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BILINEAR INTEGRATION IN TENSOR PRODUCTS

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ABSTRACT. The integration of vector valued functions with respect to vector valued measures is studied in this paper. The resulting integral take its values in a tensor product space satisfying a mild separation condition. The relationship with bilinear integrals of R. Bartle and I. Dobrakov is examined. The present integral has the advantage of allowing a treatment of integration with respect to certain measures derived from spectral measures, that is not otherwise available.

0. Introduction. The notion of integrating vector valued functions with respect to vector valued measures has been treated by a number of authors [**1, 4**–**7, 2**]. If the indefinite integral takes its values in a tensor product space, then the special nature of topological tensor products can be exploited to yield an integration theory suitable for bilinear integration with respect to spectral measures. Integrals of this nature arise in the study of random evolutions with respect to operator valued measures [**10**].

As a guide to the sort of properties we are looking for, let $1 \le p < \infty$ and $X = Y = L^p([0,1])$, and consider $X \otimes Y$ as a dense subspace of $L^p([0,1]^2)$. Let $\{y_j\}_{j=1}^{\infty}$ be an unconditionally summable sequence in Y, and set $m(A) = \sum_{j\in A} y_j$ for each subset A of $\mathbf{N}$. Then m is a Y-valued measure. A *scalar valued* function $f : \mathbf{N} \to \mathbf{C}$ is m-integrable if and only if $\{f(j)y_j\}_{j=1}^{\infty}$ is unconditionally summable in Y. It is reasonable, therefore, that an X-valued function $f : \mathbf{N} \to X$ should be m-integrable in $L^p([0,1]^2)$ whenever $\{f(j) \otimes y_j\}_{j=1}^{\infty}$ is unconditionally summable in $L^p([0,1]^2)$.

Although this looks like a natural starting point for bilinear integration, it gives rise to some unusual features. For example, suppose that $1 \le p < 2$. Then there exists an unconditionally summable sequence $\{y_j\}_{j=1}^{\infty}$ in $L^p([0,1])$ and a bounded function $f : \mathbf{N} \to L^p([0,1])$

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such that $\{f(j) \otimes y_j\}_{j=1}^{\infty}$ is not summable in $L^p([0,1]^2)$. In other words, *bounded* $L^p([0,1])$-valued functions need not be integrable, see Example 2.2. A modification of this example produces an absolutely summable sequence $\{x_j\}_{j=1}^{\infty}$ in $L^p([0,1])$ and a function $g : \mathbf{N} \to L^p([0,1])$ such that $\{g(j)\|x_j\|\}_{j=1}^{\infty}$ is unconditionally summable in $L^p([0,1])$ but $\{g(j) \otimes x_j\}_{j=1}^{\infty}$ is not summable in $L^p([0,1]^2)$, that is, g is *Pettis* integrable with respect to the variation $V(n)$ of the vector measure $n : A \mapsto \sum_{j \in A} x_j$, but not with respect to n itself; see Example 2.3. Of course, if g were *Bochner* integrable with respect to $V(n)$, then we would have $\sum_{j=1}^{\infty} \|g(j)\|_p \|x_j\|_p < \infty$, so $\{g(j) \otimes x_j\}_{j=1}^{\infty}$ would necessarily be an unconditionally summable sequence in $L^p([0,1]^2)$.

The bilinear integral of R. Bartle [**1**] is defined in terms of convergence in semivariation, thereby encompassing integration with respect to finitely additive set functions. However, convergence in semivariation is too strong to deal with integration with respect to the measures m of the above type. For example, if $\{y_j\}_{j=1}^{\infty}$ is unconditionally summable in $L^p([0,1])$, but $\sum_{j=1}^{\infty} \|y_j\|_p^p = \infty$, such sequences exist whenever $1 \le p < 2$, and $f : \mathbf{N} \to \mathbf{C}$ is an m-integrable *scalar function*, then the $L^p([0,1])$-valued function $F : j \mapsto f(j)1$ may not be integrable in the sense of [**1**, Definition 1], see Example 2.4. A sufficiently general notion of bilinear integration ought to imply that F is m-integrable in $L^p([0,1]^2)$ if and only if f is m-integrable, in the usual sense, in $L^p([0,1])$.

Such an integral is given in Definition 1.5. We can avoid using semivariation in the definition of the integral employed in the present work, at the expense of having a more restrictive setting than that of [**1**] and [**4**–**7**]. We give an example of an $L^2([0,1])$-valued measure m whose tensor product semivariation with respect to the space $L^2([0,1])$ takes only the values zero and infinity, see Example 4.1. The only $L^2([0,1])$-valued functions integrable with respect to such a measure, in the sense of [**1**] and [**4**–**7**], are the *null* functions. Nevertheless, we can identify the space of $L^2([0,1])$-valued m-integrable functions on $[0,1]$, in the sense outlined below, with the space of all integral kernels associated with trace class operators.

1. Bilinear integration. A σ-additive set function defined on a σ-algebra, with values in a locally convex Hausdorff space, is called a *vector measure*. Let X and Y be Banach spaces, and let $X \otimes Y$ denote their algebraic tensor product. The tensor product $X \otimes Y$ equipped with a norm topology τ is denoted by $X \otimes_\tau Y$. We denote the completion of $X \otimes_\tau Y$ by $X \hat{\otimes}_\tau Y$. Let $m : \mathcal{S} \to Y$ be a vector measure defined on the σ-algebra $\mathcal{S}$ of subsets of a set Ω. As usual, the integral of X-valued functions with respect to m is first defined for elementary functions. An X-valued $\mathcal{S}$*-simple function* is a function ϕ for which there exist $k = 1, 2, \ldots$, sets $A_j \in \mathcal{S}$ and vectors $c_j \in X$, $j = 1, \ldots, k$, such that $\phi = \sum_{j=1}^k c_j \chi_{A_j}$. The integral $\phi \otimes m$ of ϕ with respect to the Y-valued measure m is defined by $(\phi \otimes m)(A) = \sum_{j=1}^k c_j \otimes [m(A \cap A_j)]$, for all $A \in \mathcal{S}$. Then $(\phi \otimes m)(A) \in X \otimes Y$, for each $A \in \mathcal{S}$.

We shall make some restrictive assumptions concerning the spaces X and Y and their tensor product $X \otimes Y$, allowing us to integrate a class of functions more general than the simple functions. Suppose that τ is the topology defined on $X \otimes Y$ by a norm $\|\cdot\|_\tau$ with the property that there exists $C > 0$ such that

(T1) $\|x \otimes y\|_\tau \le C\|x\| \, \|y\|$ for all $x \in X$ and $y \in Y$, and

(T2) $X' \otimes Y'$ may be identified with a linear subspace of the continuous dual $(X \otimes_\tau Y)' = (X \hat{\otimes}_\tau Y)'$ of $X \otimes_\tau Y$ and $\|x' \otimes y'\| \le C\|x'\| \, \|y'\|$ for all $x' \in X'$ and $y \in Y'$.

If conditions (T1) and (T2) hold, then τ is merely said to be a *norm tensor product topology* on $X \otimes Y$.

Definition 1.1. A norm tensor product topology τ on $X \otimes Y$ is said to be *completely separated* if the subspace $X' \otimes Y'$ of $(X \otimes_\tau Y)'$ separates the *completion* $X \hat{\otimes}_\tau Y$ of the normed space $X \otimes_\tau Y$, that is, if $u \in X \hat{\otimes}_\tau Y$ and $\langle u, x' \otimes y' \rangle = 0$ for all $x' \in X'$ and $y' \in Y'$, then $u = 0$.

An equivalent formulation of the condition that τ is completely separated is that, if $\{\xi_n\}_{n=1}^\infty$ is any τ-Cauchy sequence in $X \otimes Y$ for which $\lim_{n\to\infty} \langle \xi_n, x' \otimes y' \rangle = 0$ for all $x' \in X'$ and $y' \in Y'$, then $\lim_{n\to\infty} \xi_n = 0$ in τ. For a completely separated, norm tensor product topology τ, the completion $X \hat{\otimes}_\tau Y$ of $X \otimes_\tau Y$ is naturally identified with a subspace of the completion of $X \otimes Y$ in the topology $\sigma(X \otimes Y, X' \otimes Y')$.

The injective tensor product topology is always completely separated, [**13**, 45.4]. If one of the Banach spaces X, Y has the approximation property, then the projective tensor product topology on $X \otimes Y$ is completely separated, [**13**, 43.2].

We begin by stating an elementary but useful condition, [**12**, 18.4].

Proposition 1.2. *Let τ be a norm tensor product topology on $X \otimes Y$. If $X \otimes_\tau Y$ has a fundamental system of neighborhoods of zero which are closed for the topology $\sigma(X \otimes Y, X' \otimes Y')$, then τ is completely separated.*

Vector valued function spaces are a central example of normed tensor products. For a σ-finite measure space $(\Gamma, \mathcal{E}, \mu)$ and $1 \le p < \infty$, the vector space of μ-equivalence classes $[\psi]_\mu$ of strongly μ-measurable functions $\psi : \Gamma \to X$ such that the scalar function $\|\psi\|_X^p$ is μ-integrable is denoted by $L^p(\Gamma, \mathcal{E}, \mu; X)$. By strongly μ-measurable, we mean the limit μ-almost everywhere of X-valued $\mathcal{E}$-simple functions. Then $L^p(\Gamma, \mathcal{E}, \mu; X)$ is a Banach space under the norm $\|[\psi]_\mu\|_p = (\int_\Gamma \|\psi\|_X^p \, d\mu)^{1/p}$. In most circumstances, we write ψ instead of $[\psi]_\mu$. If $p = \infty$, then $L^\infty(\Gamma, \mathcal{E}, \mu; X)$ is the Banach space of (equivalence classes of) strongly μ-measurable functions $\psi : \Gamma \to X$ for which the scalar function $\|\psi\|_X$ is μ-essentially bounded. The norm is the μ-essential bound $\|\psi\|_\infty$ of $\|\psi\|_X$.

If $1 \le p \le \infty$, then $L^p(\Gamma, \mathcal{E}, \mu) \otimes X$ may be identified with a subspace of $L^p(\Gamma, \mathcal{E}, \mu; X)$. The relative topology of $L^p(\Gamma, \mathcal{E}, \mu; X)$ on $L^p(\Gamma, \mathcal{E}, \mu) \otimes X$ satisfies conditions (T1),(T2); this is the tensor product of central interest to the present work. The completion of $L^p(\Gamma, \mathcal{E}, \mu) \otimes X$ in this topology may be identified with $L^p(\Gamma, \mathcal{E}, \mu; X)$ in the case $1 \le p < \infty$.

