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LIKE VANISHING HOLOMORPHIC RANDOM FUNCTIONS

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ABSTRACT. For every random function holomorphic in mean on an open connected subset D of $\mathbf{C}$ satisfying $\mathbf{P}[f(z) = 0] > 0$ for all $z \in D$, there is a measurable set $|\Delta$ satisfying $\mathbf{P}[\Delta] > 0$ and $f(z,\omega) = 0$ almost surely on Δ for every $z \in D$.

1. Introduction. The most realistic formulations of the equations arising in applied mathematics typically involve the study of random functions, which are presently a very active area of mathematical research (see [**1, 6, 10**]). On the other hand, it is of considerable interest in the stochastic analysis to know whether a sample property of a random function can be automatically derived from its behavior in mean [**2**–**4, 8**]. In [**8**] we proved that every random function holomorphic in mean on an open subset D of the complex field is equivalent to a random function whose paths are holomorphic on D. This paper is devoted to investigate the behavior of those random functions which are holomorphic in mean on an open connected subset D of $\mathbf{C}$ and vanish in a very broad sense; namely, for each $z \in D$, the event $[f(z) = 0]$ can happen, that is, each set $\Delta_z = \{\omega : f(z,\omega) = 0\}$ has a positive probability which depends on the element z. In such a case we prove that there is a measurable set Δ satisfying $\mathbf{P}(\Delta) > 0$ and $f(z,\omega) = 0$ almost surely on Δ for every $z \in D$. In particular, we obtain a surprising conclusion; namely, two holomorphic random functions f and g on D have versions with a nonzero probability of having common paths if, and only if, $\mathbf{P}[f(z) = g(z)] > 0$ for all $z \in D$.

2. The results. Throughout the paper, $(\Omega, \Sigma, \mathbf{P})$ will denote a complete probability space, and X will stand for a complex Banach space. Given a subset D of $\mathbf{C}$, a map $f : D \times \Omega \to X$ is said to be an *X-valued* (*first-order*) *random function* on D if, for each $z \in D$,

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the map $\omega \mapsto f(z,\omega)$ lies in $\mathcal{L}_1(\mathbf{P},X)$, the linear space of all X-*valued first-order Bochner random variables.* For every fixed $\omega \in \Omega$, the function $z \mapsto f(z,\omega)$ from D into X is called a *path* of f. Given $\xi \in \mathcal{L}_1(\mathbf{P},X)$, $[\xi]$ denotes the equivalence class of ξ for the usual almost surely identification. The space $L_1(\mathbf{P},X) = \{[\xi] : \xi \in \mathcal{L}_1(\mathbf{P},X)\}$ becomes a complex Banach space with the norm $\|[\xi]\|_1 = \int_\Omega \|\xi\|\, d\mathbf{P}$. For the basic information on Bochner integrability we refer to [**5**]. An X-valued random function f on D is said to be *holomorphic in mean* on D if, for every $z_0 \in D$, the quotient $(f(z,\cdot) - f(z_0,\cdot))/(z - z_0)$ has a limit in mean as z approaches z_0, which obviously means that the function $z \mapsto [f(z,\cdot)]$ from D into the complex Banach space $L_1(\mathbf{P},X)$ is holomorphic in the traditional sense. For a full discussion on holomorphic vector-valued functions, the reader is referred to [**7**; Section 3.2].

Lemma 1. *Let $\{\xi_n\}$ be a sequence of X-valued random variables converging in probability to a random variable ξ. Then* $\limsup \mathbf{P}[\xi_n = 0] \le \mathbf{P}[\xi = 0]$.

Proof. The sequence $\{\int_\Omega (\|\xi_n\|/(1+\|\xi_n\|))\, d\mathbf{P}\}$ converges to $\int_\Omega (\|\xi\|/(1+\|\xi\|))\, d\mathbf{P}$. Now we note that $\int_\Omega (\|\xi_n\|/(1+\|\xi_n\|))\, d\mathbf{P} \le \mathbf{P}[\xi_n \ne 0] = 1 - \mathbf{P}[\xi_n = 0]$, for all $n \in \mathbf{N}$, and therefore

$$\int_\Omega \frac{\|\xi\|}{1+\|\xi\|}\, d\mathbf{P} \le \liminf (1 - \mathbf{P}[\xi_n = 0]) = 1 - \limsup \mathbf{P}[\xi_n = 0].$$

Given a natural number k, $\{k\xi_n\}$ converges in probability to $k\xi$. Hence,

$$\int_\Omega \frac{k\|\xi\|}{1+k\|\xi\|}\, d\mathbf{P} \le 1 - \limsup \mathbf{P}[k\xi_n = 0] = 1 - \limsup \mathbf{P}[\xi_n = 0].$$

Letting $k \to \infty$, we deduce that $\mathbf{P}[\xi \ne 0] \le 1 - \limsup \mathbf{P}[\xi_n = 0]$ and so $\limsup \mathbf{P}[\xi_n = 0] \le \mathbf{P}[\xi = 0]$. □

Given two random variables ξ and ζ, the quantity $\mathbf{P}\{\omega \in \Omega : \xi(\omega) = \zeta(\omega)\}$ is independent of which members of $[\xi]$ and $[\zeta]$ we choose and we shall write it as $\mathbf{P}[[\xi] = [\zeta]]$.

