S}^*(\beta)$. For $\beta = 0$, we usually set $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$. Further, for $\beta < 1$, we introduce

$$\mathcal{R}_\eta(\beta) = \{f \in \mathcal{A} : \ \exists\, \eta \in \mathbf{R} \text{ s.t. } \mathrm{Re}\,[e^{i\eta}(f'(z) - \beta)] > 0, z \in \Delta\}.$$

It is well known that $\mathcal{R}_\eta(\beta) \subset \mathcal{S}$ for $0 \le \beta < 1$. If $\eta = 0$, then we denote $\mathcal{R}_0(\beta)$ simply by $\mathcal{R}(\beta)$. A standard analytic definition for close-to-convex functions states that the function $f \in \mathcal{A}$ is said to be *close-to-convex of order* $\beta < 1$ *with respect to a fixed starlike function* g if and only if

$$\mathrm{Re}\left[e^{i\eta}\left(\frac{zf'(z)}{g(z)} - \beta\right)\right] > 0, \quad z \in \Delta, \tag{1.1}$$

for some real $\eta \in (-\pi/2, \pi/2)$. The family of close-to-convex functions of order β with relative to $g \in \mathcal{S}^*$ is denoted by $\mathcal{C}_\eta(\beta; g)$. If $\eta = 0$, we simply denote it by $\mathcal{C}(\beta; g)$. Thus, we remark that the usual class of all close-to-convex functions of order β, denoted by $\mathcal{C}(\beta)$, is the set $\{\mathcal{C}_\eta(\beta; g) : g \in \mathcal{S}^*\}$. Clearly $\mathcal{C}_\eta(\beta; z) \equiv \mathcal{R}_\eta(\beta)$ for $\beta < 1$. Set $\mathcal{C}(0) = \mathcal{C}$. It is important to note that $\mathcal{K}(\beta) \subset \mathcal{S}^*(\beta) \subset \mathcal{C}(\beta) \subset \mathcal{S}$ for $0 \le \beta < 1$. For general properties of these classes of functions, we refer to the book by Pommerenke [**27**] and Goodman [**11**]. All of the inequalities in this paper involving functions of z, such as (1.1), hold uniformly in Δ. The condition "for all $z \in \Delta$" will be omitted in the remainder of the paper, although it is understood to hold.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be formal Maclaurin series. Then the *Hadamard product*, or *convolution*, of f and g is defined by the power series $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$. The *modified Hadamard product*, or *integral convolution*, is defined as

$$\begin{aligned} f \otimes g = \int_0^z \frac{(f * g)(t)}{t}\, dt &= \sum_{n=1}^{\infty} \frac{a_n b_n}{n} z^n \\ &= f(z) * g(z) * (-\log(1 - z)). \end{aligned}$$

We shall use the notation $A * B$ $(A \otimes B)$, where A and B denote two subsets of $\mathcal{H}$, to denote the set of all functions of the form $f * g$ $(f \otimes g)$ where $f \in A$ and $g \in B$.

Let a and b be two complex numbers such that both a and b assume no negative integer values and

$$A_n = \frac{(a+1)(b+1)}{(n+a)(n+b)}, \quad \text{for } n \geq 2. \tag{1.2}$$

We are concerned with certain properties of the function

$$G(a,b;z) = \sum_{n=1}^{\infty} A_n z^n, \tag{1.3}$$

where $A_1 = 1$ and A_n is defined by (1.2). If a or b equals -1, then we have $G(a,b;z) = z$ and therefore this case may be excluded. Clearly,

$$G(0,1;z) = 2\left[1 + \frac{1-z}{z}\log(1-z)\right].$$

For an extensive list of the special cases of the function $G(a,b;z)$, see [**12**].

In this section, we discuss general problems involving the geometric properties of the function $G(a,b;z)$, and properties for the convolution of $G(a,b;z)$ and functions with the derivative in a halfplane. More precisely, we find simple conditions on β and β' under which if $f \in \mathcal{R}_\eta(\beta)$ then the function $G(a,b;z) * f(z)$ is in $\mathcal{R}_\eta(\beta')$, $\mathcal{C}(\beta';g)$, $\mathcal{S}^*(\beta')$ or $\mathcal{K}(\beta')$. The key lemmas and their proofs will be presented in Section 2 and the proofs of the main results will be given in Section 3. In Section 4 we discuss some open problems related to the theory of convolution and to certain integral transforms connected to the classes discussed in this paper.

Now, we start discussing some facts which are needed in the sequel, and in the main results of this paper. Suppose that $f \in \mathcal{A}$. Then it is easy to verify that the function $H_f(a,b;z) \equiv G(a,b;z) * f(z)$ satisfies the differential equation

$$z^2 H_f''(z) + (a+b+1)zH_f'(z) + abH_f(z) = (a+1)(b+1)f(z). \tag{1.4}$$

Throughout the paper, the function $G(a,b;z)$ denotes the function defined by (1.3) and $H_f(a,b;z)$ stands for the Hadamard product $G(a,b;z) * f(z)$. Using these notations and a simple calculation with the power series of f and $G(a,b;z)$, we easily obtain that the function $H_f(a,b;z)$ for $a\neq b$ has the integral representation

$$H_f(a,b;z) := \frac{(a+1)(b+1)}{b-a}\int_0^1 t^{a-1}(1-t^{b-a})f(tz)\,dt, \quad \text{if } b\neq a.$$

Now if we choose $a=-\alpha$ and $b=2-\alpha$ in the above formula, then we have

$$\begin{aligned} F_\alpha(z) &\equiv H_f(-\alpha,2-\alpha;z) \\ &= \frac{(1-\alpha)(3-\alpha)}{2}\int_0^1 t^{-(\alpha+1)}(1-t^2)f(tz)\,dt, \end{aligned} \tag{1.5}$$

an operator considered in [**1**, Corollary 1] with an additional assumption that $0\le\alpha<1$, see Corollary 1.18. In the limiting case $b\to\infty$, the operator $H_f(a,b;z)$ reduces to

$$H_f(a,\infty;z) = B_f(a;z) := \frac{a+1}{z^a}\int_0^z t^{a-1}f(t)\,dt, \quad \operatorname{Re} a > -1, \tag{1.6}$$

which is the well-known Bernardi transform of f. Therefore, allowing $b\to\infty$ we see that the corresponding differential equation (1.4) becomes

$$zB_f'(z) + aB_f(z) = (a+1)f(z)$$

and the interaction of f and $B_f(z)$ in terms of geometric function theory has been studied on several occasions, see [**6, 40, 25, 28, 29, 30, 9**].

Let $\Phi_p(a;z)$ denote the well-known generalization of the Riemann zeta and polylogarithm functions, or simply the pth order polylogarithm function, given by

$$\Phi_p(a;z) = \sum_{n=1}^{\infty}\frac{z^n}{(n+a)^p},$$

where any term with $n+a=0$ is excluded, see Lerch [**16**] and also [**4**, Sections 1.10 and 1.12]. Using the definition of the Gamma function [**4**, p. 27] a simple transformation produces the integral formula

$$\Phi_p(a;z) = \frac{1}{\Gamma(p)}\int_0^1 z(\log 1/t)^{p-1}\frac{t^a}{1-tz}\,dt, \quad \operatorname{Re} a > -1 \quad \text{and} \quad \operatorname{Re} p > 1,$$

from which one gets the integral representation for the function $H_f(a,b;z)$ when $a=b$. Thus, we note that if we take $a=b$ in (1.3) then we have

$$G(a,a;z)=\sum_{n=1}^{\infty}\frac{(a+1)^2}{(n+a)^2}z^n,$$

which is equivalent to $(a+1)^2\Phi_2(a;z)$. Further, if $a=b=0$ in (1.3) then we get the function $G(0,0;z)$ which is the well-known dilogarithm function $\mathrm{Li}_2(z)$, defined by

$$\mathrm{Li}_2(z)=\int_0^1 z(\log 1/t)\frac{dt}{1-tz}=\sum_{n=1}^{\infty}\frac{z^n}{n^2}$$

and therefore, we easily have the integral representation

$$\mathrm{Li}_2(z)*f(z)=\int_0^1\frac{1}{t}(\log 1/t)f(tz)\,dt.$$

The pth order polylogarithm function can be used to express many sums of the reciprocal powers and to evaluate Dirichlet L' series which appears in number theory, see, for example, [**12**]. The basic reference is Lewin's book, [**17**], and we remark that the dilogarithm function, which can also be obtained as a limiting case of the Gaussian hypergeometric function

$$\mathrm{Li}_2(z)=\lim_{a\to 0}\frac{{}_2F_1(a,a;1;z)-1}{a^2},$$

has many interesting applications in mathematical physics. We mention that among the special cases of the Gaussian hypergeometric series (function)

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)(1,n)}z^n,$$

with $c\neq -1,-2,\dots$, $(a,n)=a(a+1)\cdots(a+n-1)$ for $n=1,2,\dots$ and $(a,0)=1$ for $a\neq 0$, are several generic classes of functions such as Chebyshev, Legendre, Gegenbauer, and Jacobi polynomials, so that results about ${}_2F_1(a,b;c;z)$ lead to interesting conclusions concerning these classes of functions, see [**34, 35, 38**]. In a recent paper, Jones [**14**] discussed the valence property of $\Phi_p(0;z)$ for $p<-1$. In this paper we focus our attention only for the generalized second order polylogarithm function defined by $G(a,b;z)$ and study the properties of the function $G(a,b;z)$ and its various inclusion results for the Hadamard product $f(z)*G(a,b;z)$, particularly when $f\in\mathcal{R}_\eta(\beta)$. Further, we also show that under our conditions on a and b, the function $G(a,b;z)*f(z)$ belongs to $\mathcal{H}^\infty$, the class of all bounded analytic functions in Δ, see Proposition 1.8. Thus, the conclusion of our theorems of this paper become stronger although we do not include this fact in the statement of our main theorems.

We start with some auxiliary statements. However, later in Theorem 1.20, we obtain a sufficient condition for $H_f(a,b;z)$ to belong to $\mathcal{R}(\beta)$ when f is in the class of convex functions.