It follows from the Hahn-Banach theorem and Proposition 1.2 that, for a σ-finite measure space $(\Gamma, \mathcal{E}, \mu)$ and $1 \le p \le \infty$, the $L^p(\Gamma, \mathcal{E}, \mu; X)$-topology on $L^p(\Gamma, \mathcal{E}, \mu) \otimes X$ is completely separated.

For each set $A \in \mathcal{S}$, let $\mathcal{S} \cap A = \{B \cap A : B \in \mathcal{S}\}$. A set $A \in \mathcal{S}$ is called *m-null* if $m(B) = 0$ for every subset $B \in \mathcal{S} \cap A$ of A. A statement which holds outside an m-null set is said to hold m-almost everywhere.

The following ubiquitous convergence result is a variant of Vitali's convergence theorem, [**8**, Theorem III.6.15].

Lemma 1.3. *Let $(\Omega, \mathcal{S})$ be a measurable space and $\lambda : \mathcal{S} \to \mathbf{C}$ a scalar measure. Suppose that f_k, $k = 1, 2, \dots$, are λ-integrable scalar functions converging λ-almost everywhere to a function f, with the property that the numbers $f_k\lambda(A) = \int_A f_k \, d\lambda$, $k = 1, 2, \dots$, converge for each $A \in \mathcal{S}$. Then f is λ-integrable and $f_k\lambda(A) \to f\lambda(A)$ uniformly for $A \in \mathcal{S}$ as $k \to \infty$.*

Given a function $f : \Omega \to X$ and an element $x' \in X'$, let $\langle f, x' \rangle$ denote the scalar function defined by $\omega \mapsto \langle f(\omega), x' \rangle$ for every $\omega \in \Omega$. If $y' \in Y'$, then define a scalar measure $\langle m, y' \rangle$ on $\mathcal{S}$ by $\langle m, y' \rangle(A) = \langle m(A), y' \rangle$ for every $A \in \mathcal{S}$, and its total variation is denoted by $|\langle m, y' \rangle|$.

For an X-valued $\mathcal{S}$-simple function ϕ, the integral $\phi \otimes m$ is σ-additive in $X \otimes_\tau Y$ by property (T1) of the norm tensor product topology τ. The following lemma is needed for Definition 1.5 to make sense.

Lemma 1.4. *Let τ be a completely separated, norm tensor product topology on $X \otimes Y$. Suppose that ϕ_k, $k = 1, 2, \dots$, are X-valued $\mathcal{S}$-simple functions for which $\{(\phi_k \otimes m)(A)\}_{k=1}^\infty$ is τ-Cauchy in $X \otimes_\tau Y$ for each $A \in \mathcal{S}$ and $\phi_k \to 0$, m-almost everywhere, as $k \to \infty$. Then $\lim_{k\to\infty}(\phi_k \otimes m)(A) = 0$ in $X \otimes_\tau Y$ for each $A \in \mathcal{S}$ with respect to τ.*

Proof. For each $x' \in X'$ and $y' \in Y'$, the scalars $\langle (\phi_k \otimes m)(A), x' \otimes y' \rangle = \int_A \langle \phi_k, x' \rangle \, d\langle m, y' \rangle$, $k = 1, 2, \dots$, converge to zero for every $A \in \mathcal{S}$, and $\langle \phi_k, x' \rangle \to 0$, $\langle m, y' \rangle$-almost everywhere. An appeal to Lemma 1.3 shows that $\lim_{k\to\infty} \langle (\phi_k \otimes m)(A), x' \otimes y' \rangle = 0$ for all $x' \in X'$ and $y' \in Y'$. But we know that $\{(\phi_k \otimes m)(A)\}_{k=1}^\infty$ is already τ-Cauchy in $X \otimes Y$, so the fact that τ is completely separated tells us that $\lim_{k\to\infty}(\phi_k \otimes m)(A) = 0$ in τ. □

Our bilinear integral is defined by adopting the conclusion of [**1**, Theorem 9], a translation to the bilinear context of 'Dunford's second integral,' or in modern parlance, the Pettis integral for strongly measurable functions.

Definition 1.5. Let $(\Omega, \mathcal{S})$ be a measurable space, and let X and Y be Banach spaces. Suppose that τ is a completely separated, norm tensor product topology on $X \otimes Y$. Let $m : \mathcal{S} \to Y$ be a vector measure.

A function $f : \Omega \to X$ is said to be *m-integrable* in $X\hat{\otimes}_\tau Y$ if there exist X-valued $\mathcal{S}$-simple functions ϕ_j, $j = 1, 2, \dots$, such that $\phi_j \to f$, m-almost everywhere, as $j \to \infty$, and $\{(\phi_j \otimes m)(A)\}_{j=1}^\infty$ converges in $X\hat{\otimes}_\tau Y$ for each $A \in \mathcal{S}$. Let $(f \otimes m)(A) = \int_A f(\omega) \otimes dm(\omega)$ denote this limit. Sometimes we write $m(f)$ for the definite integral $(f \otimes m)(\Omega)$.

To check that $f \otimes m$ is well-defined, suppose that we have some other X-valued $\mathcal{S}$-simple functions ϕ'_j, $j = 1, 2, \dots$, such that $\phi'_j \to f$, m-almost everywhere, as $j \to \infty$ and $\{(\phi'_j \otimes m)(A)\}_{j=1}^\infty$ converges in $X\hat{\otimes}_\tau Y$ for each $A \in \mathcal{S}$. Then $[\phi'_j - \phi_j] \to 0$, m-almost everywhere, as $j \to \infty$ and $\{([\phi'_j - \phi_j] \otimes m)(A)\}_{j=1}^\infty$ converges in $X\hat{\otimes}_\tau Y$, for each $A \in \mathcal{S}$. By Lemma 1.4, we must have $(f \otimes m)(A) = \lim_{j\to\infty}(\phi_j \otimes m)(A) = \lim_{j\to\infty}(\phi'_j \otimes m)(A)$ for each set $A \in \mathcal{S}$.

The set function $f \otimes m : \mathcal{S} \to X\hat{\otimes}_\tau Y$ is the setwise limit of σ-additive set functions $\phi_k \otimes m$, $k = 1, 2, \dots$, so by the Vitali-Hahn-Saks theorem, [**8**, Theorem IV.10.6], it is itself σ-additive for the topology τ. It is easy to see that the map $(f, m) \mapsto f \otimes m$ is bilinear, in the obvious sense.

We point out some facts that are easily established. In the case that $X = \mathbf{C}$, a function $f : \Omega \to \mathbf{C}$ is m-integrable in the sense above if and only if it is m-integrable in the sense of vector measures described in [**11**, Section II.2], see [**14**, Theorem 2.4]. For the case when $Y = \mathbf{C}$ and X is a Banach space, a function $f : \Omega \to X$ is m-integrable in the sense above if and only if it is strongly m-measurable in X and Pettis m-integrable, [**3**, Section II.3]. In both cases, the class of functions so obtained coincides with the integral of Bartle [**1**].

Moreover, if m has the $*$-property with respect to X, [**1**, Definition 2], then a function $f : \Omega \to X$ is m-integrable in $X\hat{\otimes}_\tau Y$ if and only if it is integrable in the sense of Bartle [**1**, Theorem 9]. In this case, both integrals agree. The assumption that τ is a completely separated, tensor product topology allows us to avoid using X-semivariation to define integration with respect to m; an example of a measure without finite semivariation, and so without the $*$-property, is given in Example 2.2. We mention the connection with the bilinear integral of Dobrakov [**4**] in Section 3. The proof of the next statement is straightforward and is omitted.

Proposition 1.6. *Let X and Y be Banach spaces, and let τ be a completely separated, norm tensor product topology on $X \otimes Y$. Let $m : \mathcal{S} \to Y$ be a vector measure. If a function $f : \Omega \to X$ is m-integrable in $X \hat{\otimes}_\tau Y$, then for all $x' \in X'$ and $y' \in Y'$, the scalar function $\langle f, x' \rangle$ is integrable with respect to the scalar measure $\langle m, y' \rangle$ and the equality*

$$\left\langle \int_A f \otimes dm,\ x' \otimes y' \right\rangle = \int_A \langle f, x' \rangle \, d\langle m, y' \rangle \tag{1a}$$

is valid.

Furthermore, the X-valued function f is integrable with respect to the scalar measure $\langle m, y' \rangle$, the scalar valued function $\langle f, x' \rangle$ is integrable with respect to the Y-valued measure m, and the following equalities hold for all $A \in \mathcal{S}$:

$$\begin{aligned} \left\langle \int_A f \otimes dm,\ x' \otimes y' \right\rangle &= \left\langle \int_A f \, d\langle m, y' \rangle,\ x' \right\rangle \\ &= \left\langle \int_A \langle f, x' \rangle \, dm,\ y' \right\rangle. \end{aligned} \tag{1b}$$

Corollary 1.7. *Let X, Y, τ and m be as in Proposition* 1.6. *If a function $f : \Omega \to X$ is m-integrable in $X \hat{\otimes}_\tau Y$, then the indefinite integral $f \otimes m$ is absolutely continuous with respect to m.*

Proof. Let $A \in \mathcal{S}$ be an m-null set. It follows from (1a) that $\langle [f \otimes m](B), x' \otimes y' \rangle = 0$ for all subsets $B \in \mathcal{S} \cap A$, whenever $x' \in X'$ and $y' \in Y'$. Now apply the assumption that τ is a completely separated, norm tensor product topology. $\square$

Corollary 1.8. *Let X, Y, τ and m be as in Proposition* 1.6. *If a function $f : \Omega \to X$ is m-integrable in $X \hat{\otimes}_\tau Y$ and if $g : \Omega \to \mathbf{C}$ is a bounded $\mathcal{S}$-measurable function, then gf is m-integrable in $X \hat{\otimes}_\tau Y$, the function f is $g.m$-integrable in $X \hat{\otimes}_\tau Y$ and the equalities $(gf) \otimes m = f \otimes (g.m) = g.(f \otimes m)$ hold.*

Proof. Bounded, scalar valued measurable functions are integrable with respect to a vector measure taking values in a Banach space, [**11**, Theorem II.3.1], so g is necessarily integrable for both measures m and $f \otimes m$. Then,

$$\begin{aligned}\langle g(f \otimes m), x' \otimes y' \rangle &= g.\langle (f \otimes m), x' \otimes y' \rangle \\ &= g.[\langle f, x' \rangle . \langle m, y' \rangle], \quad \text{by (1a)} \\ &= [g\langle f, x' \rangle].\langle m, y' \rangle = \langle gf, x' \rangle . \langle m, y' \rangle \\ &= \langle f, x' \rangle .[g\langle m, y' \rangle] = \langle f, x' \rangle . \langle gm, y' \rangle .\end{aligned}$$

Once we prove that gf is m-integrable and f is $g.m$-integrable, the desired equalities are seen by appealing to Proposition 1.6, and the assumption that τ is completely separated.

