ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 2, Summer 1998

AN APPLICATION OF REGULARLY VARYING FUNCTIONS

Y-T. HUANG, J. IBBOTSON AND Z. ZIELEZNY

ABSTRACT. Let $\mathcal{K}'_M$ be the space of distributions growing no faster than $e^{M(cx)}$ for some constant c, where M is a suitably defined function. We assume that the dual of M in the sense of Young is regularly varying at zero and infinity with positive indices of variation. We prove that two necessary conditions for a convolution operator to be hypoelliptic in $\mathcal{K}'_M$ are also sufficient.

0. Introduction. D.H. Pahk [**2**] studied hypoelliptic convolution equations in spaces $\mathcal{K}'_M$ of distributions growing no faster than $e^{M(cx)}$ for some positive constant c. Here M is a function defined on $[0,\infty)$ by

$$M(x) = \int_0^x \mu(t)\,dt \tag{0.1}$$

where μ is a continuous, increasing function on $[0,\infty)$ such that $\mu(0) = 0$ and $\mu(t)\to\infty$ as $t\to\infty$. We will also consider the extension of M, which we also denote by M, to all of $\mathbf{R}^n$ by $M(x) = M(|x|)$.

Pahk proved that the Fourier transform $\hat{S}$ of a distribution S which is a hypoelliptic convolution operator in $\mathcal{K}'_M$ satisfies the following conditions:

(H_r) There exist positive constants A_1 and B_1 such that

$$|\hat{S}(\xi)| \ge |\xi|^{-A_1}, \qquad \text{if } \xi\in\mathbf{R}^n \quad\text{and}\quad |\xi|\ge B_1.$$

(H_c) $(N(\eta)/\log|\zeta|)\to\infty$, if $\zeta = \xi + i\eta \in \mathbf{C}^n$, $|\zeta|\to\infty$ and $\hat{S}(\zeta) = 0$, where N is the dual of M in the sense of Young.

On the other hand, a convolution operator S in $\mathcal{K}'_M$ is hypoelliptic in $\mathcal{K}'_M$ if its Fourier transform $\hat{S}$ satisfies the seemingly stronger condition:

Received by the editors on April 6, 1996.
AMS 1980 *Mathematics Subject Classification.* Primary 45E10, Secondary 46F12.

(H) Given $\varepsilon > 0$, we can find a positive constant B such that, for every integer $m \geq 0$, there exists a constant C_m with the property that

$$\frac{1}{|\hat{S}(\zeta)|} \leq |\zeta|^B e^{\varepsilon N(\eta)},$$
$$\text{if } \zeta = \xi + i\eta \in \mathbf{C}^n,\ N(\eta) \leq m \log |\zeta| \quad \text{and} \quad |\zeta| \geq C_m.$$

The question whether conditions (H_r) and (H_c) imply (H) was left open. However, it is known [**3**, Theorem 8] that they are equivalent in the case where $M(x) = x^p/p$ for $p > 1$.

In this paper we prove that conditions (H_r) and (H_c) are indeed equivalent to (H), if we assume that the function N is regularly varying at zero and at infinity with positive indices of variation. The notion of regular variation at infinity was first introduced by J. Karamata [**1**] for application in probability theory.

In Section 1 we recall the basic facts concerning the space $\mathcal{K}'_M$ and the space $\mathcal{O}'_c(\mathcal{K}'_M : \mathcal{K}'_M)$ of convolution operators in $\mathcal{K}'_M$. In Section 2 we discuss some properties of regularly varying functions. Section 3 contains the fundamental lemma and Section 4 is devoted to the proof of our main result.

1. Preliminaries. Let M and N be functions on $[0, \infty)$ defined as in equation (0.1) by means of μ and ν, respectively. We say that M and N are dual in the sense of Young if μ and ν are mutual inverses. For example, x^p/p and x^q/q are dual in the sense of Yong when $p > 1$ and $1/p + 1/q = 1$. If $x, \eta \in \mathbf{R}^n$, we set $M(x) = M(|x|)$ and $N(\eta) = N(|\eta|)$. We denote by $\mathcal{K}_M$ the space of all C^∞-functions on $\mathbf{R}^n$ such that

$$p_k(\varphi) = \sup_{x \in \mathbf{R}^n, |\alpha| \leq k} e^{M(kx)} |D^\alpha \varphi(x)| < \infty, \quad k = 1, 2, \dots, \tag{1.1}$$

where, as usual, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n$ and $D^\alpha = \prod_{j=1}^n (i^{-1}(\partial/\partial x_j))^{\alpha_j}$.

The topology in $\mathcal{K}_M$ is defined by the seminorms (1.1).

The dual $\mathcal{K}_{M'}$ of $\mathcal{K}_M$ can be identified with a subspace of the space $\mathcal{D}'$ of distributions on $\mathbf{R}^n$. A distribution $u \in \mathcal{D}'$ is in $\mathcal{K}_{M'}$ if and only if there exists an integer $m \geq 0$, a constant $c \geq 0$, and a bounded continuous function f on $\mathbf{R}^n$ with

$$u = \frac{\partial^{mn}}{\partial x_1^m \cdots \partial x_n^m}[e^{M_c} f], \qquad \text{where } M_c(x) = M(cx).$$

Because of this property we call $\mathcal{K}'_M$ the space of distributions which "grow no faster than $e^{M(cx)}$" for some $c > 0$.