Lemma 2. *Let F be a holomorphic function from an open subset D of $\mathbf{C}$ into $L_1(\mathbf{P}, X)$ such that there exists $\delta > 0$ satisfying $\delta \leq \mathbf{P}[F(z) = 0]$ for all $z \in D$. Then $\delta \leq \mathbf{P}[F(z) = F'(z) = 0]$ for all $z \in D$.*

Proof. Fix $z \in D$ and consider the holomorphic function G on D given by $G(w) = (w - z)^{-1}(F(w) - F(z))$ if $w \neq z$ and $G(z) = F'(z)$. Then $F(w) = F(z) + (w - z)G(w)$ for all $w \in D$ and, for $0 < |w - z|$ small enough, we have

$$\begin{aligned}\delta &\leq \mathbf{P}[F(w) = 0]\\ &= \mathbf{P}[F(z) + (w - z)G(w) = 0]\\ &\leq \mathbf{P}[\|F(z)\| = \|G(w)\| = 0]\\ &\quad + \mathbf{P}[\|F(z)\| = |w - z|\,\|G(w)\|, \|F(z)\| \neq 0]\end{aligned}$$

and letting $w \to z$ we have

$$\begin{aligned}\delta &\leq \limsup_{w\to z} \mathbf{P}[\|F(z)\| = \|G(w)\| = 0]\\ &\quad + \limsup_{w\to z} \mathbf{P}[\|F(z)\| = |w - z|\|G(w)\|, \|F(z)\| \neq 0].\end{aligned}$$

Further, applying Lemma 1, we get

$$\limsup_{w\to z} \mathbf{P}[\|F(z)\| = \|G(w)\| = 0] \leq \mathbf{P}[\|F(z)\| = \|F'(z)\| = 0]$$

and

$$\limsup_{w\to z} \mathbf{P}[\|F(z)\| = |w - z|\|G(w)\|, \|F(z)\| \neq 0] = 0.$$

Therefore,

$$\delta \leq \mathbf{P}[\|F(z)\| = \|F'(z)\| = 0]. \qquad \square$$

For a subset Δ of Ω, let χ_Δ denote the characteristic function of Δ.

Theorem 1. *Let F be a holomorphic function from an open connected subset D of $\mathbf{C}$ into $L_1(\mathbf{P}, X)$ such that there exists $\delta > 0$ satisfying $\delta \leq \mathbf{P}[F(z) = 0]$ for all $z \in D$. Then there exists a measurable set Δ with $\mathbf{P}[\Delta] \geq \delta$ such that $F(z)[\chi_\Delta] = 0$ for all $z \in D$.*

Proof. By the above lemma, $\mathbf{P}[F(z) = F'(z) = 0] \geq \delta$ for all $z \in D$. Assume inductively that $\mathbf{P}[F(z) = F'(z) = \cdots = F^{(n)}(z) = 0] \geq \delta$ for all $z \in D$. Then the function $z \mapsto (F(z), \ldots, F^{(n)}(z))$ may be viewed as a holomorphic function, G, from D into $L_1(\mathbf{P}, X \times \overset{n}{\cdots} \times X)$ satisfying $\mathbf{P}[G(z) = 0] \geq \delta$ for all $z \in D$. On account of the above lemma, we have

$$\begin{aligned}\delta &\leq \mathbf{P}[G(z) = G'(z) = 0] \\ &= \mathbf{P}[F(z) = F'(z) = \cdots = F^{(n)}(z) = F^{(n+1)}(z) = 0]\end{aligned}$$

for every $z \in D$.

Thus we get $\delta \leq \mathbf{P}[F(z) = \cdots = F^{(n)}(z) = 0]$ for all $z \in D$ and $n \in \mathbf{N} \cup \{0\}$, and fixing $z_0 \in D$, this clearly forces the existence of a measurable set Δ with $\mathbf{P}[\Delta] \geq \delta$ and $F^{(n)}(z_0)[\chi_\Delta] = 0$ for every $n \in \mathbf{N} \cup \{0\}$. Consequently, the function $z \mapsto F(z)[\chi_\Delta]$ is a holomorphic function from D into $L_1(\mathbf{P}, X)$ having zero derivatives of all orders in z_0 and therefore equals zero on a suitable open disc contained in D. From the uniqueness theorem [**7**, Theorem 3.11.5], it may be concluded that $F(z)[\chi_\Delta] = 0$ for every $Z \in D$, which is the desired conclusion. □

Lemma 3. *Let F be a continuous function from a subset D of $\mathbf{C}$ into $L_1(\mathbf{P}, X)$. Then, for every $\delta > 0$, the set $C_\delta = \{z \in D : \mathbf{P}[F(z) = 0] \geq \delta\}$ is closed in D.*

Proof. Let $\{z_n\}$ be a sequence in C_δ converging to an element z in D. Then the sequence $\{F(z_n)\}$ converges in probability to $F(z)$ and, applying Lemma 1, it follows that $\delta \leq \limsup \mathbf{P}[F(z_n) = 0] \leq \mathbf{P}[F(z) = 0]$. □