Let g_j, $j = 1, 2, \dots$, be $\mathcal{S}$-simple functions converging uniformly to g on Ω, and suppose that ϕ_k, $k = 1, 2, \dots$, satisfy the assumptions of Definition 1.5. Since the $X \otimes_\tau Y$-valued measures $\phi_k \otimes m$, $k = 1, 2, \dots$, are uniformly τ-bounded on $\mathcal{S}$ by the Nikodym boundedness theorem, [**3**, Theorem I.3.1], it follows that as $j \to \infty$, the sequence $\{(g_j.[\phi_k \otimes m])(A)\}_{j=1}^\infty$ converges to $(g.[\phi_k \otimes m])(A)$, uniformly in both $A \in \mathcal{S}$ and $k \in \mathbf{N}$. In particular, $g_k \phi_k \to gf$, m-almost everywhere, and $[(g_k\phi_k) \otimes m](A)$ converges in $X\hat{\otimes}_\tau Y$ as $k \to \infty$, for each $A \in \mathcal{S}$. Only a glance at Definition 1.5 is needed to see that gf is m-integrable in $X\hat{\otimes}_\tau Y$. Finally f is $g.m$-integrable because

$$\begin{aligned}\lim_{k\to\infty} \big([\phi_k \otimes (g.m)]\big)(A) &= \lim_{k\to\infty} \lim_{j\to\infty} \big([\phi_k \otimes (g_j.m)]\big)(A) \\ &= \lim_{j\to\infty} \lim_{k\to\infty} \big([\phi_k \otimes (g_j.m)]\big)(A)\end{aligned}$$

exists for each $A \in \mathcal{S}$. □

The following bounded convergence result will be useful later.

Lemma 1.9. *Let X,Y and τ be as in Proposition* 1.6*. Suppose that $m : \mathcal{S} \to Y$ is a vector measure. If $f : \Omega \to X$ is a bounded function which is m-integrable in $X\hat{\otimes}_\tau Y$, then for every $\varepsilon > 0$, there exist X-valued $\mathcal{S}$-simple functions ϕ_k, $k = 1, 2, \dots$, which converge m almost everywhere to f such that $\|\phi_k(\omega)\| \le \|f\|_\infty + \varepsilon$ for all $\omega \in \Omega$ and $k = 1, 2, \dots$, and $(\phi_k \otimes m)(A) \to (f \otimes m)(A)$ as $k \to \infty$ for each $A \in \mathcal{S}$.*

Proof. Because f is m-integrable, there exist X-valued $\mathcal{S}$-simple functions ψ_k, $k = 1, 2, \ldots$, such that $\psi_k \to f$, m almost everywhere, and $(\psi_k \otimes m)(A) \to (f \otimes m)(A)$ as $k \to \infty$ for each $A \in \mathcal{S}$. Let ε be a positive number and p a continuous seminorm on X. For each $k = 1, 2, \ldots$, let

$$A_k = \bigcap_{j=k}^{\infty} \{\omega : \|\psi_j(\omega)\| \leq \|f\|_\infty + \varepsilon\}$$

and

$$\phi_k = \psi_k \chi_{A_k}.$$

If $\omega \in \Omega$ is a point such that $\lim_{k\to\infty} \psi_k(\omega) = f(\omega)$, then the triangle inequality implies that $\lim_{k\to\infty} \phi_k(\omega) = f(\omega)$, and hence $\cup_{k=1}^{\infty} A_k$ is a set of full m-measure. The equality

$$(\phi_k \otimes m)(A) = (\psi_k \otimes m)(A \cap A_k)$$

is valid for all $k = 1, 2, \ldots$ and all $A \in \mathcal{S}$.

It follows from the Vitali-Hahn-Saks theorem, [**8**, Theorem IV.10.6], that $\{\psi_k \otimes m\}_{k=1}^{\infty}$ is a uniformly σ-additive family of $X \otimes_\tau Y$-valued measures, so that

$$\lim_{j\to\infty} \sup_k \|(\psi_k \otimes m)(A \cap A_j) - (\psi_k \otimes m)(A)\|_\tau = 0.$$

Hence, if $A \in \mathcal{S}$, then $(\phi_k \otimes m)(A) \to (f \otimes m)(A)$ as $k \to \infty$. □

Another standard property of vector integrals is that continuous linear maps can be dragged inside the integral to act on the integrand–a property which takes the following form in the present context.

Suppose that X_j, Y_j, $j = 1, 2$, are Banach spaces, and that τ_1 and τ_2 are norm tensor product topologies on the tensor products $X_1 \otimes Y_1$ and $X_2 \otimes Y_2$, respectively. The *tensor product* of two linear maps $S : X_1 \to X_2$ and $T : Y_1 \to Y_2$, is the linear map $S \otimes T : X_1 \otimes Y_1 \to X_2 \otimes Y_2$ defined by $(S \otimes T)(x \otimes y) = (Sx) \otimes (Ty)$ for each $x \otimes y \in X_1 \otimes Y_1$. There is no guarantee that $S \otimes T$ is continuous with respect to τ_1 and τ_2 even if S and T are continuous. However, if $S \otimes T : X_1 \otimes_{\tau_1} Y_1 \to X_2 \otimes_{\tau_2} Y_2$ *is* continuous, then the same symbol $S \otimes T$ denotes the associated continuous linear map between the completions $X_1 \widehat{\otimes}_{\tau_1} Y_1$ and $X_2 \widehat{\otimes}_{\tau_2} Y_2$.

Proposition 1.10. *Suppose that X_j, Y_j, $j = 1, 2$, are Banach spaces, and that τ_1 and τ_2 are completely separated, norm tensor product topologies on the tensor products $X_1 \otimes Y_1$ and $X_2 \otimes Y_2$, respectively. Let $m : \mathcal{S} \to Y_1$ be a measure, and suppose that $S : X_1 \to X_2$ and $T : Y_1 \to Y_2$ are continuous linear maps whose tensor product $S \otimes T : X_1 \otimes_{\tau_1} Y_1 \to X_2 \otimes_{\tau_2} Y_2$ is continuous.*

If $f : \Omega \to X_1$ is m-integrable in $X_1 \widehat{\otimes}_{\tau_1} Y_1$, then $S \circ f$ is $T \circ m$-integrable in $X_2 \widehat{\otimes}_{\tau_2} Y_2$ and

$$(S \otimes T) \int_A f \otimes dm = \int_A [S \circ f] \otimes d[T \circ m],$$

for every $A \in \mathcal{S}$.

Proof. Let ϕ_k, $k = 1, 2, \ldots$, be X_1-valued $\mathcal{S}$-simple functions satisfying the assumptions of Definition 1.5. Then $S \circ \phi_k \to S \circ f$, m almost everywhere, as $k \to \infty$, because S is continuous. The continuity of T guarantees that $T \circ m$ is a Y_2-valued measure. Since $S \otimes T$ is continuous, the sequence $\{(S \otimes T)([\phi_k \otimes m](A))\}_{k=1}^{\infty}$ converges in $X_2 \widehat{\otimes}_{\tau_2} Y_2$ for each $A \in \mathcal{S}$. A glance at Definition 1.5 is enough to complete the proof. □

2. Semi-variation. In the context of bilinear integration, Bartle [**1**] worked with a concept related to semivariation originally introduced in [**9**]; it is needed in the proof of the bounded convergence theorem for bilinear integrals.

Let $\mathcal{S}$ be a σ-algebra of subsets of a nonempty set Ω. Let X and Y be Banach spaces, and suppose that τ is a norm tensor product topology on $X \otimes Y$. Let $m : \mathcal{S} \to Y$ be a vector measure. The *X-semivariation* $\beta_X(m) : \mathcal{S} \to [0, \infty]$ of m in $X \otimes_\tau Y$ is defined by

$$\beta_X(m)(A) = \sup \left\{ \left\| \sum_{j=1}^{k} x_j \otimes m(A_j) \right\|_\tau \right\}$$

for every $A \in \mathcal{S}$; the supremum is taken over all pairwise disjoint sets $A_1, \ldots, A_k$ from $\mathcal{S} \cap A$ and vectors $x_1, \ldots, x_k$ from X, such that $\|x_j\| \le 1$ for all $j = 1, \ldots, k$ and $k = 1, 2, \ldots$. A similar notion applies if the canonical bilinear map $(x, y) \mapsto x \otimes y$ from $X \times Y$ into $X \otimes Y$ is replaced by some continuous bilinear map $(x,y) \mapsto xy$ into a Banach space Z. If $X = \mathbf{C}$, then the $\mathbf{C}$-semivariation $\beta_{\mathbf{C}}$ of m in $Y = \mathbf{C} \otimes Y$ coincides with the usual notion of semivariation $\|m\|$ of a vector valued measure, [**3**, Section I.1].

If, for all sets $A_k \in \mathcal{S}$ decreasing to the empty set, we have $\beta_X(m)(A_k) \to 0$ as $k \to \infty$, then we say that the X-semivariation $\beta_X(m)$ of m in $X \otimes_\tau Y$ is *continuous*. A study of continuity for semivariation has been conducted by Dobrakov [**4**–**7**]. If $\beta_X(m)$ is continuous, then $\beta_X(m)(\Omega) < \infty$. In fact, an equivalent formulation for the continuity of $\beta_X(m)$ is that the set of $X \otimes Y$-valued measures $\phi \otimes m$ as ϕ ranges over all $\mathcal{S}$-simple functions with values in the unit ball of X, is bounded and uniformly σ-additive for the norm $\|\cdot\|_\tau$. A result of Bartle-Dunford-Schwartz, [**3**, Theorem I.2.4], then ensures that there exists a finite nonnegative measure λ on $\mathcal{S}$ such that $\lambda \le \beta_X(m)$ and $\lim_{\lambda(A)\to 0} \beta_X(m)(A) = 0$, see [**6**, Lemma 2]. In the paper [**1**], continuity of the semivariation is called, unhelpfully, the $*$-*property*.