If $S \in \mathcal{K}'_M$ and the function $g(y) = \langle S_x, \varphi(y-x)\rangle$ is in $\mathcal{K}_M$ for every $\varphi \in \mathcal{K}_M$, then S is a convolution operator in $\mathcal{K}'_M$. In this case one can define the convolution $S * u$ of S with every distribution $u \in \mathcal{K}'_M$. We denote by $\mathcal{O}'_c(\mathcal{K}'_M : \mathcal{K}'_M)$ the space of all convolution operators in $\mathcal{K}'_M$. If $S \in \mathcal{O}'_c(\mathcal{K}'_M : \mathcal{K}'_M)$ then the Fourier transform $\hat{S}$ of S is an entire function having the following Paley-Wiener type property:

(PW) For every $\varepsilon > 0$ there exist positive constants A_2 and B_2 such that

$$|\hat{S}(\zeta)| \le |\zeta|^{B_2} e^{\varepsilon N(\eta)}, \qquad \text{if } \zeta = \xi + i\eta \in \mathbf{C}^n \quad \text{and} \quad |\zeta| \ge A_2,$$

where N is the dual to M in the sense of Young.

We denote by $\mathcal{E}_M$ the space of all C^∞-functions f on $\mathbf{R}^n$ such that

$$D^\alpha f(x) = O(e^{M(ax)}) \quad \text{as } |x| \to \infty,$$

for all multi-indices α and some constant $a \ge 0$ depending on f. If $S \in \mathcal{O}'_c(\mathcal{K}'_M : \mathcal{K}'_M)$ and $u \in \mathcal{E}_M$ then $S * u \in \mathcal{E}_M$. The distribution S is said to be hypoelliptic in $\mathcal{K}'_M$ if every solution $u \in \mathcal{K}'_M$ of the convolution equation $S * u = v$ is in $\mathcal{E}_M$ whenever $v \in \mathcal{E}_M$.

2. Regularly varying functions. We consider real-valued, increasing functions ψ defined on $[0, \infty)$ with $\psi(0) = 0$.

Definition. The function ψ is *regularly varying at infinity* if, for each $x > 0$ and for some $\rho \in \mathbf{R}$,

$$\lim_{r\to\infty} \frac{\psi(rx)}{\psi(r)} = x^\rho. \tag{2.1}$$

The number ρ is called the index of regular variation at infinity.

The function ψ is regularly varying at zero if $\psi(x^{-1})$ is regularly varying at infinity. In other words, ψ is regularly varying at zero if, for each $x > 0$ and some $\sigma \in \mathbf{R}$ we have

$$\lim_{r\to 0^+} \frac{\psi(rx)}{\psi(r)} = x^\sigma. \tag{2.2}$$

We call σ the index of regular variation at zero.

Remark 1. Since ψ is increasing and positive for $x > 0$, it is regularly varying at infinity if we assume only that the limit in (2.1) exists and is finite for two positive values of x, say x_1 and x_2, such that $x_1 \neq 1$ and $x_2 \neq 1$ and $\log x_1/\log x_2$ is irrational, see [**4**, Theorem 1.8].

One of the fundamental theorems on regularly varying functions concerns uniform convergence.

Uniform convergence theorem [**4**, Theorem 1.1]. *If ψ is regularly varying at infinity* (*or at zero*), *then for every interval* $[a, b]$, $0 < a < b < \infty$, *the convergence in* (2.1) (*in* (2.2), *respectively*) *is uniform for* $x \in [a, b]$.

Another useful result concerns the growth of a regularly varying function at infinity.

Theorem [**4**, p. 18]. *If ψ is regularly varying at infinity with index ρ, then for every $\varepsilon > 0$,*

$$\lim_{x\to\infty} x^{-\rho-\varepsilon}\psi(x) = 0.$$

In particular, if $p > \rho$, then there is a constant C such that

$$\psi(x) \leq C(1 + x^p), \quad 0 \leq x < \infty. \tag{2.3}$$

If ψ is regularly varying at infinity with index $\rho > 0$, we can improve the uniform convergence theorem in the following way.

Theorem 1. *If ψ is regularly varying at infinity with index $\rho > 0$, then, for every $h > 0$ the convergence in* (2.1) *is uniform for* $x \in [0, h]$.