Proposition 1.7. *If* $\mathrm{Re}\,a>-1$ *and* $\mathrm{Re}\ b>-1$, *then we have*

$$f\in\mathcal{K}\,(\mathcal{C}\ \text{or}\ \mathcal{S}^*)\quad\Longrightarrow\quad H_f(a,b;z)\in\mathcal{K}\,(\mathcal{C}\ \text{or}\ \mathcal{S}^*).$$

Proof. First we recall that the function

$$(1+a)\Phi_1(a;z)=\sum_{n=1}^{\infty}\frac{1+a}{n+a}z^n=z({}_2F_1(1,a+1;a+2;z))$$

is convex in Δ for $\mathrm{Re}\,a>-1$, see, for example, [**39**], and therefore from the fact [**41**] that $\mathcal{S}^*\otimes\mathcal{S}^*\subset\mathcal{S}^*$, we deduce that the function $G(a,b;z)$ defined by (1.3) is convex for $\mathrm{Re}\,a>-1$ and $\mathrm{Re}\,b>-1$. The last fact and the Pólya-Schoenberg conjecture proved in [**41**] immediately yield the required implication. $\square$

Proposition 1.8. *Suppose that* a,b *are related by any one of the following*:

(1) $\mathrm{Re}\,a>-2$ *and* $\mathrm{Re}\,b>-2$,

(2) $a\in\mathbf{C}$ *is such that* a *assumes no negative integer values and* $b=\bar{a}$.

If $f\in\mathcal{R}_\eta(\beta)$ *or* $\mathcal{K}$, *then the function* $H_f(a,b;z)$ *is in* $\mathcal{H}^\infty$.

Proof. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{R}_\eta(\beta)$. Then as in [**20**] we easily find that

$$|a_n|\le\frac{2(1-\beta)\cos\eta}{n}. \tag{1.9}$$

First we assume that $\operatorname{Re} a > -2$ and $\operatorname{Re} b > -2$. Consider $H_f(a,b;z) = G(a,b;z) * f(z)$, where G is defined by (1.3). Applying (1.9) and (1.3) we obtain

$$\begin{aligned}|H_f(a,b;z)| &\le |z| + |(a+1)(b+1)| \sum_{n=2}^{\infty} \frac{1}{|(n+a)(n+b)|} |a_n||z|^n \\ &\le 1 + 2(1-\beta)\cos\eta |(a+1)(b+1)| \\ &\quad \cdot \sum_{n=2}^{\infty} \frac{1}{n|(n+a)(n+b)|} \\ &< \infty, \qquad |z| \le 1;\ \operatorname{Re} a > -2, \quad \operatorname{Re}\ b > -2.\end{aligned}$$

The above observation shows that if f is in the class $\mathcal{R}_\eta(\beta)$ then the function $H_f(a,b;z)$ is in $\mathcal{H}^\infty$ whenever $\operatorname{Re} a > -2$ and $\operatorname{Re} b > -2$.

If $f \in \mathcal{K}$, then it is known that $|a_n| \le 1$ for $n \ge 2$ and therefore using this estimate we have that the function $H_f(a,b;z)$ is in $\mathcal{H}^\infty$ whenever $\operatorname{Re} a > -2$ and $\operatorname{Re} b > -2$.

The second part follows similarly. □

Now we state some results which give the geometric properties of the function $G(a,b;z)$.

Theorem 1.10. *Let a, b satisfy either $a, b > -1$, or $a, b \in (-2,-1)$. Let A_n be defined by* (1.2)*. Then, for $ab \le 2$, the function $G(a,b;z) = \sum_{n=1}^{\infty} A_n z^n$ is close-to-convex with respect to $-\log(1-z)$.*

Taking $a = -\alpha$ and $b = 2 - \alpha$ in Theorem 1.10 we easily have the following result.

Corollary 1.11. *For $\alpha \in [-\sqrt{3}+1, 1)$, the function $G(-\alpha, 2-\alpha; z)$ is close-to-convex with respect to $-\log(1-z)$.*

We state our next result which gives a condition for $G(a,b;z)$ to be in $\mathcal{S}^* \cap \mathcal{R}(1/2)$.

Theorem 1.12. *Let a, b satisfy either $a, b > -1$, or $a, b \in (-2,-1)$. Let A_n be defined by* (1.2)*. If a and b are such that $ab \le 6/(6+a+b)$,* *then the function* $G(a,b;z)=\sum_{n=1}^{\infty}A_nz^n$ *belongs to* $\mathcal{S}^*\cap\mathcal{R}(1/2)$ *and is also close-to-convex with respect to* $-\log(1-z)$.

Taking $a=0$ we see that if $b>-2$ then the function $G(0,b;z)$ is in $\mathcal{S}^*\cap\mathcal{R}(1/2)$ and is also close-to-convex with respect to $-\log(1-z)$.

Our next theorem considers the situation when $a\in\mathbf{C}\backslash\{-1,-2,\dots\}$ and $b=\bar{a}$.

Theorem 1.13. *For* $a\in\mathbf{C}\backslash\{-1,-2,\dots\}$, *let*

$$G(a,\bar{a};z)=z+\sum_{n=2}^{\infty}\frac{|a+1|^2}{|n+a|^2}z^n.$$

(1) *If* $|a|\le\sqrt{2}$, *then the function* $G(a,\bar{a};z)$ *is close-to-convex with respect to* $-\log(1-z)$.

(2) *If* $|a|\le\sqrt{11/3}$ *and* $|a|^2(3+\operatorname{Re}a)\le 3$, *then the function* $G(a,\bar{a};z)$ *belongs to* $\mathcal{R}(1/2)$.

(3) *If* $|a|\le\sqrt{2}$ *and* $|a|^2(3+\operatorname{Re}a)\le 3$, *then the function* $G(a,\bar{a};z)$ *belongs to* $\mathcal{S}^*\cap\mathcal{R}(1/2)$.

Definition 1.14. Let $\mathcal{I}_f$ be an operator acting on the function f. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are two subclasses of $\mathcal{A}$. We say that a class $\mathcal{F}_1$ is $\mathcal{F}_2$-*admissible* with respect to the operator $\mathcal{I}_f$ if

$$f\in\mathcal{F}_1\quad\Longrightarrow\quad\mathcal{I}_f\in\mathcal{F}_2.$$

If $\mathcal{F}_1$ is $\mathcal{F}_1$-admissible then we call $\mathcal{I}_f$, a *class preserving* operator.

There are several theorems which are related to Definition 1.14, and most of them deal with certain well-known operators in geometric function theory, for example, the Bernardi operator defined by (1.6) and its various generalizations [**25**]. We consider the so-called Alexander transform Λ_f of f defined by

$$\Lambda_f(z)=\int_0^1\frac{f(tz)}{t}\,dt=f*(-\log(1-z)). \tag{1.15}$$

Note also that $\mathrm{Li}_2(z) = \Lambda_f(z)$, $f(z) = z[{}_2F_1(1,1;2;z)]$. The Alexander transform, which is the starting point for the study of many other general operators in the theory of univalent functions, gives a one-to-one correspondence between the families $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$. Now we state our next result.

Theorem 1.16. *For $\beta < 1$, let $\mathcal{R}_\eta(\beta)$ be $\mathcal{S}^*(\gamma)$-admissible for the Alexander transform Λ_f defined by* (1.15). *Let $f \in \mathcal{R}_\eta(\beta)$, $\beta' = \beta[2 - 2\log 2] + 2\log 2 - 1$ and $H_f(a, b; z) = f(z) * G(a, b; z)$.*

(1) *Let a and b satisfy either $a, b > -1$ or $a, b \in (-2, -1)$. If $ab \le 6/(6 + a + b)$, then the function $H_f(a, b; z)$ is in $\mathcal{S}^*(\gamma) \cap \mathcal{R}_\eta(\beta')$.*

(2) *If $a \in \mathbf{C}$ is such that $|a| \le \sqrt{2}$ and $|a|^2(3 + \mathrm{Re}\, a) \le 3$, then the function $H_f(a, \bar{a}; z)$ is in $\mathcal{S}^*(\gamma) \cap \mathcal{R}_\eta(\beta')$.*

Corollary 1.17. *Suppose that β_j and β_j', $j = 0, 1, 2$, satisfy any one of the following relations*:

(i) $\beta_0 = (1 - 2\log 2)/(2 - 2\log 2) \approx -0.629$ *and* $\beta_0' = \gamma_1 = 0$;

(ii) $\beta_1 = -(2 - \sqrt{3})(2\log 2 - 1)/(2 - (2 - \sqrt{3})(2\log 2 - 1)) \approx -0.054$ *with* $\gamma_1 \approx 0.409$ *and* $\beta_1' = \beta_1[2 - 2\log 2] + 2\log 2 - 1 \approx 0.353$;

(iii) $\beta_2 = -(2\log 2 - 1)/(3 - 2\log 2) \approx -0.239$ *with* $\gamma_2 \approx 0.083$ *and* $\beta_2' = \beta_2[2 - 2\log 2] + 2\log 2 - 1 = -\beta_2 \approx 0.239$.

Suppose that $\alpha < 1$ satisfies the condition

$$\alpha^3 - 6\alpha^2 + 8\alpha + 8 \ge 0,$$

and the function F_α is defined by (1.5). *Then we have*

(a) $f \in \mathcal{R}_\eta(\beta_0) \Rightarrow F_\alpha \in \mathcal{S}^* \cap \mathcal{R}_\eta(0)$,

(b) $f \in \mathcal{R}(\beta_1) \Rightarrow F_\alpha \in \mathcal{S}^*(\gamma_1) \cap \mathcal{R}(\beta_1')$,

(c) $f \in \mathcal{R}(\beta_2) \Rightarrow F_\alpha \in \mathcal{S}^*(\gamma_2) \cap \mathcal{R}(\beta_2')$.

We include the following result, which applies only to the case $0 \le \alpha < 1$, obtained by Ali and Singh [**1**].

Corollary 1.18 [**1**, Corollary 1]. *Let* $0 \le \alpha < 1$ *and* β_α *be defined by*

$$\frac{\beta_\alpha - 1/2}{1-\beta_\alpha} = -\frac{(1-\alpha)(3-\alpha)}{2}\int_0^1 t^{-\alpha}\frac{1-t}{1+t}\,dt.$$

Then, for $f \in \mathcal{R}_\eta(\beta)$, *the function* $F_\alpha(z)$ *defined by* (1.5) *is convex if* $\beta \ge \beta_\alpha$. *For* $\beta < \beta_\alpha$, F_α *need not be convex.*

We compare the above two corollaries. For example, if we choose $\alpha = 0$ in Corollary 1.18 then, after some computation, we get that the function F_0 is convex if $\beta \ge \beta_0 = (4-6\log 2)/(5-6\log 2) \approx -0.188$. For $\beta < \beta_0 \approx -0.188$, we see from Corollary 1.18 that the function F_0 need not be convex. On the other hand, Corollary 1.17 (a) shows that if $\beta \ge (1-2\log 2)/(2-2\log 2) \approx -0.629$ then the function F_0 not only is starlike but also is in $\mathcal{R}_\eta(0)$. We remark that a convex function need not be in $\mathcal{R}_\eta(0)$ and conversely. Further, in the starlike case, the range for α can be extended as is clear from Corollary 1.17.

By taking $a = b = 0$ in Theorem 1.16, we have the following example which is related to the dilogarithm function $\mathrm{Li}_2(z)$. The proof of this example follows from the idea of the proof of Corollary 1.17.