Another of Dobrakov's results [**4**, $*$-Theorem] implies that if $X \hat{\otimes}_\tau Y$ contains no subspace isomorphic to c_0, then the X-semivariation $\beta_X(m)$ of m in $X \otimes_\tau Y$ is continuous once it is finite. In the case that τ is equal to the injective tensor product topology, the X-semivariation $\beta_X(m)$ of m in $X \otimes_\tau Y$ is always continuous, hence finite, [**16**, Lemma 15]. However, the X-semivariation $\beta_X(m)$ of m in $X \otimes_\tau Y$ need not even be *finite* if τ is the projective tensor product topology, see Example 4.1.

A Y-valued measure m, with finite variation $V(m) : \mathcal{S} \to [0,\infty)$, necessarily has finite X-semivariation in $X \otimes_\tau Y$, by virtue of the separate continuity (T1) of the canonical map $X \times Y \to X \otimes_\tau Y$. Moreover, the X-semivariation of m in $X \otimes_\tau Y$ is continuous. We state here a result that follows immediately from [**1**, Theorem 5].

Proposition 2.1. *Let τ be a completely separated, norm tensor product topology on $X \otimes Y$. Suppose that $m : \mathcal{S} \to Y$ is a measure for which there exist sets $\Omega_k \in \mathcal{S}$, $k = 1,2,\dots$, increasing to Ω such that the total variation $V(m)(\Omega_k)$ of m on Ω_k is finite for each $k = 1,2,\dots$, that is, m has σ-finite variation.*

If a function $f : \Omega \to X$ is Bochner integrable with respect to the σ-finite measure $V(m)$, then f is m-integrable in $X \hat{\otimes}_\tau Y$.

The nature of bilinear integration is unlike the cases in which either the measure or function is scalar; for example, *bounded* vector valued functions need not be integrable with respect to a vector valued measure.

Example 2.2. A measure $m : \mathcal{S} \to Y$ can have σ-finite variation without having continuous X-semivariation, or even finite X-semivariation. Let $1 \le p < 2$, and $X = Y = L^p([0,1])$. We give $X \otimes Y$ the relative topology τ of $L^p([0,1]^2)$, so that $X \hat{\otimes}_\tau Y = L^p([0,1]^2)$.

Suppose that $\{y_j\}_{j=1}^\infty$ is an unconditionally summable sequence in $L^p([0,1])$ such that $\sum_{j=1}^\infty \|y_j\|_p^p = \infty$, [**15**, Theorem 1.c.2]. Let $\Omega = \mathbf{N}$ and let $\mathcal{S}$ be the family of all subsets of Ω. Let $m : \mathcal{S} \to Y$ be the vector measure defined by $m(A) = \sum_{k \in A} y_k$ for every $A \in \mathcal{S}$.

Let A_j, $j = 1, 2, \dots$, be pairwise disjoint subsets of $[0,1]$ with positive Lebesgue measure $|A_j|$. Set $f_j = \chi_{A_j}/|A_j|^{1/p}$ for each $j = 1, 2, \dots$; then $\|f_j\|_p = 1$. As $k \to \infty$, we have

$$\int_0^1 \left\| \sum_{j=1}^k f_j y_j(t) \right\|_p^p dt = \sum_{j=1}^k \int_0^1 |y_j(t)|^p \, dt = \sum_{j=1}^k \|y_j\|_p^p \to \infty.$$

Therefore the vector measure m has infinite X-semivariation but σ-finite variation. A function $G : \Omega \to L^p([0,1])$ is m-integrable in $L^p([0,1]^2)$ if and only if the sequence $\{G(j) \otimes y_j\}_{j=1}^\infty$ is unconditionally summable in $L^p([0,1]^2)$. This is obviously guaranteed by the condition $\sum_{j=1}^\infty \|G(j)\|_p \|y_j\|_p < \infty$ of the above proposition. However, the bounded $L^p([0,1])$-valued function $j \mapsto f_j$ on Ω is not m-integrable.

Example 2.3. Let the notation be as in the above example. Then the sequence $\{f_j \otimes y_j\}_{j=1}^\infty$ is not summable in $L^p([0,1]^2)$. Now let $\{a_j\}_{j=1}^\infty$ be a summable sequence of positive scalars and set $g_j = a_j f_j$ for every $j = 1, 2, \dots$. The $L^p([0,1])$-valued measure $m : A \mapsto \sum_{j \in A} g_j$ on $\mathcal{S}$ has finite variation $V(m) : A \mapsto \sum_{j \in A} a_j$. Set $x_j = y_j / a_j$ for each $j = 1, 2, \dots$. Then $\{a_j x_j\}_{j=1}^\infty$ is unconditionally summable in $L^p([0,1])$, but $\{x_j \otimes g_j\}_{j=1}^\infty$ is not summable in $L^p([0,1]^2)$. In other words, the function $j \mapsto x_j$ on $\mathbf{N}$ is Pettis integrable with respect to $V(m)$ in $L^p([0,1])$, but it is *not* m-integrable in $L^p([0,1]^2)$. Of course, an $L^p([0,1])$-valued function on $\mathbf{N}$ that is *Bochner* integrable with respect to $V(m)$ is necessarily m-integrable.

Example 2.4. Let $1 \le p < 2$ and $0 < a < 1/p - 1/2$. Then there exists an unconditionally summable sequence $\{z_j\}_{j=1}^\infty$ in $L^p([0,1])$ such that $\|z_j\|_p = 1/(j^{1/2}\ln(j+1))$, [**15**, Theorem 1.c.2]. For each $j = 1, 2, \dots$, set $y_j = j^{-a} z_j$. Then

$$\sum_{j=1}^\infty \|y_j\|_p^p = \sum_{j=1}^\infty \frac{1}{j^{ap} j^{p/2} \ln(j+1)^p} = \infty.$$

Nevertheless, $\{j^a y_j\}_{j=1}^\infty$ is unconditionally summable in $L^p([0,1])$.

Le $\mathcal{S}$ be as in Example 2.2. Let $m : \mathcal{S} \to L^p([0,1])$ be the measure given by $m(A) = \sum_{j\in A} y_j$ for every subset A of $\mathbf{N}$.

Claim. *The function $f : j \mapsto j^a 1$ on $\mathbf{N}$, with values in $L^p([0,1])$, is m-integrable in $L^p([0,1]^2)$, but it is not m-integrable in $L^p([0,1]^2)$ in the sense of* [**1**].

Proof. We already know that f is m-integrable because $\{f(j)y_j\}_{j=1}^\infty$ is unconditionally summable in $L^p([0,1])$. Let ϕ be an $L^p([0,1])$-valued simple function $\mathbf{N}$. Then, for each $j \in \mathbf{N}$,

$$\|f(j) - \phi(j)\|_p \ge \big|\|f(j)\|_p - \|\phi(j)\|_p\big| = \big|j^a - \|\phi(j)\|_p\big|.$$

Let J be an integer greater than $(\max_j \|\phi(j)\|_p + 1)^{1/a}$. Then for all $j \ge J$, the inequality $\|f(j) - \phi(j)\|_p \ge 1$ holds. Let $A = \{j : \|f(j) - \phi(j)\|_p \ge 1\}$. The argument used in Example 2.2 now shows that the $L^p([0,1])$-semivariation of m in $L^p([0,1]^2)$ of the set A is greater than or equal to $\sum_{j \ge J} \|y_j\|_p^p = \infty$. Consequently, f cannot be approximated in $L^p([0,1])$-semivariation of m by $L^p([0,1])$-valued simple functions. □

The following result is a direct consequence of Lemma 1.9 and the definition of semivariation, so that the proof is omitted.

Lemma 2.5. *Let τ be a completely separated, norm tensor product topology on $X \otimes Y$. If $m : \mathcal{S} \to Y$ is a vector measure and if g is a Y-valued bounded function which is m-integrable in $X\hat{\otimes}_\tau Y$, then*

$$\|(g \otimes m)(A)\|_\tau \le \|g\|_\infty \beta_X(m)(A), \quad A \in \mathcal{S}.$$

Let $m : \mathcal{S} \to Y$ be a measure. Let τ be a norm tensor product topology on $X \otimes Y$. If there exists an increasing family of sets $\Omega_k \in \mathcal{S}$, $k = 1, 2, \dots$, such that $\cup_{k=1}^{\infty} \Omega_k = \Omega$ and $\beta_X(m)(\Omega_k) < \infty$ for every $k = 1, 2, \dots$, then we say that m has *σ-finite X-semivariation* in $X \otimes_\tau Y$. If $A \in \mathcal{S}$ and the restriction of m to the σ-algebra $\mathcal{S} \cap A$ has σ-finite X-semivariation, then we say that m has σ-finite X-semivariation *on* A.

Because bilinear integration at the present level of generality does not have the same features as usual integration theories, it is prudent to prove basic properties carefully. The following result is a useful consequence of σ-finite X-semivariation.

Theorem 2.6. *Let τ be a completely separated, norm tensor product topology on $X \otimes Y$. Suppose that a vector measure $m : \mathcal{S} \to Y$ has σ-finite X-semivariation in $X \otimes_\tau Y$.*

If f_j, $j = 1, 2, \dots$, are X-valued functions on Ω which are m-integrable in $X \hat{\otimes}_\tau Y$ such that $\{f_j\}_{j=1}^{\infty}$ converges m-almost everywhere to an X-valued function f, and $\{(f_j \otimes m)(A)\}_{j=1}^{\infty}$ converges in $X \hat{\otimes}_\tau Y$ for each $A \in \mathcal{S}$, then f is m-integrable in $X \hat{\otimes}_\tau Y$ and $\{(f_j \otimes m)(A)\}_{j=1}^{\infty}$ converges in $X \hat{\otimes}_\tau Y$ to $(f \otimes m)(A)$, uniformly for $A \in \mathcal{S}$.

Proof. Let $\|\cdot\|_\tau$ denote the norm of $X \hat{\otimes}_\tau Y$. As usual, $\|\mu\|_\tau : \mathcal{S} \to [0, \infty)$ denotes the semivariation of a measure $\mu : \mathcal{S} \to X \hat{\otimes}_\tau Y$. Because the sequence $\{f_j \otimes m\}_{j=1}^{\infty}$ of vector measures converges setwise in $X \hat{\otimes}_\tau Y$, it is uniformly bounded by the Nikodym boundedness theorem, and uniformly σ-additive by the Vitali-Hahn-Saks theorem, so there exists a nonnegative measure $\lambda : \mathcal{S} \to [0, \infty)$ with the property that $\lim_{\lambda(A) \to 0} \|f_j \otimes m\|_\tau(A) = 0$, uniformly for $j = 1, 2, \dots$. Moreover, λ may be chosen with the property that $0 \le \lambda(A) \le \sup_j \|f_j \otimes m\|_\tau(A)$ for all $A \in \mathcal{S}$, [**3**, Corollary I.2.5].

