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# A SIMPLIFIED PROOF OF OSHIRO'S THEOREM FOR co-$H$ RINGS

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The purpose of this note is to give an alternative proof of the following result originally due to Oshiro [**7**]:

**Theorem.** *For a ring, $R$, the following are equivalent*:

(i) *$R$ is a right co-$H$ ring.*

(ii) *Every right $R$-module is the direct sum of a projective and a singular.*

(iii) *The class of projective right $R$-modules is closed under essential extensions.*

(iv) (a) *Every right $R$-module which is not singular has a nonzero projective direct summand.*

(b) *$R$ has* ACC *on the right annihilators of subsets of $R$.*

*Note.* In [**7**] it was also shown that such a ring is semi-primary $QF-3$.

**Definition 1.** A module $M$ is said to be CS if each submodule is essential in some direct summand of $M$, or equivalently, if every essentially closed submodule is a direct summand. A module $M$ is said to be $\sum$-CS if every direct sum of copies of $M$ is CS. Sometimes CS modules are called *extending* modules.

**Definition 2** [**7**]. A ring $R$ is said to be *right co-Harada*, abbreviated to *right co-$H$*, if every projective right module is CS. It is easy to see that $R$ is right co-$H$ if and only if $R$ is $\sum$-CS as a right $R$-module.

In [**7**], the proof of the above theorem, though revealing much about the structure of such rings, was very long and, in places, rather difficult

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to follow. A second proof of this result was obtained in [**2**], as a corollary to a larger theorem. Although shorter, this one was also a little technical. A composite version is described below, some parts of which are new, while others come from the above sources. This proof has the advantages of being short and of using mainly well-known module theoretic ideas.

From now on, we will use the notation $X \subseteq_{\text{ess}} Y$ to indicate that $X$ is essential in $Y$, and $X \subseteq_{\text{cl}} Y$ to indicate that $X$ is essentially closed in $Y$, i.e., there are no proper essential extensions of $X$ in $Y$. $X \subseteq^{\oplus} Y$ will mean that $X$ is a direct summand of $Y$.

**Lemma 3.** *If $P$ is a projective $R$-module and $X$ is a submodule of $P$, then $P/X$ is singular if and only if $X \subseteq_{\text{ess}} P$. In particular, a projective singular module is zero.*

*Proof.* Well known. □

**Lemma 4** [**5**, Proposition 1.4]. *If $C \subseteq B \subseteq_{\text{ess}} A$ and $C \subseteq_{\text{cl}} A$, then $B/C \subseteq_{\text{ess}} A/C$.*

**Lemma 5** [**3**, 20.3A]. *The following are equivalent for an injective module $E$:*

(i) *$E$ is $\sum$-injective.*

(ii) *$E$ is countably $\sum$-injective.*

(iii) *$R$ satisfies* ACC *on annihilators of subsets of $E$.*

**Lemma 6** [**4**]. *For any cardinal $\xi$, there exists a cardinal $\sigma$ such that every $\xi$-generated module has at most $\sigma$ submodules.*

*Proof.* Every $\xi$-generated right $R$-module is an epimorphic image of $F = R_R^{(\xi)}$, so it cannot have more submodules than $F$. □

**Definition 7.** If $A \subseteq B$ and $A = \oplus_{\lambda\in\Lambda} A_\lambda$, then $A$ is said to be a *local direct summand* of $B$ if, for any finite subset $F$ of $\Lambda$, $\oplus_{\lambda\in F} A_\lambda \subseteq^{\oplus} B$.

The next result is essentially proved in [**6**].