Example 1.19. For $\beta < 1$, let $\mathcal{R}_\eta(\beta)$ be $\mathcal{S}^*(\gamma)$-admissible for the Alexander transform Λ_f defined by (1.15). Then, for $f \in \mathcal{R}_\eta(\beta)$, the function $\mathrm{Li}_2(z) * f(z)$ is in $\mathcal{S}^*(\gamma) \cap \mathcal{R}_\eta(\beta')$. In particular, we have the following:

(a) $f \in \mathcal{R}_\eta(\beta_0) \Rightarrow \mathrm{Li}_2(z) * f(z) \in \mathcal{S}^* \cap \mathcal{R}_\eta(0)$,

(b) $f \in \mathcal{R}(\beta_1) \Rightarrow \mathrm{Li}_2(z) * f(z) \in \mathcal{S}^*(\gamma_1) \cap \mathcal{R}(\beta_1')$,

(c) $f \in \mathcal{R}(\beta_2) \Rightarrow \mathrm{Li}_2(z) * f(z) \in \mathcal{S}^*(\gamma_2) \cap \mathcal{R}(\beta_2')$,

where β_j, β_j' and γ_j, $j = 0, 1, 2$, are as in Corollary 1.17.

We note that in Corollary 1.17 and in Example 1.19, the constants β_j, β_j' and γ_j are all independent of the choice of the parameter α. Here it is interesting to make the following remark. Clearly $\mathcal{K} \not\subset \mathcal{R}(0)$ as the convex function $z/(1-z)$ demonstrates. However, it is well known that the inclusion $\mathcal{K}(1/2) \subset \mathcal{R}(1/2)$ holds while the converse is not true. In fact, MacGregor [**21**, Theorem 2] has shown that the best possible radius of convexity for functions in $\mathcal{R}(1/2)$ is $1/\sqrt{2}$. Further, if $\mathrm{Re}\, f'(z) > 0$ in Δ then f is convex in $|z| < r$ for $r \le \sqrt{\sqrt{2}-1}$

and not in any larger disc, see [**20**, Theorem 2] and [**42**]. Because of these observations, we let $f \in \mathcal{K}$ and determine a simple condition on the parameters so that $H_f(a,b;z)$ will be in $\mathcal{R}(\beta)$, compare with Proposition 1.7, as in

Theorem 1.20. *Suppose that a, b and β are related by any one of the following conditions*:

(i) *a, b satisfy either $a, b > -1$, or $a, b \in (-2,-1)$, and*

$$\beta = \frac{1}{2} + \frac{6 - ab(a+b+6)}{(2+a)(2+b)(3+a)(3+b)} < 1$$

such that $ab \leq 24/(9+a+b)$,

(ii) *$a \in \mathbf{C}\backslash\{-1,-2,\dots\}$, $b = \bar{a}$ such that $|a| \leq \sqrt{26/3}$, $|a|^2(9 + 2\mathrm{Re}\, a) \leq 24$ and*

$$\beta = \frac{1}{2} + \frac{6 - 2|a|^2(3+\, a)}{|(2+a)(3+a)|^2} < 1.$$

Then for $f \in \mathcal{K}$ we have $H_f(a,b;z)$ belongs to $\mathcal{R}(\beta)$.

Corollary 1.21. *Let $f \in \mathcal{K}$. Then we have $\mathrm{Li}_2(z) * f(z) \in \mathcal{K}(1/2) \cap \mathcal{R}(2/3)$. More generally, if $b \in (-1,\infty)$ then we have*

$$H_f(0,b;z) \in \mathcal{K}(1/2) \cap \mathcal{R}(\beta_0), \quad \textit{with } \beta_0 = \frac{1}{2} + \frac{1}{(2+b)(3+b)}.$$

Proof. Let $f \in \mathcal{K}$. On taking $a = b = 0$ in Theorem 1.20, we have $\mathrm{Li}_2(z) * f(z) \in \mathcal{R}(2/3)$. As the function $-\log(1-z)$ is convex of order $1/2$ and since the inclusion $\mathcal{K} * \mathcal{K}(1/2) \subset \mathcal{K}(1/2)$ holds, we deduce that $\mathrm{Li}_2(z) = (-\log(1-z)) * (-\log(1-z)) \in \mathcal{K}(1/2)$. Finally, because $f \in \mathcal{K}$, the conclusion follows by applying the last inclusion once again.

The general result for $b \in (-1,\infty)$ follows similarly and so we omit the details of the proof. □

We recall that the function $H_f(a,b;z)$ satisfies second order differential equation (1.4) and therefore we can make use of the method of differential subordination as a tool in the study of analytic properties of the function $H_f(a,b;z)$. For instance, since $\mathcal{K} \not\subset \mathcal{R}_\eta(0)$, it will be of interest to find a general condition on a and b so that, for $f \in \mathcal{R}_\eta(\beta)$, the function $H_f(a,b;z)$ will be at least in $\mathcal{R}_\eta(0)$ as in the following

Theorem 1.22. *Suppose that $a, b > -1$, $\delta < 1$ and $f \in \mathcal{A}$ satisfies the condition*

$$\text{Re}\left\{e^{i\eta}\left(\frac{f(z)}{z} - \delta_1\right)\right\} > 0 \tag{1.23}$$

where

$$\delta_1 = \delta - \frac{(1-\delta)(a+b+2)}{2(1+a)(1+b)}.$$

Let the function $G(a,b;z)$ be defined by (1.2) *and $H_f(a,b;z) = f(z) * G(a,b;z)$. Then we have*

$$\left\{e^{i\eta}\left(\frac{H_f(a,b;z)}{z} - \delta\right)\right\} > 0,$$

or, equivalently,

$$f \in \mathcal{R}_\eta(\delta_1) \quad \Longrightarrow \quad H_f(a,b;z) \in \mathcal{R}_\eta(\delta).$$

Taking $a = -\alpha$ and $b = 2 - \alpha$ in Theorem 1.22, we have the following

Example 1.24. Let $\alpha < 1$, $\delta < 1$, $f \in \mathcal{R}_\eta(\delta_1)$ with

$$\delta_1 = \delta - \frac{(1-\delta)(2-\alpha)}{(1-\alpha)(3-\alpha)},$$

and let F_α be defined by (1.5). Then, by Theorem 1.22, we see that the function $F_\alpha(z) \equiv H_f(-\alpha, 2-\alpha; z)$ is in $\mathcal{R}_\eta(\delta)$. In particular, if $\alpha < 1$ and β' is defined by

$$\beta' = -\frac{(2-\alpha)}{(1-\alpha)(3-\alpha)} \tag{1.25}$$

then for $f \in \mathcal{R}_\eta(\beta')$ the function F_α defined by (1.5) satisfies $\operatorname{Re}(e^{i\eta}F'_\alpha(z)) > 0$ in Δ, and hence $F_\alpha(z)$ is univalent in Δ. This example, in particular, shows that we have a simple condition which ensures the univalence of $F_\alpha(z)$ in Δ for any α such that $\alpha < 1$. Further, a simple calculation shows that β_α defined in Corollary 1.18 is such that $\beta_\alpha \geq \beta'$, where β' is defined by (1.25). Therefore, the range of α is enlarged and, since $\beta_\alpha \geq \beta'$, the close-to-convexity, with respect to $g(z) = z$, of F_α follows under a much weaker hypothesis.

Corollary 1.26. *Let $a > -1$, $\beta < (1-a)/2$, $\delta = 2/(5-4\beta) < 1$,*

$$\delta_1 = \delta - \frac{(1-\delta)(1-\beta)}{(1+a)(1-a-2\beta)} \tag{1.27}$$

and let $f \in \mathcal{A}$ satisfy the condition

$$\operatorname{Re}\left\{e^{i\eta}\left(\frac{f(z)}{z} - \delta_1\right)\right\} > 0.$$

Then we have

$$\operatorname{Re}\left\{e^{i\eta}\left(\frac{H_f(a, -a-2\beta; z)}{z} - \frac{2}{5-4\beta}\right)\right\} > 0.$$

Proof. The proof of this corollary follows from Theorem 1.22 by choosing $a + b = -2\beta$ and $\delta = 2/(5-4\beta)$. □

Our next result improves Corollary 1.26 under an additional condition on β and a.

Theorem 1.28. *For $a > -1$, let $\beta < (1-a)/2$ satisfy the condition*

$$4\beta^2 + \beta(4a-3) + 2a^2 + 1 \geq 0, \tag{1.29}$$

and let δ_1 be defined by (1.27) *with $\delta = 2/(5-4\beta) < 1$. If $f \in \mathcal{A}$ satisfies the condition*

$$\operatorname{Re}\left\{e^{i\eta}\left(\frac{f(z)}{z} - \delta_1\right)\right\} > 0$$

then we have $H_f(a, -a-2\beta; z) \in \mathcal{S}^*(\beta)$ *and*

$$\mathrm{Re}\left\{e^{i\eta}\left(\frac{H_f(a,-a-2\beta;z)}{z} - \frac{2}{5-4\beta}\right)\right\} > 0.$$

The following theorem deals with the case when the parameter a is a complex number and $b = \bar{a}$, the complex conjugate of a.

Theorem 1.30. *Let* $a \in \mathbf{C}$ *be such that* $\mathrm{Re}\ a > -1$, $\delta < 1$ *and*

$$\delta_1 = \delta - \frac{(1-\delta)(1+\mathrm{Re}\, a)}{|1+a|^2}. \tag{1.31}$$

If $f \in \mathcal{A}$ *satisfies the condition*

$$\mathrm{Re}\left\{e^{i\eta}\left(\frac{f(z)}{z} - \delta_1\right)\right\} > 0, \tag{1.32}$$

then we have

$$\mathrm{Re}\left\{e^{i\eta}\left(\frac{H_f(a,\bar{a};z)}{z} - \delta\right)\right\} > 0,$$

or equivalently,

$$f \in \mathcal{R}_\eta(\delta_1) \quad \Longrightarrow \quad H_f(a,\bar{a};z) \in \mathcal{R}_\eta(\delta).$$

Example 1.33. Let $\beta \geq -1$ and $f \in \mathcal{R}_\eta(\beta)$. Then the function

$$H_f(0,0;z) \equiv \mathrm{Li}_2(z) * f(z) = \int_0^1 \frac{1}{t}(\log 1/t) f(tz)\, dt$$

is in $\mathcal{R}_\eta(0)$, and hence the function $\mathrm{Li}_2(z) * f(z)$ is univalent, see Corollary 1.21.

We next improve Theorem 1.30 for certain values of a and δ.