Let N be an m-null set outside which $f_j \to f$ pointwise. Corollary 1.7 then implies that $\|f_j \otimes m\|_\tau(N) = 0$ for all $j = 1, 2, \dots$. It follows from the inequality above that N is λ-null.

Let $\varepsilon > 0$. Choose $\delta > 0$ such that, for every set $A \in \mathcal{S}$ with the property that $\lambda(A) < \delta$, the inequality $\|f_j \otimes m\|_\tau(A) < \varepsilon/4$ holds, whenever $j = 1, 2, \dots$. There exists an increasing sequence of sets

$\Omega_k \in \mathcal{S}$, $k = 1, 2, \ldots$, on which the X-semivariation $\beta_X(m)$ of m is finite and whose union is Ω. The σ-additivity of the measure λ guarantees that, for some $K \in \mathbf{N}$, we have $\lambda(\Omega \setminus \Omega_K) < \delta/2$.

An appeal to Egorov's theorem, [**3**, Theorem III.6.12], ensures that there exists a set B_δ such that $\lambda(\Omega \setminus B_\delta) < \delta/2$ and $\|f_k - f\|_X \to 0$ uniformly on B_δ, as $k \to \infty$. Let $A_\delta = B_\delta \cap \Omega_K$. Then $\lambda(\Omega \setminus A_\delta) < \delta$, $\|f_k - f\|_X \to 0$ uniformly on A_δ as $k \to \infty$, and $\beta_X(A_\delta) < \infty$. Choose $K_\varepsilon \in \mathbf{N}$ such that

$$\sup_{\omega \in A_\delta} \|f(\omega) - f_k(\omega)\|_X < \frac{\varepsilon}{4\beta_X(m)(A_\delta) + 1},$$

for all $k \geq K_\varepsilon$. It follows, from Lemma 2.5 and the definition of semivariation that, given $A \in \mathcal{S}$,

$$\begin{aligned} \|(f_j \otimes m)(A) - (f_k \otimes m)(A)\|_\tau & \\ &\leq \|([f_j - f_k] \otimes m)(A \cap A_\delta)\|_\tau + \varepsilon/2 \\ &\leq \| \, \|f_j - f_k\|_X \chi_{A_\delta}\|_\infty \beta_X(m)(A_\delta) + \varepsilon/2 < \varepsilon, \end{aligned}$$

for all $j, k \geq K_\varepsilon$. Thus $\{f_k \otimes m(A)\}_{k=1}^\infty$ converges in $X\hat{\otimes}_\tau Y$, uniformly for $A \in \mathcal{S}$.

It remains to prove that f is m-integrable in $X\hat{\otimes}_\tau Y$. Each function f_k is integrable, so applying the same process to f_k, and choosing a subsequence $\{f_{k_j}\}_{j=1}^\infty$ of $\{f_k\}_{k=1}^\infty$, if necessary, we obtain X-valued $\mathcal{S}$-simple functions ϕ_j, $j = 1, 2, \ldots$, and an increasing family of sets $D_j \in \mathcal{S}$, $j = 1, 2, \ldots$, such that

(1) $\cup_{j=1}^\infty D_j$ is a set of full λ-measure,

(2) $\sup_{\omega \in D_j} \|f(\omega) - f_{k_j}(\omega)\|_X < 1/j$,

(3) $\|(f_{k_l} \otimes m)(A) - (f_{k_j} \otimes m)(A)\|_\tau < 1/j$, for all $l \geq j$ and all $A \in \mathcal{S}$,

(4) $\sup_{\omega \in D_j} \|f_{k_j}(\omega) - \phi_j(\omega)\|_X < 1/j$,

(5) $\|(f_{k_j} \otimes m)(A) - (\phi_j \otimes m)(A)\|_\tau < 1/j$, for all $A \in \mathcal{S}$,

for all $j = 1, 2, \ldots$. Hence, $\phi_j \to f$, m-almost everywhere, and $\{(\phi_j \otimes m)(A)\}_{j=1}^\infty$ converges uniformly for $A \in \mathcal{S}$ to $\lim_{k\to\infty}(f_k \otimes m)(A)$. According to Definition 1.5 the function f is m-integrable in $X\hat{\otimes}_\tau Y$ and $\lim_{k\to\infty}(f_k \otimes m)(A) = (f \otimes m)(A)$, uniformly for $A \in \mathcal{S}$. □

As a consequence of the proof above, it is evident that if $f : \Omega \to X$ is m-integrable and m has σ-finite X-semivariation on the set $\{\omega \in \Omega :$

$f(\omega) \neq 0\}$, then the measure $f \otimes m$ has σ-finite X-semivariation. The proof above shows that the condition that $\{(f_j \otimes m)(A)\}_{j=1}^{\infty}$ converges in $X\hat{\otimes}_\tau Y$ for each $A \in \mathcal{S}$ may be replaced by the condition that $\{f_j \otimes m\}_{j=1}^{\infty}$ is a bounded and uniformly countably additive family of $X\hat{\otimes}_\tau Y$-valued measures.

Under further technical assumptions involving continuity of semivariation and related concepts, I. Dobrakov [**6**, Theorem 17] has obtained an analog of the Lebesgue dominated convergence theorem. We state here the bounded convergence theorem of Bartle [**1**, Theorem 7, Lemma 3] in our setting.

Theorem 2.7. *Let τ be a completely separated, norm tensor product topology on $X \otimes Y$, and suppose that a Y-valued measure m on $\mathcal{S}$ has continuous X-semivariation in $X \otimes_\tau Y$.*

Then every strongly m-measurable, bounded function $f : \Omega \to X$ is m-integrable. Moreover, if $f_k : \Omega \to Y$, $k = 1, 2, \dots$, are uniformly bounded Y-valued functions converging to f, m-almost everywhere, then the integrals $\int_A f_k \otimes dm$, $k = 1, 2, \dots$, converge to $\int_A f \otimes dm$ in $X\hat{\otimes}_\tau Y$, uniformly for $A \in \mathcal{S}$.

Remark. Let τ be a completely separated, norm tensor product topology on the tensor product $X \otimes Y$ of the Banach spaces X and Y. Suppose that the X-semivariation in $X \otimes_\tau Y$ of a Y-valued measure m is finite. If every strongly m-measurable, bounded function $f : \Omega \to Y$ is m-integrable, then C. Swartz [**18**, Theorem 6] has proved that the X-semivariation of m is necessarily continuous.

3. Relationship with Dobrakov's integral. In this section, we show that for Banach spaces X and Y, the bilinear integral given in Definition 1.5 differs from the bilinear integral developed by I. Dobrakov [**4**] only in the case that a Y-valued measure m does not have σ-finite X-semivariation in $X \otimes_\tau Y$, a situation which is often the case for vector measures arising from spectral measures, see Example 4.1.

Throughout this section, let $\mathcal{S}$ be a σ-algebra of subsets of a nonempty set Ω, and let $m : \mathcal{S} \to Y$ be a vector measure.

To prove that a function f is integrable, the *local* integrability of f

and a candidate for the indefinite integral are required.

A function $f : \Omega \to X$ is said to be *locally m-integrable* if every non-m-null set $A \in \mathcal{S}$ contains a non-m-null set $B \in \mathcal{S}$ such that $f\chi_B$ is m-integrable in $X\hat{\otimes}_\tau Y$.

Lemma 3.1. *Let τ be a completely separated, norm tensor product topology on $X \otimes Y$, and suppose that the measure m has σ-finite X-semivariation in $X \otimes_\tau Y$. Then a function $f : \Omega \to X$ is locally m-integrable if and only if it is strongly m-measurable.*

Proof. Let $\lambda : \mathcal{S} \to [0, \infty)$ be a measure equivalent to m, [**3**, Corollary I.2.6]. Suppose first that f is locally m-integrable. By an exhaustion argument with the measure λ, [**3**, Lemma III.2.4], there exists an increasing sequence $\{\Omega_j\}_{j=1}^\infty$ of sets from $\mathcal{S}$, with union Ω, so that $f\chi_{\Omega_j}$ is m-integrable in $X\hat{\otimes}_\tau Y$ for every $j = 1, 2, \ldots$. In particular, the functions $f\chi_{\Omega_j}$, $j = 1, 2, \ldots$, are strongly m-measurable, so f, being the pointwise limit of such functions, is itself strongly m-measurable.

Conversely suppose that f is strongly m-measurable; then it is the limit m almost everywhere of X-valued $\mathcal{S}$-simple functions $\{\phi_k\}_{k=1}^\infty$. By Egorov's measurability theorem, [**8**, Theorem III.6.12], every set A of positive λ-measure contains a set B of positive λ-measure on which $\{\phi_k\}_{k=1}^\infty$ converges uniformly. But $\Omega = \cup_{j=1}^\infty \Omega_j$ for sets Ω_j on which m has finite X-semivariation, so for some $j = 1, 2, \ldots$, the set $B \cap \Omega_j$ has positive λ-measure. From Lemma 2.5, we have the estimate

$$\|[(\phi_k - \phi_l) \otimes m](T)\|_\tau \le \sup_{\omega \in B} \|(\phi_k - \phi_l)(\omega)\|_X \beta_X(m)(\Omega_j), \quad k, l \in \mathbf{N}$$

for all subsets $T \in \mathcal{S}$ of $B \cap \Omega_j$, and hence, f is m-integrable on the non-m-null set $B \cap \Omega_j$ contained in A. $\square$

Theorem 3.2. *Let τ be a completely separated, norm tensor product topology on $X \otimes Y$. Let $m : \mathcal{S} \to Y$ be a vector measure and $f : \Omega \to X$ a strongly m-measurable function such that m has σ-finite X-semivariation, in $X \otimes_\tau Y$, on the set $\{\omega \in \Omega : f(\omega) \ne 0\}$.*

Then f is m-integrable in $X\hat{\otimes}_\tau Y$ if and only if for each $x' \in X'$ and $y' \in Y'$, the scalar function $\langle f, x' \rangle$ is integrable with respect to the *scalar measure* $\langle m, y'\rangle$*, and there exists a measure* $\mu : \mathcal{S} \to X\hat{\otimes}_\tau Y$ *such that for each* $A \in \mathcal{S}$*, the equality*

$$\langle \mu(A), x' \otimes y'\rangle = \int_A \langle f(\omega), x'\rangle\, d\langle m(\omega), y'\rangle, \qquad x' \in X', \quad y' \in Y', \tag{2}$$

holds. In this case, $\mu = f \otimes m$*.*

Proof. If f is m-integrable, then the equality (2) holds for $\mu = f \otimes m$ by Proposition 1.6. In the other direction, suppose that (2) holds. There exists an increasing sequence of sets $\Omega_j \in \mathcal{S}$, $j = 1, 2, \dots$, such that m has finite X-semivariation on Ω_j and $\cup_{j=1}^{\infty}\Omega_j = \{\omega \in \Omega : f(\omega) \neq 0\}$. We may assume, by Lemma 3.1, and the usual exhaustion argument, [**3**, Lemma III.2.4], that $f_j = f\chi_{\Omega_j}$ is m-integrable for each $j = 1, 2, \dots$. By (2) and Proposition 1.6 we have $\mu(A \cap \Omega_j) = [f_j \otimes m](A \cap \Omega_j)$ for each $j = 1, 2, \dots$ and $A \in \mathcal{S}$.