Theorem 2. *Let F be a holomorphic function from an open connected subset D of $\mathbf{C}$ into $L_1(\mathbf{P}, X)$. If, for each $z \in D$, $\mathbf{P}[F(z) = 0] > 0$, then there exists a measurable set Δ with $\mathbf{P}[\Delta] > 0$ such that $F(z)[\chi_\Delta] = 0$ for every $z \in D$. Furthermore, the set $\{\mathbf{P}[\Delta] : \Delta \in \Sigma, [\chi_\Delta]F = 0\}$ attains a maximum, which coincides with the infimum of the set $\{\mathbf{P}[F(z) = 0] : z \in D\}$.*

Proof. For each $k \in \mathbf{N}$, let C_k be the closed subset of D given by

$C_k = \{z \in D : \mathbf{P}[F(z) = 0] \geq 1/k\}$ (see Lemma 3). Then $D = \cup_{k=1}^{\infty} C_k$. Since D is a locally compact Hausdorff space, from Baire's theorem [**9**, Theorem 2.2(b)] C_k contains an open disc, say D_0, for a suitable $k \in \mathbf{N}$.

Note that $1/k \leq \mathbf{P}[F(z) = 0]$ for all $z \in D_0$. From Theorem 1, $F(z)[\chi_\Delta] = 0$ for all $z \in D_0$, for a suitable measurable set Δ with $\mathbf{P}[\Delta] \geq 1/k$. According to [**7**, Theorem 3.11.5], we have $F(z)[\chi_\Delta] = 0$ for all $z \in D$.

For shortness, we denote $E_1 = \{\mathbf{P}[F(z) = 0] : z \in D\}$, $E_2 = \{\mathbf{P}[\Delta] : \Delta \in \Sigma, [\chi_\Delta]F = 0\}$, $\eta_1 = \inf E_1$ and $\eta_2 = \sup E_2$. Let $\{\Delta_n\}$ be a sequence in E_2 with $\lim \mathbf{P}[\Delta_n] = \eta_2$ and consider $\Delta = \cup_{n=1}^{\infty} \Delta_n$. Then $[\chi_\Delta]F = 0$ and $\mathbf{P}[\Delta] \in E_2$. Hence $\mathbf{P}[\Delta_n] \leq \mathbf{P}[\Delta] \leq \eta_2$ and therefore $\mathbf{P}[\Delta] = \eta_2$ and η_2 is the maximum of E_2. Clearly $\eta_1 \geq \eta_2$ and, from Theorem 1, actually it is satisfied $\eta_1 = \eta_2$. □

Corollary 1. *Let f be an X-valued random function holomorphic in mean on an open connected subset D of $\mathbf{C}$. Then the following conditions are equivalent*:

1. $\mathbf{P}[f(z,\omega) = 0] > 0$ *for all* $z \in D$.

2. *There is a* $\Delta \in \Sigma$ *with* $\mathbf{P}[\Delta] > 0$ *such that* $f(z,\omega) = 0$ *almost surely on* Δ, *for every* $z \in D$.

Moreover, the set $\{\mathbf{P}[\Delta] : f(z,\omega) = 0$ *almost surely on* Δ *for all* $z \in D\}$ *attains a maximum which coincides with the infimum of the set* $\{\mathbf{P}[f(z,\omega) = 0] : z \in D\}$.

Corollary 2. *Let f_1 and f_2 be X-valued random functions holomorphic in mean on an open connected subset D of $\mathbf{C}$. Then the following conditions are equivalent*:

1. $\mathbf{P}[f_1(z,\omega) = f_2(z,\omega)] > 0$ *for all* $z \in D$.

2. *There exist two random functions* g_1 *and* g_2 *on* D *equivalent to* f_1 *and* f_2, *respectively, whose paths are holomorphic on* D *and satisfying* $\mathbf{P}[g_1(z,\omega) = g_2(z,\omega)$ *for all* $z \in D] > 0$.

Proof. It suffices to show that the second assertion follows from the first one. If 1 holds, then Corollary 1 shows that there is a Δ satisfying $\mathbf{P}[\Delta] > 0$ and $f_1(z,\omega) = f_2(z,\omega)$ almost surely on Δ for every $z \in D$. Let g_1 and g_2 be random functions with holomorphic paths equivalent to f_1 and f_2, respectively, given by [**8**]. It is clear that, for every $z \in D$, $g_1(z,\omega) = g_2(z,\omega)$ almost surely on Δ. Accordingly, if S is a countable dense subset of D, then for each $z \in S$ there exists a negligible set Δ_z such that $g_1(z,\omega) = g_2(z,\omega)$ for all $\omega \in \Delta \backslash \Delta_z \cdot \Delta_0 = \cup_{z\in S}\Delta_z$ is a negligible set and for all $z \in D$ and $\omega \in \Delta\backslash\Delta_0$ we have $g_1(z,\omega) = g_2(z,\omega)$ since g_1 and g_2 have continuous paths. □

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