Theorem 1.34. *Let* $a \in \mathbf{C}$ *be such that* $\mathrm{Re}\, a \geq -1/2$ *and satisfy the condition*

$$\sqrt{1+16(\mathrm{Im}\, a)^2} \leq 3 + 4\mathrm{Re}\, a. \tag{1.35}$$

Let $f \in \mathcal{A}$, $\delta = 2/(5 + 4\mathrm{Re}\, a)$, *and*

$$\delta_1 = \delta - \frac{(1-\delta)(1+\mathrm{Re}\, a)}{|1+a|^2}.$$

Then for

$$\mathrm{Re}\left\{ e^{i\eta}\left(\frac{f(z)}{z} - \delta_1\right)\right\} > 0$$

we have $H_f(a, \bar{a}; z) \in \mathcal{S}^*(-\mathrm{Re}\, a)$ *and*

$$\mathrm{Re}\left\{ e^{i\eta}\left(\frac{H_f(a, \bar{a}; z)}{z} - \delta\right)\right\} > 0.$$

Next we formulate some of the interesting special cases of our results in the form of examples.

Examples 1.36. (1) If $a = 0$ and $\beta < 1/2$, then we see that the condition (1.29) is obviously satisfied and therefore, from Theorem 1.28, we obtain that

$$H_f(0, -2\beta; z) \in \mathcal{K}(\beta) \cap \mathcal{R}_\eta(2/(5-4\beta))$$

if

$$\mathrm{Re}\,\{e^{i\eta} f'(z)\} > -\frac{[(2\beta - 3/4)^2 + 7/16]\cos\eta}{(5-4\beta)(1-2\beta)}.$$

In particular, for $f \in \mathcal{R}_\eta(\beta)$ with $\beta \geq -1/5$, the function

$$H_f(0, 0; z) \equiv \mathrm{Li}_2(z) * f(z) = \int_0^1 \frac{1}{t}(\log 1/t) f(tz)\, dt$$

is convex and also belongs to $\mathcal{R}_\eta(2/5)$, compare Example 1.33 and Corollary 1.21.

(2) Taking $\beta = 1/2$ in Theorem 1.28, we find that if $a \in (-1, 0)$, then we have

$$\mathrm{Re}\,\{e^{i\eta} f'(z)\} > \frac{(2a+1)^2 \cos\eta}{6a(a+1)} \quad \Longrightarrow \quad H_f(a, -a-1; z) \in \mathcal{K}(1/2) \cap \mathcal{R}_\eta(2/3).$$

(3) Taking $\beta = -1/2$ in Theorem 1.28 we obtain that if

$$\delta_1 = \frac{2}{7} - \frac{15}{14(1+a)(2-a)},$$

then

$$f \in \mathcal{R}_\eta(\delta_1) \quad \Longrightarrow \quad H_f(a, 2-a; z) \in \mathcal{K}(-1/2) \cap \mathcal{R}_\eta(2/7),$$

or equivalently,

$$f \in \mathcal{R}_\eta(\beta) \quad \Longrightarrow \quad H_f(-\alpha, 2-\alpha; z) \in \mathcal{K}(-1/2) \cap \mathcal{R}_\eta(2/7),$$

whenever $\beta \geq (2/7) - 15/[14(1-\alpha)(2+\alpha)]$ and $-2 < \alpha < 1$. Here $\mathcal{K}(-1/2)$ denotes the family of convex functions of order $-1/2$ and we remark that [**37**] any function in $\mathcal{K}(-1/2)$ is close-to-convex of order $1/2$.

(4) Applying Theorem 1.34 with $\operatorname{Im} a = 0$ (or by considering Theorem 1.28 with $a = -\beta$), we obtain, on replacing $f(z)$ by $zf'(z)$, the following result which deals with the second order polylogarithm functions: if $a \geq -1/2$ and $\delta = -(2a+1)/[(1+a)(5+4a)]$, then

$$f \in \mathcal{R}_\eta(\delta) \quad \Longrightarrow \quad H_f(a, a; z) \in \mathcal{K}(-a) \cap \mathcal{R}_\eta(2/(5+4a)).$$

(5) Choose $\operatorname{Re} a = 1/2$ and replace $f(z)$ by $zf'(z)$ in Theorem 1.34. Then we see that if $|\operatorname{Im} a| \leq \sqrt{3/2}$ and

$$\delta_1 = \frac{2}{7} - \frac{30}{7(9+4(\operatorname{Im} a)^2)}$$

then

$$\begin{aligned} f \in \mathcal{R}_\eta(\delta_1) \quad \Longrightarrow \quad & H_f(1/2 + i\operatorname{Im} a, 1/2 - i\operatorname{Im} a; z) \\ & \in \mathcal{K}(-1/2) \cap \mathcal{R}_\eta(2/7). \end{aligned}$$

2. Preliminaries and key lemmas. Our results rely heavily on the following lemmas.

Lemma 2.1 [**8**]. *If $a_n \geq 0$, $\{na_n\}$ and $\{na_n - (n+1)a_{n+1}\}$ both are nonincreasing, i.e., $\{na_n\}$ is monotone of order 2, then $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $\mathcal{S}^*$.*

An important subclass of $\mathcal{A}$ is described in the following classical result of Fejér [**8**] which we state as a lemma.

Lemma 2.2 [**8, 40**]. *Assume $a_1 = 1$, and $a_n \geq 0$ for $n \geq 2$, such that $\{a_n\}$ is a convex decreasing sequence, i.e.,*

$$0 \geq a_{n+2} - a_{n+1} \geq a_{n+1} - a_n, \quad \textit{for all } n \in \mathbf{N}.$$

Then

$$\mathrm{Re}\left\{\sum_{n=1}^{\infty} a_n z^{n-1}\right\} > \frac{1}{2} \quad \textit{for all } z \in \Delta.$$

The next lemma gives a sufficient condition for $f \in \mathcal{A}$ to be in $\mathcal{C}(0; -\log(1-z))$.

Lemma 2.3 [**26**, Corollary 7]. *Suppose that*

$$1 \geq 2a_2 \geq \cdots \geq na_n \geq \cdots \geq 0 \tag{2.4}$$

or

$$1 \leq 2a_2 \leq \cdots \leq na_n \leq \cdots \leq 2. \tag{2.5}$$

Then $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is close-to-convex with respect to $-\log(1-z)$.

We recall that if $f \in \mathcal{K} \subset \mathcal{S}^*(1/2)$ then $\mathrm{Re}\,(f(z)/z) > 1/2$ in Δ, but the converse is not true, not even if the coefficients of f are real and positive [**21**]. On the other hand, functions in $\mathcal{A}$ with $\mathrm{Re}\,(f(z)/z) > 1/2$ have a nice property with respect to convolutions, as described in the following lemma.

Lemma 2.6. *If p is analytic in Δ, $p(0) = 1$, and $\mathrm{Re}\,p(z) > 1/2$ in Δ then for any function F, analytic in Δ, the function $p * F$ takes values in the convex hull of $F(\Delta)$.*

The conclusion of Lemma 2.6 readily follows by using the Herglotz' representation for p, and it can also be regarded as a special case of a general convolution result [**40**, Theorem 2.4].

A function f analytic in Δ is said to be *subordinate* to an analytic function g, written $f \prec g$, or $f(z) \prec g(z)$, if $f(z) = g(w(z))$ for some function w analytic in Δ, satisfying $|w(z)| \le |z|$. If g is univalent in Δ, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$, see [**27**].

Lemma 2.7 [**23,24**]. *Let $\Omega \subset \mathbf{C}$, and let q be analytic and univalent on $\bar{\Delta}$ except for those $\zeta \in \partial\Delta$ for which $\lim_{z\to\zeta} q(z) = \infty$. Suppose that $\psi : \mathbf{C}^3 \times \Delta \to \mathbf{C}$ satisfies the condition*

$$\psi(r, s, t; z) \notin \Omega, \tag{2.8}$$

when $r = q(\zeta)$ is finite, $s = m\zeta q'(\zeta)$, $\mathrm{Re}\,(1 + t/s) \ge m\mathrm{Re}\,(1 + \zeta q''(\zeta)/q'(\zeta))$, and $z \in \Delta$, for $m \ge 1$ and $|\zeta| = 1$. If p is analytic in Δ, with $p(0) = q(0)$, and if p satisfies

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega, \quad z \in \Delta,$$

then $p(z) \prec q(z)$ in Δ.

Suppose that $p \in \mathcal{H}$ with $p(z) = e^{i\eta} + p_1 z + \cdots$, $|\eta| < \pi/2$, and

$$q(z) = \frac{e^{i\eta} + e^{-i\eta}z}{1 - z}.$$

Then it is easy to see that the condition (2.8) reduces to

$$\psi(ix, y, u + iv; z) \notin \Omega \tag{2.9}$$

when x is real, $y \le -|e^{i\eta} + ix|^2)/(2\cos\eta)$ and $y + u \le 0$. We use this special case for the proofs of Theorems 1.22, 1.28, 1.30 and 1.34.

From a general result, the authors [**29, 30, 9, 34**] obtained as a special case the following result which gives the order of starlikeness of the Alexander transform.

Corollary 2.10. *Let $f \in \mathcal{A}$ and Λ_f be the Alexander transform as in* (1.15).

(i) *If $\beta_0 = (1 - 2\log 2)/(2 - 2\log 2) \approx -0.629$, then $\mathcal{R}_\eta(\beta_0)$ is $\mathcal{S}^* \cap \mathcal{R}_\eta(0)$-admissible for $\Lambda_f(z)$.*

(ii) *If* $\beta_1 = -(2-\sqrt{3})(2\log 2-1)/(2-(2-\sqrt{3})(2\log 2-1)) \approx -0.054$ *and* $\gamma_1 \approx 0.409$*, then* $\mathcal{R}(\beta_1)$ *is* $\mathcal{S}^*(\gamma_1)\cap\mathcal{R}(\beta_1')$*-admissible for* $\Lambda_f(z)$*, where* $\beta_1' = \beta_1[2-2\log 2]+2\log 2-1$.

(iii) *If* $\beta_2 = -(2\log 2-1)/(3-2\log 2) \approx -0.239$ *and* $\gamma_2 \approx 0.083$*, then* $\mathcal{R}(\beta_2)$ *is* $\mathcal{S}^*(\gamma_2)\cap\mathcal{R}(\beta_2')$*-admissible for* $\Lambda_f(z)$*, where* $\beta_2' = \beta_2[2-2\log 2]+2\log 2-1$.

(iv) *If* β_0 *and* γ *are related by*

$$\gamma = \frac{3[\beta_0+(1-\beta_0)(2\log 2-1)]}{6\beta_0+(1-\beta_0)\pi^2}, \tag{2.11}$$

where $0 \le \gamma \le 1/2$*, then* $\mathcal{R}_\eta(\beta_0)$ *is* $\mathcal{S}^*(\gamma)\cap\mathcal{R}_\eta(\beta_0')$*-admissible for* $\Lambda_f(z)$*, where* $\beta_0' = \beta_0[2-2\log 2]+2\log 2-1$.