The σ-additivity of μ ensures that $\mu(A \cap \Omega_j) \to \mu(A)$ as $j \to \infty$, uniformly for $A \in \mathcal{S}$, and since $f_j \to f$ pointwise as $j \to \infty$, it follows from Theorem 2.6 that f is m-integrable in $X\hat{\otimes}_\tau Y$ and $\mu = f \otimes m$. □

Remark. If $X\hat{\otimes}_\tau Y$ contains no subspace isomorphic to l^∞, [**3**, Corollary I.4.7], then Equation (2) implies that the set function $\mu : \mathcal{S} \to X\hat{\otimes}_\tau Y$ is necessarily σ-additive; for example, if $X\hat{\otimes}_\tau Y$ is separable.

In the introduction we mentioned features of Bartle's approach to bilinear integration that are not well adapted to integration with respect to spectral measures. An approach developed by I. Dobrakov [**4**–**7**] largely circumvents these problems. The key is the simple device of replacing convergence in X-semivariation employed in [**1**], by convergence almost everywhere, at the expense of passing from finitely additive to countably additive vector measures. However, the approximation of integrable functions is still made with simple functions based on sets with finite X-semivariation–an aspect we have managed to avoid in Section 1.

We see how Dobrakov's integral [**4**] can also be used to formulate conditions for the integrability in $X\hat{\otimes}_\tau Y$ of an X-valued function. The connection with Definition 1.5 will be a by-product. We slightly modify the presentation of C. Swartz [**18**].

Let E and F be Banach spaces, and let $\mathcal{L}(E,F)$ denote the space of all continuous linear operators from E into F. Assume that $\mathcal{L}(E,F)$ is equipped with the strong operator topology. The (E,F)-*semivariation* $\gamma(\nu):\mathcal{S}\to[0,\infty]$ of an operator valued measure ν is defined by $\gamma(\nu)(A)=\sup\|\sum_{j=1}^{k}\nu(B_j)x_j\|_F$ for every $A\in\mathcal{S}$, where the supremum is taken over all partitions $B_j\in\mathcal{S}$, $j=1,\dots,n$, of A, and $\|x_j\|_E\le 1$ for all $j=1,\dots,n$, with $n=1,2,\dots$. Let $\mathcal{S}_\nu$ be the collection of all sets $A\in\mathcal{S}$ such that $\gamma(\nu)(A)<\infty$. Then $\mathcal{S}_\nu$ is a δ-ring. The indefinite integral ${}^{\mathrm{D}}\!\int\phi\,d\nu$ of an E-valued $\mathcal{S}$-simple function ϕ with respect to ν is defined in the obvious way; it is an F-valued measure $A\mapsto {}^{\mathrm{D}}\!\int_A\phi\,d\nu$, $A\in\mathcal{S}$.

Let us consider the special case in which $F=\mathbf{C}$. Then the operator valued measure $\nu:\mathcal{S}\to\mathcal{L}(E,\mathbf{C})$ is a finitely additive set function with values in the dual Banach space E'. Denote by $V(\nu)$ the variation of this E'-valued set function ν. It then follows from the Hahn-Banach theorem that $V(\nu)=\gamma(\nu)$ on $\mathcal{S}$. It can happen that $\mathcal{S}_\nu$ is extremely small. To show this, let $\mathcal{B}([0,1])$ denote the Borel σ-algebra of the interval $[0,1]$.

Example 3.3. Let $1<p<\infty$ and $1/p+1/q=1$. Let $\mathcal{S}=\mathcal{B}([0,1])$. The $L^p([0,1])$-valued measure $m:A\mapsto\chi_A$ on $\mathcal{S}$ defines a measure $\nu:\mathcal{S}\to\mathcal{L}(L^q([0,1]),\mathbf{C})$. Since $V(\nu)(A)=\infty$ for every non-Lebesgue-null set $A\in\mathcal{S}$, [**3**, Example I.1.16], it follows that $\mathcal{S}_\nu$ consists of only those Lebesgue-null sets in $\mathcal{S}$.

The sometimes pernicious character of semivariation is concealed in the following definition of Dobrakov [**4**, Definition 2].

A function $f:\Omega\to E$ is said to be (D) ν-*integrable* if it is the limit ν-almost everywhere of E-valued $\mathcal{S}_\nu$-simple functions ϕ_k, $k=1,2,\dots$, with the property that the F-valued indefinite integrals ${}^{\mathrm{D}}\!\int\phi_k\,d\nu$, $k=1,2,\dots$, are uniformly σ-additive on $\mathcal{S}$. In this case, the integral ${}^{\mathrm{D}}\!\int_A f\,d\nu$ of f over a set $A\in\mathcal{S}$ is defined by ${}^{\mathrm{D}}\!\int_A f\,d\nu=\lim_{k\to\infty}{}^{\mathrm{D}}\!\int_A\phi_k\,d\nu$, and the so-defined indefinite integral ${}^{\mathrm{D}}\!\int f\,d\nu:\mathcal{S}\to F$ is σ-additive by the Vitali-Hahn-Saks theorem.

It is proved in [**4**, Theorem 2] that this definition of ${}^{\mathrm{D}}\!\int f\,d\nu$ makes sense, and that the limit ${}^{\mathrm{D}}\!\int_A f\,d\nu = \lim_{k\to\infty} {}^{\mathrm{D}}\!\int_A \phi_k\,d\nu$ in the Banach space F is actually uniform in $A \in \mathcal{S}$ (a similar argument appears in Theorem 2.6). A (D) ν-integrable function is called *ν-null* if its indefinite integral is the zero vector measure. The preceding cautionary example, shows that the only (D) ν-integrable functions may be the ν-null functions. We hasten to add that Example 3.3 is excluded in the setting of Section 1.

Let τ be a completely separated, norm tensor product topology on $X\otimes Y$. To the vector measure $m : \mathcal{S} \to Y$ there is an associated measure $\tilde{m} : \mathcal{S} \to \mathcal{L}(X, X\hat{\otimes}_\tau Y)$ defined by $\tilde{m}(A)x = x \otimes m(A)$, $A \in \mathcal{S}$. With the notation above, we have $E = X$, $F = X\hat{\otimes}_\tau Y$ and $\gamma(\tilde{m}) = \beta_X(m)$. Moreover, the $\tilde{m}$-null and m-null sets coincide.

Consider the special case in which $\beta_X(m)(\Omega)$ is finite. Then a strongly $\mathcal{S}$-measurable function $f : \Omega \to X$ is m-integrable in $X\hat{\otimes}_\tau Y$ if and only if it is (D) $\tilde{m}$-integrable, in which case

$$(f \otimes m)(A) = {}^{\mathrm{D}}\!\int_A f\,d\tilde{m}, \quad A \in \mathcal{S}. \tag{3}$$

This is a consequence of the Vitali-Hahn-Saks theorem.

Given $\zeta \in (X\otimes_\tau Y)'$, the measure $m_\zeta = \zeta \circ \tilde{m} : \mathcal{S} \to \mathcal{L}(X, \mathbf{C})$ satisfies that $\gamma(m_\zeta) = V(m_\zeta)$. Let $\|\cdot\|_\tau$ denote the dual norm of $(X \otimes_\tau Y)'$. Then we have the equality

$$\beta_X(m)(A) = \sup_{\|\zeta\|_\tau \le 1} V(m_\zeta)(A), \quad A \in \mathcal{S}.$$

A strongly $\mathcal{S}$-measurable function $f : \Omega \to X$ is said to be *scalarly* (D) *m-integrable* if f is (D) m_ζ-integrable for each $\zeta \in (X \otimes_\tau Y)'$. In this case the indefinite integral ${}^{\mathrm{D}}\!\int f\,dm_\zeta : \mathcal{S} \to \mathbf{C}$ is σ-additive for every $\zeta \in (X \otimes_\tau Y)'$.

If f is (D) $\tilde{m}$-integrable, then by composing the approximating sequence for f with $\zeta \in (X \otimes_\tau Y)'$, we see that the function f is necessarily scalarly (D) m_ζ-integrable and the equality ${}^{\mathrm{D}}\!\int f\,dm_\zeta = \langle {}^{\mathrm{D}}\!\int f\,d\tilde{m}, \zeta\rangle$ holds on $\mathcal{S}$.

In the case that m has finite X-semivariation, the following result has been proved by C. Swartz [**17**, Proposition 2] as an application of the dominated convergence theorem, [**6**, Theorem 17].