**Lemma 8.** *If $M$ is a module for which $R$ has* ACC *on right ideals of the form $\mathbf{r}(m)$ where $m \in M$, then all local direct summands of $M$ are essentially closed in $M$.*

**Lemma 9.** *If $A = \oplus_{\lambda\in\Lambda} A_\lambda$ is a local direct summand of $B$, $A \subseteq_{\text{ess}} B$ and $R$ satisfies the* ACC *on right ideals of the form $\mathbf{r}(S)$, where $S$ is a subset of $A$, then $A = B$.*

*Proof.* By Lemma 8, it is enough to show that $R$ satisfies ACC on right ideals of the form $\mathbf{r}(b)$, where $b \in B$. Let $\mathcal{F}$ be the set of all finite sets $F$ of $\Lambda$. For each $F \in \mathcal{F}$, let $A_F = \oplus_{\lambda\in F} A_\lambda$, and fix $K_F$ so that $B = A_F \oplus K_F$. Set $K = \cap_{F\in\mathcal{F}} K_F$. For each $k \in K \cap A$, there exists $F \in \mathcal{F}$ with $k \in A_F \cap K_F = 0$, so $K \cap A = 0$ and $K = 0$.

Let $b \in B$. Then for each $F \in \mathcal{F}$ let $a_F \in A_F$ and $k_F \in K_F$ with $b = a_F + k_F$. Let $\Omega_b = \{a_F : F \in \mathcal{F}\} \subseteq A$. Then $x \in \mathbf{r}(\Omega_b) \Leftrightarrow bx = k_F x$ $(\forall\, F \in \mathcal{F}) \Leftrightarrow bx \in K \Leftrightarrow x \in \mathbf{r}(b)$. That is, $\mathbf{r}(b) = \mathbf{r}(\Omega_b)$, so $R$ satisfies the ACC on right ideals of the form $\mathbf{r}(b)$. □

We are now ready to proceed with the proof of the theorem. Note that in [**7**] the statement of condition (iv)(a) is slightly different, namely, that all nonco-small modules have a nonzero projective direct summand. In fact, a module is nonco-small if and only if it is not singular. Oshiro was considering both co-$H$ and, dually, $H$ rings, so used the dual notation, but here we consider only co-$H$ rings; so we will use the more usual notation.

Of the implications in the proof, (i) $\Rightarrow$ (ii) is taken directly from [**7**] and is included for completeness, and (ii) $\Rightarrow$ (iii) is a straightforward proof pointed out to me by Alberto del Valle Robles. (ii) $\Rightarrow$ (iv) (b) is an application of [**4**, Theorem 1.12] to the case where $M = R$, used in the same way as in [**2**]. Once again, the whole argument has been included here for completeness.

*Proof of the theorem.* (i) $\Rightarrow$ (ii). Let $M$ be a right $R$-module. There is a free module, $F$, and an epimorphism $\theta : F \twoheadrightarrow M$. By the hypothesis, there exist $P$ and $Q$ such that $F = P \oplus Q$ with $\ker\theta \subseteq_{\text{ess}} P$. We have $M = \theta(P) \oplus \theta(Q)$, $\theta(P) \cong P/\ker\theta$ is singular and $\theta(Q) \cong Q$ is projective. Hence, condition (ii).

(ii) $\Rightarrow$ (iii). Let $P$ be a projective module and $M$ be an essential extension of $P$. By condition (ii), we have $M = Q \oplus S$, where $Q$ is projective and $S$ is singular. Now $P/(P \cap Q) \cong (P + Q)/Q \hookrightarrow (Q \oplus S)/Q \cong S$. By Lemma 3, $P \cap Q \subseteq_{\text{ess}} P$, so $P \cap Q \subseteq_{\text{ess}} M$ and hence $Q \subseteq_{\text{ess}} M$. Thus, $M = Q$, and the result is shown.

(iii) $\Rightarrow$ (i). Let $C$ be a closed submodule of a projective module $P$. Now $C = E(C) \cap P$, so $P/C = P/(E(C) \cap P) \cong (E(C) + P)/E(C)$. By condition (iii), $E(C) + P$ is projective, and since $E(C) \subseteq^{\oplus} E(C) + P$, we have that $P/C$ is isomorphic to a direct summand of $E(C) + P$, so $P/C$ is projective and hence $C \subseteq^{\oplus} P$.

(ii) $\Rightarrow$ (iv)(a) is trivial.