Proof. From [**29**, Lemma B], see also [**32**, Lemma 1], we obtain that for the Alexander transform $\Lambda_f(z)$ of f defined by (1.15) we have

$$f\in\mathcal{R}_\eta(\beta) \quad\Longrightarrow\quad \Lambda_f(z)\in\mathcal{R}_\eta(\beta'), \tag{2.12}$$

where $\beta' = \beta[2-2\log 2]+2\log 2-1$. This implication shows that the class $\mathcal{R}_\eta(\beta)$ is $\mathcal{R}_\eta(\beta')$-admissible with respect to the Alexander transform $\Lambda_f(z)$. Further, the conclusion that $\mathcal{R}_\eta(\beta_j)$, $j=0,1,2$, is $\mathcal{S}^*(\gamma_j)$-admissible ($j=0,1,2$) follows from the results of [**9**, Corollary 1], [**30**, Corollary 3] and [**29**, Corollary 3], respectively. Part (iv) follows from [**34**, Corollary 1.13]. As noticed in [**34**], part (iv) is sharp only for γ defined by (2.11) equals zero and $\beta_0 \approx -0.629$ whereas a simple computation shows that (ii) and (iii) improve (iv). On the other hand, the part (iv) gives information for $0\le\gamma\le 1/2$, see [**34, 35**]. □

The following lemmas play a key role in the proof of some of our main results.

Lemma 2.13. *Let* a,b *satisfy either* $a,b>-1$*, or* $a,b\in(-2,-1)$*. Suppose that* A_n *is defined by* (1.2)*. If* a *and* b *are related by the condition*

$$ab \le 6/(6+a+b) \tag{2.14}$$

then $\{nA_n\}$ *is a convex decreasing sequence for* $n\ge 1$*.*

Proof. Let A_n be defined by (1.2) for $n \geq 1$. Suppose that a, b satisfy either $a, b > -1$, or $a, b \in (-2, -1)$. Therefore, the conditions on a and b imply that $A_1 = 1$ and $A_n > 0$ for $n \geq 2$. According to Lemma 2.2, we need to show that the hypotheses imply that

$$(n+1)A_{n+1} \geq (n+2)A_{n+2}, \quad \text{for all } n \geq 1, \tag{2.15}$$

and

$$nA_n - 2(n+1)A_{n+1} + (n+2)A_{n+2} \geq 0, \quad \text{for all } n \geq 1. \tag{2.16}$$

For convenience, we define $B_n = nA_n - (n+1)A_{n+1}$. By (1.2), it is a simple exercise to see that

$$B_{n+1} = (1+a)(1+b)\left[\prod_{k=1}^{2} \frac{1}{(n+k+a)(n+k+b)}\right] X(n), \tag{2.17}$$

where

$$X(n) = n^2 + 3n + 2 - ab.$$

Clearly

$$X(n) \geq 5n + 1 - ab \geq 6 - ab, \quad \text{for all } n \geq 1.$$

By (2.14) we notice that $6 - ab > 0$ and therefore, $X(n) > 0$ for $n \geq 1$. From the last fact and (2.17), we deduce that the inequality (2.15) holds. Next we show that (2.16) also holds. To this end, we show that the sequence $\{B_n\}$ is nonincreasing. On replacing n by $n-1$ in (2.17), we find, after some computation, that

$$B_n = (a+1)(b+1)\left[\prod_{k=0}^{1} \frac{1}{(n+k+a)(n+k+b)}\right](n^2 + n - ab). \tag{2.18}$$

Further, by an elementary calculation, (2.18) and (2.17), we can easily obtain that

$$B_n - B_{n+1} = 2(a+1)(b+1)\left[\prod_{k=0}^{2} \frac{1}{(n+k+a)(n+k+b)}\right] U(n), \tag{2.19}$$

where

$$U(n) = n^3 + 3n^2 + n(2 - 3ab) - ab(3 + a + b). \tag{2.20}$$

Therefore, the fact that

$$n^3 \geq 3n^2 - 3n + 1, \quad \text{for all } n \geq 1,$$

and (2.20) yield

$$U(n) \geq 6n^2 - (1 + 3ab)n + 1 - ab(3 + a + b) := V(n).$$

Since the inequality $n^2 \geq 2n - 1$ holds for all $n \geq 1$, we deduce that

$$V(n) \geq (11 - 3ab)n - 5 - ab(3 + a + b) := W(n).$$

It is easy to verify that the condition (2.14), in particular, implies that $3ab \leq 11$ and therefore the coefficient of n in $W(n)$ is nonnegative. This observation immediately gives that

$$W(n) \geq W(1) = 6 - ab(6 + a + b), \quad \text{for all } n \geq 1.$$

However, the condition (2.14) is equivalent to $W(1) \geq 0$. Thus, we proved a chain of inequalities

$$U(n) \geq V(n) \geq W(n) \geq W(1) \geq 0, \quad \text{for } n \geq 1.$$

Equation (2.19) and the last inequalities imply that $\{B_n\}$, i.e., $\{nA_n - (n+1)A_{n+1}\}$, is a nonincreasing sequence. We have already shown that the sequence $\{nA_n\}$ is nonincreasing. Hence, $\{nA_n\}$ is a convex decreasing sequence for $n \geq 1$. $\square$

Lemma 2.21. *For $a \in \mathbf{C}\backslash\{-1, -2, \dots\}$, let*

$$A_n = \frac{|a+1|^2}{|n+a|^2}, \quad n \geq 1.$$

If $|a| \leq \sqrt{11/3}$ and $|a|^2(3 + \operatorname{Re} a) \leq 3$, then $\{nA_n\}$ is a convex decreasing sequence for $n \geq 1$.

Proof. Follows easily from the proof of Lemma 2.13. $\square$

Lemma 2.22. *Suppose that a, b and β are related by either* (i) *or* (ii) *of Theorem* 1.20. *Then we have $G(a, b; z) \in \mathcal{R}(\beta)$.*

Proof. We first consider the case when a, b satisfy either $a, b > -1$ or $a, b \in (-2, -1)$. Suppose that the condition (i) of Theorem 1.20 holds. Since the second part follows similarly, we give the details only for the first part.

The function $G(a, b; z)$ belongs to $\mathcal{R}(\beta)$ if and only if $\operatorname{Re} G'(a, b; z) > \beta$, or equivalently

$$\operatorname{Re}\left(1 + \frac{1}{2(1-\beta)} \sum_{n=2}^{\infty} n A_n z^{n-1}\right) > \frac{1}{2}$$

where A_n is defined by (1.2). Define $C_1 = 1$ and

$$C_n = \frac{(a+1)(b+1)n}{2(1-\beta)(n+a)(n+b)}, \quad \text{for } n \geq 2.$$

Therefore, to prove $G(a, b; z) \in \mathcal{R}(\beta)$, in view of Lemma 2.2, it suffices to show that $\{C_n\}$ is a convex decreasing sequence for $n \geq 1$. As in the proof of Lemma 2.13, it is easy to see that the inequality

$$C_{n+1} \geq C_{n+2}$$

holds for all $n \geq 1$, because by hypothesis we have $ab < 6$. Therefore, we need only to show that the hypotheses imply that

$$C_n - 2C_{n+1} + C_{n+2} \geq 0, \quad \text{for all } n \geq 1. \tag{2.23}$$

It can be easily seen that the condition

$$\beta = \frac{1}{2} + \frac{6 - ab(a+b+6)}{(2+a)(2+b)(3+a)(3+b)},$$

where $\beta < 1$, is equivalent to $C_1 - 2C_2 + C_3 = 0$ and therefore it suffices to verify the inequality (2.23) for $n \geq 2$. From the proof of Lemma 2.13, it is clear that the inequality (2.23) holds for all $n \geq 2$ provided $U(n) \geq 0$ for all $n \geq 2$, where

$$U(n) = n^3 + 3n^2 + n(2 - 3ab) - ab(3 + a + b).$$

Using the inequalities

$$n^3 \geq 6n^2 - 12n + 8 \quad \text{and} \quad n^2 \geq 4n - 4$$

which is true for $n \geq 2$, and the method of proof of Lemma 2.13, we easily deduce that $U(n) \geq 0$ for all $n \geq 2$. Hence, under the hypothesis, $\{C_n\}$ is a convex decreasing sequence for $n \geq 1$. Therefore, by Lemma 2.2, we obtain that the function $G(a,b;z)$ satisfies $\operatorname{Re} G'(a,b;z) > \beta$ in Δ. $\square$

A simple version of the following lemma has been used in [**28**] to study the starlikeness of the Bernardi operator and the proof of this lemma follows by using the same method as in the proof of [**32**] or [**36**]. However, for the sake of completeness, we give the details.

Lemma 2.24. *For $\beta < 1$, let $\mathcal{R}_\eta(\beta)$ be $\mathcal{S}^*(\gamma)$-admissible for the Alexander transform Λ_f defined by (1.15). Suppose that $f \in \mathcal{R}_\eta(\beta_1)$ and $g \in \mathcal{R}(\beta_2)$, where $\beta_1 < 1$ and $\beta_2 < 1$ are such that*

$$1 - \beta = 2(1-\beta_1)(1-\beta_2). \tag{2.25}$$

Then the function $f * g$ is in $\mathcal{S}^*(\gamma) \cap \mathcal{R}_\eta(\beta')$, where $\beta' = \beta[2 - 2\log 2] + 2\log 2 - 1$.

Proof. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $\mathcal{R}_\eta(\beta_1)$, then we have

$$1 + \frac{e^{i\eta}}{2(1-\beta_1)\cos\eta} \sum_{n=2}^{\infty} n a_n z^{n-1} \prec \frac{1}{1-z}.$$

Similarly, for $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{R}(\beta_2)$, we have

$$1 + \frac{1}{2(1-\beta_2)} \sum_{n=2}^{\infty} n b_n z^{n-1} \prec \frac{1}{1-z}.$$

Direct application of Lemma 2.6 gives

$$L_1(z) = 1 + \frac{e^{i\eta}}{4(1-\beta_1)(1-\beta_2)\cos\eta} \sum_{n=2}^{\infty} n^2 a_n b_n z^{n-1} \prec \frac{1}{1-z}.$$

If

$$L_2(z) = 1 - 4(1-\beta_1)(1-\beta_2) + 4(1-\beta_1)(1-\beta_2)\frac{1}{1-z},$$

then, by (2.25), we have

$$\mathrm{Re}\, L_2(z) > 1 - 2(1-\beta_1)(1-\beta_2) = \beta.$$

Since $\mathrm{Re}\, L_1(z) > 1/2$, on applying Lemma 2.6 once again, we find that

$$\mathrm{Re}\,(L_1 * L_2)(z) > \beta$$

which is equivalent to

$$\mathrm{Re}\, e^{i\eta}[(f*g)'(z) + z(f*g)''(z)] > \beta \cos\eta. \tag{2.26}$$

Since $\mathcal{R}_\eta(\beta)$ is $\mathcal{S}^*(\gamma)$-admissible for the Alexander transform Λ_f defined by (1.15), we have $f*g$ is in $\mathcal{S}^*(\gamma)$. By (2.12), the inequality (2.26) implies that the convolution $f*g$ is in $\mathcal{R}_\eta(\beta')$ where $\beta' = \beta[2-2\log 2] + 2\log 2 - 1$. The proof is complete. □

Using Corollary 2.10 (i) we obtain the following result. We remark that if we use part (ii)–(iv) of Corollary 2.10 we have a stronger conclusion but under a stronger condition.