Lemma 3.4. *Let X and Y be Banach spaces, and suppose that τ is a norm tensor product topology on $X \otimes Y$. Let $m : \mathcal{S} \to Y$ be a vector measure.*

If $f : \Omega \to X$ is a scalarly (D) *m-integrable function such that m has σ-finite X-semivariation, in $X \otimes_\tau Y$, on the set $\{\omega \in \Omega : f(\omega) \neq 0\}$, then*

(i) *for each $A \in \mathcal{S}$, the linear functional $\int_A^* f\, d\tilde{m} : \zeta \mapsto {}^{\mathrm{D}}\!\int_A f\, dm_\zeta$, $\zeta \in (X \otimes_\tau Y)'$, is an element of $(X \otimes_\tau Y)'' = \mathcal{L}((X \hat{\otimes}_\tau Y)', \mathbf{C})$; and*

(ii) *the $(X \otimes_\tau Y)''$-valued set function $\int^* f\, d\tilde{m} : A \mapsto \int_A^* f\, d\tilde{m}$, $A \in \mathcal{S}$, is σ-additive with respect to the topology $\sigma\big((X \otimes_\tau Y)'', (X \otimes_\tau Y)'\big)$ on $(X \otimes_\tau Y)''$.*

Proof. Let $\{\Omega_k\}_{j=1}^\infty$ be an increasing sequence of sets, with finite X-semivariation in $X \otimes_\tau Y$ such that their union is $\{\omega \in \Omega : f(\omega) \neq 0\}$. Let $A \in \mathcal{S}$. Let $k = 1, 2, \dots$ and $B_k = \Omega_k \cap \{\omega \in \Omega : \|f(\omega)\|_X \le k\}$ and $A \in \mathcal{S}$. Then we have

$$\Big|{}^{\mathrm{D}}\!\int_{A \cap B_k} f\, dm_\zeta\Big| \le kV(m_\zeta)(\Omega_k) \le k\beta_X(m)(\Omega_k)\|\zeta\|_\tau, \qquad \zeta \in (X \otimes_\tau Y)'.$$

Hence, $\int_{A\cap B_k}^* f\, d\tilde{m} \in (X \otimes_\tau Y)''$. But, as noted above, the indefinite integral ${}^{\mathrm{D}}\!\int f\, dm_\zeta$ is σ-additive for each $\zeta \in (X \otimes_\tau Y)'$, and hence the sequence of continuous linear functionals $\int_{A\cap B_k}^* f\, d\tilde{m}$, $k = 1, 2, \dots$, converge in $(X \otimes_\tau Y)''$, with respect to the topology $\sigma((X \otimes_\tau Y)'', (X \otimes_\tau Y)')$ by the uniform boundedness principle. Hence, the limit of this sequence must be $\int_A^* f\, d\tilde{m}$, which proves statement (i). It is evident that $\int^* f\, d\tilde{m}$ is σ-additive for the topology $\sigma((X \otimes_\tau Y)'', (X \otimes_\tau Y)')$, because ${}^{\mathrm{D}}\!\int f\, dm_\zeta$ is σ-additive on $\mathcal{S}$ for each $\zeta \in (X \otimes_\tau Y)'$. Thus statement (ii) holds. □

The next result shows that we are dealing with 'weak integrals', in the fashion of the Pettis integral of a strongly measurable vector valued function with respect to a scalar measure.

Theorem 3.5. *Let X and Y be Banach spaces, and suppose that τ is a completely separated, norm tensor product topology on $X \otimes Y$.*

Suppose that $m : \mathcal{S} \to Y$ is a vector measure. Let $\tilde{m} : \mathcal{S} \to \mathcal{L}(X, X \otimes_\tau Y)$ be the measure defined by $\tilde{m}(A)x = x \otimes m(A)$ for every $x \in X$ and $A \in \mathcal{S}$. Let $f : \Omega \to X$ be a strongly m-measurable function such that m has σ-finite X-semivariation, in $X \otimes_\tau Y$, on the set $\{\omega \in \Omega : f(\omega) \neq 0\}$.

If f is scalarly (D) *m-integrable, then the following statements are equivalent*:

(i) $\int_A^* f\, d\tilde{m} \in X \hat{\otimes}_\tau Y$ *for each* $A \in \mathcal{S}$;

(ii) *the* $(X \otimes_\tau Y)''$*-valued set function* $\int^* f\, d\tilde{m} : A \mapsto \int_A^* f\, d\tilde{m}$, $A \in \mathcal{S}$, *defined in Lemma* 3.4 *is σ-additive with respect to the norm topology of* $(X \otimes_\tau Y)''$;

(iii) *f is m-integrable in* $X \hat{\otimes}_\tau Y$;

(iv) *f is* (D) *$\tilde{m}$-integrable.*

If any of the above statements holds, then $\int^* f\, d\tilde{m} = {}^{\mathrm{D}}\!\int f\, d\tilde{m} = f \otimes m$ *on* $\mathcal{S}$.

Proof. (i) $\Rightarrow$ (ii). This is a consequence of the Orlicz-Pettis lemma [**3**, Corollary I.4.4].

(ii) $\Rightarrow$ (iii). By Lemma 3.1, we know that f is locally m-integrable. Let $B \in \mathcal{S}$ be a set such that $\beta_X(m)(B) < \infty$ and $f\chi_B$ is m-integrable in $X \hat{\otimes}_\tau Y$. Then $f\chi_B$ is (D) $\tilde{m}$-integrable. By applying (3) to the function $f\chi_B$ we have

$$\begin{aligned} \langle (f\chi_B \otimes m)(A), \zeta \rangle &= \langle {}^{\mathrm{D}}\!\int_A f\chi_B\, d\tilde{m}, \zeta \rangle \\ &= \int_A^* f\, dm_\zeta \\ &= \langle \int_A^* f\chi_B\, d\tilde{m}, \zeta \rangle \end{aligned}$$

for every $\zeta \in (X \hat{\otimes}_\tau Y)'$, and hence

$$\int_A^* f\chi_B\, d\tilde{m} = (f\chi_B \otimes m)(A) \in X \hat{\otimes}_\tau Y, \quad A \in \mathcal{S}. \tag{4}$$

By an exhaustion argument, [**3**, Lemma III.2.4], there exists an increasing sequence $\{\Omega_j\}_{j=1}^\infty$ of sets with finite X-semivariation such that $f\chi_{\Omega_j}$ is m-integrable in $X\hat{\otimes}_\tau Y$ and such that the set $\{\omega \in \Omega : f(\omega) \neq 0\} \setminus (\cup_{j=1}^\infty \Omega_j)$ is m-null. Let A be an arbitrary set in $\mathcal{S}$, and let $f_j = f\chi_{\Omega_j}$ for each $j = 1, 2, \ldots$. By (ii) we have

$$\lim_{j\to\infty} \int_A^* f_j \, d\tilde{m} = \int_A^* f \, d\tilde{m} \tag{5}$$

in $(X\hat{\otimes}_\tau Y)''$ with respect to the norm topology. Now apply (4) to $B = \Omega_j$, $j = 1, 2, \ldots$, to conclude that the sequence $\{f_j \otimes m)(A)\}_{j=1}^\infty$ in $X\hat{\otimes}_\tau Y$ is convergent in the norm topology τ. Since $f_j \to f$, m almost everywhere, as $j \to \infty$, statement (ii) holds and (5) implies that $f \otimes m = \int^* f \, d\tilde{m}$ on $\mathcal{S}$.

(iii) $\Rightarrow$ (iv). Let $\{\phi_j\}_{j=1}^\infty$ be a sequence of X-valued $\mathcal{S}$-simple functions satisfying the conditions in Definition 1.5. Let $\{\Omega_j\}_{j=1}^\infty$ be an increasing sequence of sets with finite X-semivariation in $X\hat{\otimes}_\tau Y$ such that their union is $\{\omega \in \Omega : f(\omega) \neq 0\}$. Since $\lim_{j\to\infty}(\phi_j \otimes m)(A) = (f \otimes m)(A)$ uniformly for $A \in \mathcal{S}$ by Theorem 2.6 and since $f \otimes m$ is σ-additive on $\mathcal{S}$, it follows that

$$\lim_{j\to\infty} [(\phi_j \chi_{\Omega_j}) \otimes m](A) = (f \otimes m)(A), \quad A \in \mathcal{S}.$$

Again the Vitali-Hahn-Saks theorem implies that the function f is (D) $\tilde{m}$-integrable and

$$^{\mathrm{D}}\!\!\int_A f \, d\tilde{m} = (f \otimes m)(A), \quad A \in \mathcal{S},$$

because $\phi_j \chi_{\Omega_j} \to f$, $\tilde{m}$ almost everywhere, and $\phi_j \chi_{\Omega_j}$ is $\mathcal{S}_{\tilde{m}}$-simple by the fact that $\gamma(\Omega_j) = \beta(\Omega_j) < \infty$ whenever $j = 1, 2, \ldots$.

(iv) $\Rightarrow$ (i). Let $A \in \mathcal{S}$. Since $\langle {}^{\mathrm{D}}\!\!\int_A f \, d\tilde{m}, \zeta \rangle = {}^{\mathrm{D}}\!\!\int_A f \, dm_\zeta = \langle \int_A^* f \, d\tilde{m}, \zeta \rangle$ for every $\zeta \in (X\hat{\otimes}_\tau Y)'$, statement (i) holds. □

The equivalence of (i), (ii) and (iv) under the assumption that m possesses finite X-semivariation in $X \otimes_\tau Y$ has been proved in [**17**, Theorem 4].

Corollary 3.6. *Let X and Y be Banach spaces, and suppose that τ is a completely separated, norm tensor product topology on $X \otimes Y$.*

Let $m : \mathcal{S} \to Y$ be a measure. Then a function $f : \Omega \to Y$ is (D) *$\tilde{m}$-integrable if and only if it is m-integrable in $X\hat{\otimes}_\tau Y$, (Definition* 1.5*), and the set $\{\omega \in \Omega : f(\omega) \neq 0\}$ has σ-finite X-semivariation. If this is the case, then the integrals so defined are equal.*

Proof. The only thing that remains to be proved is that, if f is (D) $\tilde{m}$-integrable, then the measure m necessarily has σ-finite X-semivariation on $\{\omega \in \Omega : f(\omega) \neq 0\}$. The function f is the pointwise limit m almost everywhere of X-valued $\mathcal{S}_{\tilde{m}}$-simple functions ϕ_j, $j = 1, 2 \ldots$. Since m has finite X-semivariation on the set $A_j = \{\omega \in \Omega : \phi_j(\omega) \neq 0\}$ for each $j = 1, 2, \ldots$, and since $\{\omega \in \Omega : f(\omega) \neq 0\} \setminus (\cup_{j=1}^{\infty} A_j)$ is m-null, the conclusion follows. □

4. An example. In this section we consider the Hilbert space $L^2([0, 1])$. Let $\mathcal{B}([0, 1])$ denote the Borel σ-algebra of $[0, 1]$. The following example shows that the X-semivariation of a Y-valued measure in $X \otimes_\tau Y$ may take only the values zero and infinity. The example arises by taking the most basic spectral measure with values in $\mathcal{L}(L^2([0, 1])) = \mathcal{L}(L^2([0, 1]), L^2([0, 1]))$ and making it act on the constant function with value one. A Hilbert space inner product, linear in the first variable and antilinear in the second, will be denoted by $(\cdot | \cdot)$.

Example 4.1. Let $m : \mathcal{B}([0, 1]) \to L^2([0, 1])$ be the vector measure defined by $m(B) = \chi_B$for every $B \in \mathcal{B}([0, 1])$. Then the $L^2([0, 1])$-semivariation of m in the projective tensor product $L^2([0, 1]) \otimes_\pi L^2([0, 1])$, see [**13**, Section 41.2], is infinite on any Borel set A with positive Lebesgue measure $|A|$.