(iii) $\Rightarrow$ (iv)(b). We use Lemma 5. Let $\xi$ be an uncountably infinite cardinal such that $E(R)$ can be generated by $\xi$ or less elements. By Lemma 6, there exists a cardinal $\sigma$, such that every $\xi$-generated module has no more than $\sigma$ submodules. Take $\tau > \sigma$, and let $E = E(E(R)^{(\tau)})$. Then, since $E$ is projective, $E = \oplus_{\lambda\in\Lambda} E_\lambda$, where each $E_\lambda$ is countably generated, by the application of Kaplansky's theorem to projective modules, e.g., [**1**].

Take any copy of $E(R)$ in $E$, $E(R)'_1$, say. Then $E(R)'_1 \subseteq \oplus_{\omega\in\Omega_1} E_\omega$, where $|\Omega_1| \le \xi$. Since $\oplus_{\omega\in\Omega_1} E_\omega$ is $\xi$-generated, it has no more than $\sigma$ submodules. Hence, there is a copy of $E(R)$, $E(R)_2$ say, in $E$ such that $E(R)_2 \cap \oplus_{\Omega_1} E_\omega = 0$, otherwise there would be at least $\tau$ nonzero submodules of $\oplus_{\Omega_1} E_\omega$, formed by its intersections with the $\tau$ copies of $E(R)$ in $E$, contradicting the previous statement. Hence there exists $E(R)'_2 \subseteq \oplus_{\Lambda\setminus\Omega_1} E_\lambda$, and so $E(R)'_2 \subseteq \oplus_{\Omega_2} E_\omega$, where $|\Omega_2| \le \xi$, $\Omega_2 \subseteq \Lambda\setminus\Omega_1$.

Repeating, we can find $E(R)'_3 \subseteq \oplus_{\Lambda\setminus(\Omega_1\cup\Omega_2)} E_\lambda$, etc., to obtain a countably infinite set of copies of $E(R)$ in $E$, each contained in its own independent subset of $\{E_\lambda\}_\Lambda$. Note that, for every $i$, $E(R)'_i$ is injective so is a direct summand of $\oplus_{\Omega_i} E_\lambda$, and hence $\oplus_{i\in\mathbf{N}} E(R)'_i$ is a direct summand of $E$, so is injective. Thus, by Lemma 5, we have that $R$ has the ACC on right annihilators of subsets of $E(R)$ and hence on those of subsets of $R$.

(iv) $\Rightarrow$ (i). Let $P$ be a projective module and $C$ be an essentially closed submodule of $P$. We wish to prove that $C \subseteq^{\oplus} P$. If $C \subseteq_{\text{ess}} P$, then clearly $C = P$ and we are done. Otherwise, by Lemma 3 and condition (iv) (a), $P/C$ has a nonzero projective direct summand, $X/C$.

Consider a set of nonzero independent projective submodules of $P/C$ which form a local direct summand of $P/C$. The union of any chain of such sets is also a set of nonzero independent projectives which form a local direct summand, and $\{X/C\}$ is a nonempty example of such a set, so by Zorn's lemma, we can find a maximal such set, $\{Q_\lambda/C\}_{\lambda\in\Lambda}$. Let $Q/C = \oplus_{\lambda\in\Lambda} Q_\lambda/C$.

Say $Q$ is not essential in $P$. Then, by Lemma 3 and (iv)(a) we have $P/Q = A/Q \oplus B/Q$, where $A/Q$ is a nonzero projective. Now $P/B \cong (P/Q)/(B/Q) \cong A/Q$ is projective, and so $P = B \oplus K$, where $K \neq 0$. But, since $C \subseteq B$, we have $P/C = B/C \oplus (K \oplus C)/C$ and, moreover, it is easy to see that $\{Q_\lambda/C\}_{\lambda\in\Lambda}$ is a local direct summand of $B/C$, which implies that we can add $(K\oplus C)/C$ to our set, contradicting its maximality. Thus, $Q$ is essential in $P$, and so by Lemma 4, we have $Q/C$ is essential in $P/C$.

The annihilator of an element of a projective module is the annihilator of a subset of $R$, since all projective modules are contained in free modules, so by (iv) (b), $R$ satisfies ACC on the right annihilators of the projective module $Q/C$. Hence, by Lemma 9, we have $Q/C = P/C$, and thus $P/C$ is projective, showing that $C$ is a direct summand of $P$. $\square$

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