Corollary 2.27. *For* $\beta_0 = (1-2\log 2)/(2-2\log 2) \approx -0.629$, *we have*

$$\mathcal{R}(1/2) * \mathcal{R}_\eta(\beta_0) \subset S^* \cap \mathcal{R}_\eta(0).$$

Proof. The result follows from part (i) of Corollary 2.10 on taking $\beta_0 = (1-2\log 2)/(2-2\log 2) = -0.629\cdots$, $\beta_1 = \beta_0$ and $\beta_2 = 1/2$ in Lemma 2.24. □

3. Proofs of the main theorems.

3.1. *Proof of Theorem* 1.10. Let a, b be such that $a, b > -1$ or $a, b \in (-2,-1)$ and satisfy the inequality $ab \le 2$. Consider the function $G(a,b;z) = \sum_{n=1}^{\infty} A_n z^n$, where A_n is defined by (1.2). From the condition on a, b and (1.2), we see that A_n is positive for each $n \ge 2$ and $A_1 = 1$. By Lemma 2.3, see Equation (2.4), we need to show that the sequence $\{nA_n\}$ is decreasing. We recall, from (2.18), that

$$(3.2)\quad nA_n - (n+1)A_{n+1} = (a+1)(b+1)\left[\prod_{k=0}^{1} \frac{1}{(n+k+a)(n+k+b)}\right] X(n),$$

where

$$X(n) = n^2 + n - ab.$$

To show that the sequence $\{nA_n\}$ is decreasing, from (3.2), it suffices to prove that $X(n) \geq 0$ for all $n \geq 1$ which is clearly true, since $X(n)$ is an increasing function of n so that $X(n) \geq X(1) = 2 - ab \geq 0$, as $ab \leq 2$. Hence, the sequence $\{nA_n\}$ is decreasing and by Lemma 2.3 we obtain that the function $G(a,b;z)$ is close-to-convex with respect to $-\log(1-z)$. □

3.3. *Proof of Theorem* 1.12. Let $a, b > -1$ or $a, b \in (-2,-1)$, and let A_n be defined by (1.2). Then, from the proof of Theorem 1.10, we note that the condition $ab \leq 6/(6+a+b)$ implies that the sequence $\{nA_n\}$ is nonincreasing while from the proof of Lemma 2.13 we obtain that the sequence $\{nA_n - (n+1)A_{n+1}\}$ is nonincreasing. Therefore, by Lemma 2.1, the function $G(a,b;z) = \sum_{n=1}^{\infty} A_n z^n$ is starlike in Δ and by Lemma 2.3, the function $G(a,b;z)$ is close-to-convex with respect to the convex function $-\log(1-z)$. By Lemma 2.13, we see that the sequence $\{nA_n\}$ is also convex decreasing and therefore using Lemma 2.2 we have $\operatorname{Re} G'(a,b;z) > 1/2$ in Δ. The desired conclusion follows. □

3.4. *Proof of Theorem* 1.13. (1) From the proof of Theorem 1.10, it follows that if $|a| \leq \sqrt{2}$ then the function $G(a,\bar{a};z)$ is close-to-convex with respect to the convex function $-\log(1-z)$.

(2) Proof follows from Lemmas 2.21 and 2.2.

(3) Follows from the proof of Lemmas 2.13 and 2.1. □

3.5. *Proof of Theorem* 1.16. (1) Let $a, b > -1$ or $a, b \in (-2,-1)$. Suppose that $ab \leq 6/(6+a+b)$. Then from Theorem 1.12, we have $G(a,b;z) \in \mathcal{R}(1/2)$. Since $f \in \mathcal{R}_\eta(\beta)$, by the hypothesis, the desired conclusion follows upon substituting $\beta_2 = 1/2$ and $\beta_1 = \beta$ in Lemma 2.24.

(2) Let $a \in \mathbf{C}$ be such that a assumes no negative integer values and satisfy the condition $|a| \leq \sqrt{2}$ and $|a|^2(3+\operatorname{Re} a) \leq 3$. Then from Lemmas 2.21 and 2.2, we see that $G(a,\bar{a};z) \in \mathcal{R}(1/2)$ and the conclusion follows similarly, as an application of Lemma 2.24. □

3.6. *Proof of Corollary* 1.17. Follows from Corollary 2.10 and by taking $a=-\alpha$ and $b=2-\alpha$ in Theorem 1.16. □

3.7. *Proof of Theorem* 1.20. Let $f \in \mathcal{K}$. From the hypothesis and Lemma 2.22, we see that the function $G(a,b;z)$ belongs to $\mathcal{R}(\beta)$. From a result in [**33**], we have

$$\mathcal{K} * \mathcal{R}(\beta) \subset \mathcal{R}(\beta).$$

The conclusion now follows from the above inclusion. □

3.8. *Proof of Theorem* 1.22. Let $G(a,b;z)$ be defined by (1.2) and $H_f(a,b;z) = f(z) * G(a,b;z)$. For convenience, we let $H_f(a,b;z) = H(z)$. Define

$$P(z) = e^{i\eta}\left(\frac{H(z)}{z} - \delta\right)\frac{1}{1-\delta}. \tag{3.9}$$

Then P is analytic in Δ, $P(0)=e^{i\eta}$, and $\operatorname{Re} P(0) = \cos\eta > 0$, since $|\eta| < \pi/2$. Suppose that f satisfies the condition (1.23). Therefore, to prove that the function H satisfies the condition $\operatorname{Re}\{e^{i\eta}(H(z)/z-\delta)\} > 0$ in Δ, it suffices to show that $\operatorname{Re} P(z) > 0$ in Δ. Writing (3.9) as

$$e^{i\eta}H(z) = z[\delta e^{i\eta} + (1-\delta)P(z)],$$

and then by differentiating the above equation, we easily find that the differential equation (1.4), that is

$$zH''(z) + (a+b+1)H'(z) + ab\left(\frac{H(z)}{z}\right) = (a+1)(b+1)\left(\frac{f(z)}{z}\right),$$

is equivalent to

$$\psi(P(z), zP'(z), z^2P''(z)) = e^{i\eta}\left(\frac{f(z)}{z} - \delta\right)\frac{1}{1-\delta}, \tag{3.10}$$

where

$$\psi(r,s,t) = r + [t + (a+b+3)s]\left(\frac{1}{(a+1)(b+1)}\right).$$

If we let

$$\Omega = \left\{ w \in \mathbf{C} : \operatorname{Re} w > -\frac{(a+b+2)\cos\eta}{2(1+a)(1+b)} \right\},$$

then, by (3.10), we note that the condition (1.23) is equivalent to

$$\psi(P(z), zP'(z), z^2P''(z)) \in \Omega.$$

We now use it and apply Lemma 2.7 to conclude that $\operatorname{Re} P(z) > 0$ in Δ. For this, according to Lemma 2.7, see Equation (2.9), we need to show that

$$\psi(ix, y, u+iv) \notin \Omega \tag{3.11}$$

when x is real, $y \le -|e^{i\eta} + ix|^2/(2\cos\eta)$ and $y + u \le 0$. For real x, y, u, v such that $y \le -|e^{i\eta} + ix|^2/(2\cos\eta)$ and $y + u \le 0$, we have

$$\begin{aligned}
\operatorname{Re} \psi(ix, y, u+iv) &\le \frac{1}{(a+1)(b+1)}[(u+y) + (a+b+2)y] \\
&\le \frac{(a+b+2)y}{(a+1)(b+1)} \qquad (\text{since } a, b > -1) \\
&\le -\frac{(a+b+2)}{(a+1)(b+1)}\left[\frac{1 + 2x\sin\eta + x^2}{2\cos\eta}\right] \\
&\le -\frac{(a+b+2)}{2(a+1)(b+1)}\cos\eta
\end{aligned}$$

which, by the definition of the choice of Ω, shows that the condition (3.11) holds. Therefore, by Lemma 2.7, we infer that the function P defined by (3.10) satisfies $\operatorname{Re} P(z) > 0$ in Δ. Thus we have,

$$\operatorname{Re}\left\{ e^{i\eta}\left(\frac{f(z)}{z} - \delta_1\right)\right\} > 0 \quad \Longrightarrow \quad \operatorname{Re}\left\{ e^{i\eta}\left(\frac{H_f(a,b;z)}{z} - \delta\right)\right\} > 0,$$

where δ_1 is defined by (1.27). The second equivalent assertion, that is

$$f \in \mathcal{R}_\eta(\delta_1) \quad \Longrightarrow \quad H_f(a,b;z) \in \mathcal{R}_\eta(\delta),$$

follows upon replacing f by zf' and from the fact that

$$zH_f'(a,b;z) = zf'(z) * G(a,b;z) = H_{zf'}(a,b;z).$$

This completes the proof. □

3.12. *Proof of Theorem* 1.28. For the sake of convenience, we assume $b = -a - 2\beta$ and $H(z) = H_f(a,b;z)$. From Theorem 1.22 and the condition on f, we see that the function $H(z)/z$ does not vanish in the unit disc Δ. Define

$$p(z) = \left(\frac{zH'(z)}{H(z)} - \beta\right)\frac{1}{1-\beta}. \tag{3.13}$$

Then p is analytic in Δ and $p(0) = 1$. Therefore, to show that the function H is starlike of order β, it suffices to prove that $\mathrm{Re}\, p(z) > 0$ in Δ. We write (3.13) as

$$zH'(z) = H(z)[\beta + (1-\beta)p(z)]. \tag{3.14}$$

By differentiating (3.14) with respect to z and by an easy calculation, we obtain

$$zH''(z) = \left(\frac{H(z)}{z}\right)(1-\beta)\Big[(p(z)-1)(\beta + (1-\beta)p(z)) + zp'(z)\Big].$$