For, let n be any positive integer and suppose that $A_j, j = 1, \ldots, n$, are pairwise disjoint subsets of A, with Lebesgue measure $|A|/n$. Let $\phi_j = (n/|A|)^{1/2}\chi_{A_j}$ for each $j = 1, \ldots, n$. Then $\Phi : L^2([0, 1]) \otimes_\pi L^2([0, 1]) \to \mathbf{C}$, defined by $\Phi(f \otimes g) = (f|g)$ for every $f \in L^2([0, 1])$ and every $g \in L^2([0, 1])$, is continuous but

$$\Phi\bigg(\sum_{j=1}^{n} \phi_j \otimes m(A_j)\bigg) = \sum_{j=1}^{n} (\phi_j|m(A_j)) = |A|^{1/2} n^{1/2}.$$

Because n is any positive integer, the $L^2([0, 1])$-semivariation of m in $L^2([0, 1]) \otimes_\pi L^2([0, 1])$ is infinite on A.

Nevertheless, it is natural to consider the integration in $L^2([0,1]) \widehat{\otimes}_\pi L^2([0,1])$ of functions with values in $L^2([0,1])$, with respect to the $L^2([0,1])$-valued measure m. Because the $L^2([0,1])$-semivariation of m in $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$ is infinite, the conditions of Theorem 2.7 do not hold and bounded operator valued functions need not be integrable, see Example 4.3.

Remark. The only $L^2([0,1])$-valued functions which are (D) $\tilde{m}$-integrable in the tensor product space $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$, and hence, $\tilde{m}$-integrable in the sense of [**1**], are the null functions.

The space $\mathcal{L}^{(2)}(L^2([0,1]))$ of Hilbert-Schmidt operators acting on $L^2([0,1])$ is endowed with the Hilbert-Schmidt norm, [**13**, Section 42.4]. Let $\mathcal{K}$ denote the isometric isomorphism from $L^2([0,1]^2)$ onto $\mathcal{L}^{(2)}(L^2([0,1]))$, which sends an element k of $L^2([0,1]^2)$ to the Hilbert-Schmidt operator $T_k : L^2([0,1]) \to L^2([0,1])$ with *kernel* k, [**19**, Theorem 6.11], that is, $(T_k\phi)(x) = \int_0^1 k(x,y)\phi(y)\,dy$ for almost all $x \in [0,1]$ and for all $\phi \in L^2([0,1])$. For all ϕ, $\psi \in L^2([0,1])$ and $k \in L^2([0,1]^2)$, we have the equality

$$([\mathcal{K}k]\phi \mid \psi) = (k \mid \overline{\phi} \otimes \psi) = \int_{[0,1]^2} k(x,y)\phi(y)\overline{\psi(x)}\,dx\,dy. \tag{6}$$

The projective tensor product $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$ may be identified with a subspace of $L^2([0,1]^2)$ in the obvious way. Then the image $\mathcal{L}^{(1)}(L^2([0,1]))$ of $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$ under $\mathcal{K}$ is the space of nuclear operators on $L^2([0,1])$. The nuclear and trace class operators on the Hilbert space $L^2([0,1])$ are the same. If we equip $\mathcal{L}^{(1)}(L^2([0,1]))$ with the nuclear norm, [**13**, 42.5.(8)], then $\mathcal{K}$ induces an isometry from $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$ onto $\mathcal{L}^{(1)}(L^2([0,1]))$. These observations indicate that for practical reasons the projective tensor product $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$ is a worthy object of study.

The following proposition illustrates our claim that, in the present context, we have devised the 'right' definition of integration of vector valued functions with respect to vector valued measures in tensor product spaces. In the following proposition, given a function $k \in L^2([0,1]^2)$ and a point $x \in [0,1]$, let $[k(x,\cdot)]$ denote the equivalence class in $L^2([0,1])$ containing $k(x,\cdot)$. Moreover, let $Q : \mathcal{B}([0,1]) \to$

$\mathcal{L}(L^2([0,1]))$ denote the measure given by $Q(A)\phi = \chi_A\phi$ for every $A \in \mathcal{B}([0,1])$ and $\phi \in L^2([0,1])$. Then Q is a spectral measure; that is, $Q(A \cap B) = Q(A)Q(B)$ for all $A, B \in \mathcal{B}([0,1])$, and $Q([0,1])$ is the identity operator.

Proposition 4.2. *Let $m : \mathcal{B}([0,1]) \to L^2([0,1])$ be the vector measure given by $m(B) = \chi_B$, $B \in \mathcal{B}([0,1])$. A function $f : [0,1] \to L^2([0,1])$ is m-integrable in $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$ if and only if there exists a function $k : [0,1]^2 \to \mathbf{C}$ such that*

(i) *k is the kernel of a trace class operator; and*

(ii) *the set $\{x \in [0,1] : f(x) = [k(x,\cdot)] \text{ in } L^2([0,1])\}$ is a set of full measure.*

If f is m-integrable, then $[f \otimes m](A)$ is equal to the equivalence class in $L^2([0,1]^2)$ of the function

$$(x,y) \longmapsto \chi_A(x)k(x,y), \quad (x,y) \in [0,1]^2.$$

Moreover, the equality $\mathcal{K}\big([f \otimes m](A)\big) = Q(A)\mathcal{K}k$ is valid for each $A \in \mathcal{B}([0,1])$.

Proof. Suppose first that the conditions (i) and (ii) are satisfied. According to [**13**, 42.5.(5)], the operator T_k has a representation

$$T_k\phi = \sum_{j=1}^{\infty} \eta_j(\phi \,|\, g_j)h_j, \quad \phi \in L^2([0,1]),$$

where $\{\eta_j\}_{j=1}^\infty$ is an absolutely summable scalar sequence, and $\{g_j\}_{j=1}^\infty$ and $\{h_j\}_{j=1}^\infty$ are orthonormal sequences in the Hilbert space $L^2([0,1])$. By the Beppo Levi convergence theorem, $\sum_{j=1}^\infty |\eta_j|\,|g_j(x)|\,|h_j(y)| < \infty$, so that we can define $\xi(x,y) = \sum_{j=1}^\infty \eta_j g_j(x)h_j(y)$, for almost all $(x,y) \in [0,1]^2$. Then the function ξ belongs to $L^2([0,1]^2)$. The operators T_k and T_ξ, defined by the kernels k and ξ, respectively, are equal. However, $L^2([0,1]) \otimes L^2([0,1])$ separates elements of $L^2([0,1]^2)$, so $k = \xi$ almost everywhere on $[0,1]^2$.

Given $j = 1, 2, \ldots$, the functions g_j and h_j may be expressed as $g_j = \sum_{l=1}^\infty \phi_{jl}$ and $h_j = \sum_{n=1}^\infty \psi_{jn}$, both almost everywhere in $[0,1]$

and with respect to the norm topology of $L^2([0,1])$, for some scalar valued sequences $\{\phi_{jl}\}_{l=1}^{\infty}$ and $\{\psi_{jn}\}_{n=1}^{\infty}$ of $\mathcal{B}([0,1])$-simple functions with $\sum_{l=1}^{\infty}\|\phi_{jl}\|_2 \le 2$ and $\sum_{n=1}^{\infty}\|\psi_{jn}\|_2 \le 2$, respectively. It then follows from the condition (ii) that, for almost all $x \in [0,1]$, we have $\sum_{j,l,n=1}^{\infty}|\lambda_j|\,|\phi_{jl}(x)|\,\|\psi_{jn}\|_2 < \infty$, and so the identity $f(x) = \sum_{j,l,n=1}^{\infty}\eta_j\phi_{jl}(x)\psi_{jn}$ holds in the norm topology of $L^2([0,1])$.

Because $\int_A(\phi_{jl}(x)\psi_{jn}) \otimes dm(x) = \chi_A\phi_{jl} \otimes \psi_{jn}$ as elements of $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$ for all $j,l,n \in \mathbf{N}$ and because the triple sequence $\{\eta_j[\chi_A\phi_{jl}] \otimes \psi_{jn}\}_{j,l,n=1}^{\infty}$ is summable in the projective tensor topology for every $A \in \mathcal{S}$, the function f is m-integrable according to Definition 1.5 and

$$\int_A f \otimes dm = \sum_{j,l,n=1}^{\infty} \eta_j[\chi_A\phi_{jl}] \otimes \psi_{jn}, \quad A \in \mathcal{B}([0,1]).$$

Conversely, suppose that f is m-integrable in $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$. Then the element $(f \otimes m)(A)$ of $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$ is expressed as a function $k \in \mathcal{L}^2([0,1])$ so that $\mathcal{K}(\int_0^1 f \otimes dm)$ is a trace class operator with kernel k. In terms of the inner product $(\,\cdot\,|\,\cdot\,)$ of $L^2([0,1])$, we have $\big(\int_0^1 f \otimes dm \,|\, \overline{\phi} \otimes \psi\big) = \int_0^1 (f \,|\, \overline{\phi})\, d(m|\psi)$ by Equation (1a), and so an appeal to Equation (6) shows that,

$$\begin{aligned}\int_0^1 (f \,|\, \overline{\phi})\, d(m \,|\, \psi) &= \left(\mathcal{K}\left[\int_0^1 f \otimes dm\right]\phi \middle| \psi\right) \\ &= \int_{[0,1]^2} k(x,y)\phi(y)\overline{\psi(x)}\, dx\, dy,\end{aligned}$$

for all $\phi, \psi \in L^2([0,1])$. On taking ψ to be the characteristic function of a Borel set, we see that for each $\phi \in L^2([0,1])$, the equality $(f(x) \,|\, \overline{\phi}) = \int_0^1 k(x,y)\phi(y)\, dy$ holds for almost all $x \in [0,1]$. The separability of $L^2([0,1])$ ensures that (ii) holds. Finally the formula $\mathcal{K}\big([f \otimes m](A)\big) = Q(A)\mathcal{K}k$ is valid for each $A \in \mathcal{B}([0,1])$ by (6). □

Example 4.3. Let m be the vector measure defined in Proposition 4.2. Let

$$k(x,y) = \sum_{n=1}^{\infty} n\chi_{[1/(n+1),1/n)}(x)\,\chi_{[1/(n+1),1/n)}(y)$$

for all $x, y \in [0,1]$.

Then $\int_0^1 k(x,y)^2\,dy \leq 1$ for all $x \in [0,1]$, but a straightforward calculation shows that k is not the kernel of a trace class operator. The function $f : [0,1] \to L^2([0,1])$ defined by $f(x) = [k(x,\cdot)]$ for all $x \in [0,1]$ is therefore a bounded $L^2([0,1])$-valued function which is *not* m-integrable in $L^2([0,1])\widehat{\otimes}_\pi L^2([0,1])$.

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