Using the above equation, (3.13) and (3.14), it is easy to see that the differential equation (1.4) is equivalent to

$$\psi(p(z), zp'(z); z) = e^{i\eta}\left(\frac{f(z)}{z}\right) \tag{3.15}$$

where $\psi := \psi(r,s;z)$ is given by

$$\psi = e^{i\eta}\frac{H(z)}{z}\left[\frac{(1-\beta)s + (1-\beta)^2 r^2}{(1+a)(1+b)} + \frac{(1-\beta)(2\beta + a + b)r + \beta(a+b+1) + ab}{(1+a)(1+b)}\right].$$

We remark that our case in this theorem concerns the situation when a is real and $b = -a - 2\beta$. Further, the condition on a and β of the hypotheses implies that the factor $(1+a)(1+b)$ is positive. Therefore, the function $\psi(r,s;z)$ simplifies to $\psi_1(r,s;z)$, where

$$\psi_1(r,s;z) = e^{i\eta}\frac{H(z)}{z}\left[\frac{(1-\beta)s+(1-\beta)^2r^2+\beta(1-2\beta)-a(2\beta+a)}{(1+a)(1-a-2\beta)}\right].$$

Let δ_1 be defined by (1.27), and set

$$\Omega = \{w \in \mathbf{C} : \operatorname{Re} w > \delta_1 \cos\eta\}.$$

Suppose that $f \in \mathcal{R}_\eta(\delta_1)$, where δ_1 is defined by (1.27). Then, by (3.15), we note that the condition $f \in \mathcal{R}_\eta(\delta_1)$ is equivalent to

$$\psi_1(p(z), zp'(z); z) \in \Omega$$

and therefore, from Theorem 1.22, we have

$$\operatorname{Re}\left\{e^{i\eta}\frac{H_f(a,b;z)}{z}\right\} > \delta\cos\eta, \quad \text{with} \quad \delta = \frac{2}{5-4\beta} > 0.$$

Next we use this inequality and apply Lemma 2.7 to conclude that $\operatorname{Re} p(z) > 0$ in Δ. By Lemma 2.7, see Equation (2.9), we only need to show that $\psi_1(ix,y;z) \notin \Omega$ whenever x is real and $y \le -(1+x^2)/2$. Now, for real x and y, we have

$$\operatorname{Re}\psi_1(ix,y;z) = \operatorname{Re}\left(e^{i\eta}\frac{H(z)}{z}\right)\frac{1}{(1+a)(1-a-2\beta)}M(x,y),$$

where

$$M(x,y) = (1-\beta)y-(1-\beta)^2x^2+\beta(1-2\beta)-a(2\beta+a).$$

For real x and $y \le -(1+x^2)/2$, we note that

$$M(x,y) \le -\frac{1}{2}[4\beta^2+\beta(4a-3)+1+2a^2] \le 0,$$

by (1.29). Using this inequality we deduce that for real x and $y \le -(1+x^2)/2$,

$$\begin{aligned}\operatorname{Re}\psi_1(ix,y;z) &\le -\operatorname{Re}\left(e^{i\eta}\frac{H(z)}{z}\right)\left[\frac{4\beta^2+\beta(4a-3)+1+2a^2}{2(1+a)(1-a-2\beta)}\right]\\ &\le -\delta\cos\eta\left[\frac{4\beta^2+\beta(4a-3)+1+2a^2}{2(1+a)(1-a-2\beta)}\right]\\ &= \delta_1\cos\eta.\end{aligned}$$

The above inequality, by the definition of Ω, shows that $\psi_1(ix, y; z) \notin \Omega$. Therefore, by Lemma 2.7, we get that the function p defined by (3.5) satisfies $\mathrm{Re}\, p(z) > 0$ in Δ which is equivalent to saying that the function $H_f(a, -a-2\beta; z)$ is starlike of order β and the conclusion follows. □

3.16. *Proof of Theorem* 1.30. Proof of this theorem follows the same lines as the proof of Theorem 1.22 and so we just sketch it. Choose $a \in \mathbf{C}$ such that $\mathrm{Re}\, a > -1$ and $b = \bar{a}$. Let δ_1 be defined by (1.31). Consider $P(z)$ exactly as in (3.9), where the function $H(z)$ now takes the form $H_f(a, b; z) = H_f(a, \bar{a}; z)$. Then P is analytic in Δ, $P(0) = e^{i\eta}$, and it is easy to see that the condition (1.32) is equivalent to

$$\psi(P(z), zP'(z), z^2P''(z)) \in \Omega = \left\{ w \in \mathbf{C} : \mathrm{Re}\, w > -\frac{(1 + \mathrm{Re}\, a)}{|1+a|^2} \cos\eta \right\}$$

where the function $\psi(r, s, t)$ in our present notation takes the form

$$\psi(r, s, t) = r + \left[\frac{t + s + 2(\mathrm{Re}\, a + 1)s}{|1+a|^2} \right].$$

Therefore, for all real x, y, u such that $y \leq -|e^{i\eta} + ix|^2)/(2\cos\eta)$ and $y + u \leq 0$, we have

$$\begin{aligned} \mathrm{Re}\, \psi(ix, y, u + iv) &\leq \frac{2(\mathrm{Re}\, a + 1)y}{|1+a|^2} \\ &\leq -\frac{2(\mathrm{Re}\, a + 1)}{|1+a|^2} \left(\frac{\cos\eta}{2} \right), \end{aligned}$$

that is, $\psi(ix, y, u + iv) \notin \Omega$. Thus, by Lemma 2.7, we have $\mathrm{Re}\, P(z) > 0$ in Δ, or equivalently to

$$\mathrm{Re} \left\{ e^{i\eta} \left(\frac{H_f(a, \bar{a}; z)}{z} - \delta \right) \right\} > 0$$

and the conclusion follows. □

3.17. *Proof of Theorem* 1.34. Since the proof of this theorem follows the same lines as the proof of Theorem 1.28, we include the necessary details only. Let $\mathrm{Re}\, a \geq -1/2$ satisfy the condition (1.35) and let p be exactly as in Theorem 1.28 with $\beta = -\mathrm{Re}\, a$, and because $b = \bar{a}$, in the present case $H_f(a, b; z) = H_f(a, \bar{a}; z)$. Then, after some computation and by the proof of Theorem 1.34, we see that the condition $f \in \mathcal{R}_\eta(\delta_1)$ is equivalent to

$$\psi(p(z), zp'(z); z) \in \Omega = \{w \in \mathbf{C} : \mathrm{Re}\, w > \delta_1 \cos\eta\},$$

where δ_1 is defined as in Theorem 1.34 and the function $\psi(r, s; z)$ in this case is

$$\psi(r, s; z) = e^{i\eta} \frac{H_f(a, \bar{a}; z)}{z|1+a|^2} [(1 + \mathrm{Re}\, a)s + (1 + \mathrm{Re}\, a)^2 r^2 \\ - (1 + 2\mathrm{Re}\, a)\mathrm{Re}\, a + |a|^2].$$

We now use this relation and apply Lemma 2.7 to conclude that $\mathrm{Re}\, p(z) > 0$ in Δ.

For real x and y such that $y \le -(1 + x^2)/2$, we find

$$\mathrm{Re}\, \psi(ix, y; z) \le \mathrm{Re}\left(e^{i\eta} \frac{H_f(a, \bar{a}; z)}{z}\right) \frac{1}{|1+a|^2} N(x, y),$$

where

$$N(x, y) = (1 + \mathrm{Re}\, a)y - (1 + \mathrm{Re}\, a)^2 x^2 - (1 + 2\mathrm{Re}\ a)\mathrm{Re}\, a + |a|^2.$$

By Theorem 1.30, we deduce that

$$f \in \mathcal{R}_\eta(\delta_1) \quad \Longrightarrow \quad \mathrm{Re}\left\{e^{i\eta} \frac{H_f(a, \bar{a}; z)}{z}\right\} > \\ \delta \cos\eta = \frac{2\cos\eta}{5 + 4\mathrm{Re}\, a} > 0. \tag{3.18}$$

For real x and y such that $y \le -(1 + x^2)/2$, we note that

$$N(x, y) \le -\frac{1}{2}\left[4(\mathrm{Re}\, a)^2 + 3\mathrm{Re}\ a + 1 - 2|a|^2\right] \le 0,$$

by (1.35). Using this observation and the implication (3.18), we obtain

$$\mathrm{Re}\, \psi(ix, y; z) \le -\frac{1}{2|1+a|^2} \left(\frac{2\cos\eta}{5 + 4\mathrm{Re}\, a}\right) [4(\mathrm{Re}\, a)^2 + 3\mathrm{Re}\, a + 1 - 2|a|^2]$$

which, after some elementary computation, is seen to be equivalent to

$$\operatorname{Re}\psi(ix,y;z) \le \delta_1 \cos\gamma.$$

In other words, for real x and y such that $y \le -(1+x^2)/2$ we obtain that $\psi(ix,y;z) \notin \Omega$. Therefore, by Lemma 2.7, we have $\operatorname{Re} p(z) > 0$ and hence the function $H_f(a,\bar{a};z)$ is starlike of order β with $\beta = -\operatorname{Re} a$, which completes the proof. □

4. Concluding remarks. In this section we give a brief history of the problems which we shall state below. In [**15**] Krzyż and Lewandowski constructed a counterexample to the Biernacki conjecture that the Alexander transform Λ_f belongs to $\mathcal{S}$ for each $f \in \mathcal{S}$. Therefore, it follows that the inclusion $\mathcal{S} \otimes \mathcal{K} \subset \mathcal{S}$ is not true. This provides another counterexample to the Mandelbrojt-Schiffer conjecture, namely $\mathcal{S} \otimes \mathcal{S} \subset \mathcal{S}$. Previous counterexamples constructed in [**13, 7, 19**] show that $f \otimes g$ need not be locally univalent if f and g are in $\mathcal{S}$. In all these constructions, the corresponding functions have complex coefficients. Bshouty [**5**] showed that $f \otimes g$ need not be univalent even if f and g are functions in $\mathcal{S}$ with real coefficients. This answers the problem raised by Krzyż, see [**2**]. In [**40**], Ruscheweyh illustrates that Mandelbrojt-Schiffer conjecture is incorrect, even in a weaker form. However, if $f \in \mathcal{C}$ then the Alexander transform Λ_f defined by (1.15) belongs to $\mathcal{C}$ [**22**]. On the other hand, the author in [**29, 30**] determined conditions on $\delta > 0$ and $0 \le \gamma < 1$ and proved that

$$\operatorname{Re} f'(z) > -\delta \quad \Longrightarrow \quad \Lambda_f \in \mathcal{S}^*(\gamma).$$

The best possible value of δ for the case $\gamma = 0$ has been obtained recently in [**9**, Corollary 1]. We remark that the results of [**29, 30**] are connected to $\mathcal{R}(\beta)$ and give applications to the theory of differential subordination, see [**28, 32, 36**]. We observe that one can extend the results of [**28, 29, 30**] to become applicable for $\mathcal{R}_\eta(\beta)$ as was considered in [**9**, Corollary 1] for the case when $\gamma = 0$. We deal with this general situation elsewhere which extends parts (b) and (c) of Corollary 1.17. However, the correct order of starlikeness remains unknown. Thus we raise the following

Problem 4.1. *For a given $\beta > \beta_0 \approx -0.629$ and $f \in \mathcal{R}_\eta(\beta)$, determine the correct order of starlikeness, as a function of β and a,* *for Bernardi transform*

$$B_f(z) = \left(\sum_{n=1}^{\infty} \frac{1+a}{n+a} z^n\right) * f(z), \quad \mathrm{Re}\, a > -1,$$

and in particular to the Alexander transform $\Lambda_f(z)$ *of* f *defined by* (1.15).

Some of the recent results have been used to derive some new information on convolution theory, see [**30, 31, 1, 9, 34, 35, 36**], and the results obtained in these papers, in particular, establish the existence of a family of functions containing nonunivalent functions which is transformed into $\mathcal{S}^*(\gamma)$ under certain integral transforms, in particular to the Alexander transform (see Corollary 2.10). Some of these observations give rise to the following question:

Problem 4.2. Define a modified integral convolution by

$$f \otimes_p g = z + \sum_{n=2}^{\infty} \frac{a_n b_n}{n^p} z^n$$

where p is a real number such that $p \geq 0$. The interesting problem is to determine $\sup\{p : \mathcal{S} \otimes_p \mathcal{S} \subset \mathcal{S}\}$.

It is not difficult to find a $p > 1$ so that the modified integral convolution is in $\mathcal{S}$ whenever $f, g \in \mathcal{S}$. Clearly $\mathcal{S} * \mathcal{S} \equiv \mathcal{S} \otimes_0 \mathcal{S} \not\subset \mathcal{S}$ and $\mathcal{S} \otimes \mathcal{S} \equiv \mathcal{S} \otimes_1 \mathcal{S} \not\subset \mathcal{S}$. From the Pólya-Schoenberg conjecture proved in [**41**], namely the inclusion $\mathcal{S}^* \otimes \mathcal{S}^* \subset \mathcal{S}^*$, we have

$$f \otimes_p \left(\frac{z}{(1-z)^2}\right) \in \mathcal{S}^* \quad \text{whenever } f \in \mathcal{S}^*,$$

or equivalently,

$$f \otimes_p \left(\frac{z}{1-z}\right) \in \mathcal{K} \quad \text{whenever } f \in \mathcal{K},$$

for all $p \geq 0$. If $p < 1$, we see that $\mathcal{S} \otimes_p \mathcal{S} \not\subset \mathcal{S}$, otherwise it would violate the de Branges theorem: $f \in \mathcal{S} \Rightarrow |a_n| \leq n$. Lewis [**18**] showed that if each of f and g equals the convex function $z/(1-z)$, then in this special case $f \otimes_p g \in \mathcal{K}$ for all $p \geq 0$. Again, the order of convexity (in terms of p) for $f \otimes_p g$ remains unknown even for the case $f(z) = g(z) = z/(1-z)$. In general, it will be interesting to know the order of convexity for the pth order polylogarithm functions defined in the introduction.

Next, we briefly give details for our next problem. For this, we first note that the admissibility questions have not been considered for other subclasses of functions unlike Corollary 2.10 stating the admissible properties relating to the classes $\mathcal{R}_\eta(\beta)$ and $\mathcal{S}^*(\gamma)$. In this respect, it is interesting to mention the classical result of Frideman. According to Frideman [**10**], there exist only nine functions of the class $\mathcal{S}$ whose coefficients are rational integers. These are

$$z, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z + z^2}. \tag{4.3}$$

It is easy to see from the analytic characterization for starlike functions that each of these functions maps the disc Δ onto a starlike domain. Furthermore, each of these functions plays an important role in function theory since they together with rotations are extremal for well-known sub families of $\mathcal{S}$. We recall that if we set the starlike function g to be the identity function z, then the class $\mathcal{C}_\eta(\beta; g)$ coincides with $\mathcal{R}_\eta(\beta)$. The class $\mathcal{R}_\eta(\beta)$ and its various generalizations have been well studied in the recent years, see for details [**1, 9, 34, 35, 31, 32, 36**]. The above reasoning motivates us to pose the following

Problem 4.4. Let $\beta < 1$ and $f \in \mathcal{C}_\eta(\beta; g)$ where $g(z)$ is given by any of the nine functions described in (4.3). Find a condition on $\beta < 1$ so that $\mathcal{C}_\eta(\beta; g)$ is $\mathcal{S}^*$-admissible, $\mathcal{S}^*(\gamma)$-admissible, $\mathcal{K}(\gamma)$-admissible, etc., with respect to the $H_f(a, b; z)$ and, in particular, to the Bernardi operator defined by (1.6) which is the limiting case of $H_f(a, b; z)$ as $b \to \infty$.

A partial answer to Problem 4.4 is available in [**35**] only when $g(z)$ is the identity function.

Many authors have studied the appropriate minimum radii for the operator B_f on various subclasses of $\mathcal{S}$ and the sharpness of the various radii follow by using the standard extremal functions for the specific subclasses, see the work of Barnard and Kellog [**3**]. Therefore, if

$\operatorname{Re} a > -1$ and $\operatorname{Re} b > -1$, an interesting problem is to know the interaction of $f(z)$ and $H_f(a,b;z)$ among the special classes and to study the corresponding radii problems for the operator $H_f(a,b;z)$.

Problem 4.5. Suppose that $f \in \mathcal{K}(\beta)$, $\mathcal{S}^*(\beta)$, $\mathcal{C}(\beta)$, respectively, $\operatorname{Re} a > -1$ and $\operatorname{Re} b > -1$. Find the exact order of convexity, starlikeness, close-to-convexity, respectively, for the operator $H_f(a,b;z)$.

When discussing the interaction between $f(z)$ and $H_f(a,b;z)$ it is natural to consider the Ruscheweyh's convolution operator [**39, 40**] defined by

$$D^\delta f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z), \quad \delta > -1, f \in \mathcal{A}.$$

By assuming $M_\delta(\beta) = \{f \in \mathcal{A} : \operatorname{Re}(D^{\delta+1}f(t)/D^\delta f(t)) > \beta, z \in \Delta\}$, Ruscheweyh proved among other results the inclusion $M_{n+1}(1/2) \subset M_n(1/2)$ for $n \in \mathbf{N} \cup \{0\}$, and the various generalizations of this inclusion appear in the literature. The work in [**39**] motivates one to look for the properties of $H_f(a,b;z)$ in association with the class $M_\delta(\beta)$ and, in particular, it will be interesting to consider the following

Problem 4.6. Suppose that $f \in M_\delta(\beta)$, where $\delta > -1$ and $\beta < 1$ are fixed real numbers. Determine the exact relationship between the parameters a, b and β' such that $H_f(a,b;z) \in M_\delta(\beta')$.

Again we remark that the solution to Problems 4.5 and 4.6 are known only for the limiting case $b \to \infty$, see [**3, 11**].

For convenience, we let $\mathcal{C}_1 = \{f \in \mathcal{A} : \operatorname{Re}\ [(1-z)f'(z)] > 0, z \in \Delta\}$ and we conclude the paper by listing the values of a and b for which the properties of G and H_f are known:

(i) If $\operatorname{Re} a > -1$ and $\operatorname{Re} b > -1$, then G is convex in Δ but not necessarily in $\mathcal{R}(1/2)$.

(ii) If $ab \le 2$ and either $a, b > -1$, or $a, b \in (-2,-1)$, or $a \in \mathbf{C}$ with $b = \bar{a}$, then $G \in \mathcal{C}_1$, see Theorems 1.10 and 1.13.

(iii) If $ab \le 2$, $ab(6+a+b) \le 6$ and either $a, b > -1$, or $a, b \in (-2,-1)$, or $a \in \mathbf{C}$ with $b = \bar{a}$, then G belongs to $\mathcal{S}^* \cap \mathcal{R}(1/2) \cap \mathcal{C}_1$, see Theorems 1.12 and 1.13.

(iv) Define

$$M(a,b) = \max\{a^2b^2+3ab(a+b)+ab+6(a+b)^2+30(a+b), -2ab(a+b+9)\}.$$

If $M(a,b) \geq -48$ and either $a,b > -1$, or $a,b \in (-2,-1)$ hold, then $G \in \mathcal{R}(0)$, and more over $f \in \mathcal{K}$ implies $H_f(a,b;z) \in \mathcal{R}(0)$, see Lemma 2.22 and Theorem 1.20 (i).

(v) Suppose that $a \in \mathbf{C}\backslash\{-1,-2,\cdots\}$, $|a| \leq \sqrt{26/3}$ and $M(a,\bar{a}) \geq -48$. Then $G(a,\bar{a};z) \in \mathcal{R}(0)$, and $f \in \mathcal{K}$ implies $H_f(a,\bar{a};z) \in \mathcal{R}(0)$, see Lemma 2.22 and Theorem 1.20 (ii).

(vi) If $a,b > -1$, $\beta = -(a+b+2)/(2(1+a)(1+b))$, and if $f \in \mathcal{R}_\eta(\beta)$ then we have $H_f(a,b;z) \in \mathcal{R}_\eta(0)$, see Theorem 1.22.

(vii) If $\operatorname{Re} a > -1$, then

$$f \in \mathcal{R}_\eta(-(1+\operatorname{Re} a)/|1+a|^2) \quad \Longrightarrow \quad H_f(a,\bar{a};z) \in \mathcal{R}_\eta(0)$$

holds, see Theorem 1.30.

(viii) If $a \in (-1,1)$, $\beta = -(1+2a^2)/(5(1-a^2))$ and if $f \in \mathcal{R}_\eta(\beta)$ then we have

$$H_f(a,-a;z) \in \mathcal{R}_\eta(2/5) \cap \mathcal{K},$$

see Theorem 1.28.

(ix) If $a \in (-1,2)$, $\beta = 2/7 - 15/(14(1+a)(2-a))$, and if $f \in \mathcal{R}_\eta(\beta)$ then we have $H_f(a,1-a;z) \in \mathcal{R}_\eta(2/7) \cap \mathcal{K}(-1/2) \subset \mathcal{R}_\eta(2/7) \cap \mathcal{C}(1/2)$, see Theorem 1.28.

Therefore, an open problem is to obtain the properties of $G(a,b;z)$ and $H_f(a,b;z)$ for other values of a and b for which nothing is known in the literature.

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