Proof. For a given $\varepsilon > 0$, choose $x_0 > 0$ so that $x_0^\rho < \varepsilon/3$. Next, choose r_0 such that

$$\left|\frac{\psi(rx)}{\psi(r)} - x^\rho\right| < \frac{\varepsilon}{3} \qquad \text{for } x_0 \le x \le h \quad \text{and} \quad r \ge r_0. \tag{2.4}$$

This is possible by the uniform convergence theorem. It follows that

$$\frac{\psi(rx_0)}{\psi(r)} \le \left|\frac{\psi(rx_0)}{\psi(r)} - x_0^\rho\right| + x_0^\rho < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} \qquad \text{for } r \ge r_0,$$

and so

$$\psi(rx_0) < \frac{2\varepsilon}{3}\psi(r) \qquad \text{for } r \ge r_0.$$

Since ψ is increasing, we have

$$\psi(rx) \le \psi(rx_0) < \frac{2\varepsilon}{3}\psi(r) \qquad \text{for } 0 \le x \le x_0 \quad \text{and} \quad r \ge r_0.$$

Also, since $\rho > 0$,

$$x^\rho \le x_0^\rho < \frac{\varepsilon}{3} \qquad \text{for } 0 \le x \le x_0.$$

Therefore,

$$\left|\frac{\psi(rx)}{\psi(r)} - x^\rho\right| \le \frac{\psi(rx)}{\psi(r)} + x^\rho < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \tag{2.5}$$

for $0 \le x \le x_0$ and $r \ge r_0$.

Now, combining (2.4) and (2.5), we obtain

$$\left|\frac{\psi(rx)}{\psi(r)} - x^p\right| < \varepsilon \qquad \text{for } 0 \le x \le h \quad \text{and} \quad r \ge r_0,$$

which proves the theorem. □

In a similar way we can extend the uniform convergence theorem for functions regularly varying at zero.

Theorem 2. *If ψ is regularly varying at zero with index $\sigma > 0$, then, for every $h > 0$ the convergence in (2.2) is uniform for $x \in [0, h]$.*

Corollary 1. *If ψ is regularly varying at both zero and infinity with positive indices of variation, then, for any $h > 0$, $\psi(rx)/\psi(r)$ is bounded when $0 \leq x \leq h$ and $0 < r < \infty$.*

Proof. By Theorem 2, there are $\sigma > 0$ and $r_1 > 0$ such that

$$\frac{\psi(rx)}{\psi(r)} \leq 1 + x^\sigma \leq 1 + h^\sigma \qquad \text{for } 0 \leq x \leq h \quad \text{and} \quad 0 < r < r_1.$$

Similarly, by Theorem 1, there are $\rho > 0$ and $r_2 > r_1$ such that

$$\frac{\psi(rx)}{\psi(r)} \leq 1 + x^\rho \leq 1 + h^\rho \qquad \text{for } 0 \leq x \leq h \quad \text{and} \quad r \geq r_2.$$

If we have $r_1 \leq r \leq r_2$, then

$$\frac{\psi(rx)}{\psi(r)} \leq \frac{\psi(r_2 x)}{\psi(r_1)} \leq \frac{\psi(r_2 h)}{\psi(r_1)} \qquad \text{for } 0 \leq x \leq h.$$

Consequently,

$$\frac{\psi(rx)}{\psi(r)} \leq \max\left\{1 + h^\sigma, 1 + h^\rho, \frac{\psi(r_2 h)}{\psi(r_1)}\right\}$$

for $0 \leq x \leq h$ and $0 < r < \infty$. □

So far we have only assumed that ψ is an increasing function on $[0, \infty)$ with $\psi(0) = 0$. Now suppose that ψ is also continuous on $[0, \infty)$. Then, given $h > 0$ and $r_0 > 0$, $r \to r_0$ implies that $\psi(r) \to \psi(r_0) > 0$ and $\psi(rx) \to \psi(r_0 x)$ uniformly for $x \in [0, h]$. We therefore have the following.

Corollary 2. *If ψ is continuous and regularly varying at both zero and infinity with positive indices of variation, and if $\psi(rx)/\psi(r) \to L(x)$ as $r \to 0^+$, or $r \to \infty$, or $r \to r_0$, $0 < r_0 < \infty$, then $L(x) \leq C(1 + x^p)$, $0 \leq x < \infty$ for some positive constants C and p.*

Proof. If $r \to 0^+$ or $r \to \infty$, the conclusion follows immediately from the definition of regular variation of ψ at zero or infinity. Also, since

ψ is regularly varying at infinity, there are positive constants C and p such that

$$\psi(x) \le C(1+x^p) \qquad \text{for } 0 \le x < \infty,$$

by (2.3). Hence, if $r \to r_0$, $0 < r_0 < \infty$, we have

$$L(x) = \frac{\psi(r_0 x)}{\psi(r_0)} \le C^*(1+x^p), \quad 0 \le x < \infty,$$

where $C^* = (C(1+r_0^p)/\psi(r_0))$.

3. The basic lemma. We assume that ψ is a continuous, increasing function on $[0,\infty)$ with $\psi(0) = 0$, and we extend ψ to $\mathbf{R}$ by setting $\psi(-x) = \psi(x)$ for $x > 0$.

Lemma. *Let ψ be regularly varying at both zero and infinity with positive indices of variation. Then, for given positive constants A, B and b, we can find a constant H such that, if u is a harmonic function for $x^2+y^2 < R^2$ and satisfies the inequalities*

$$u(x,0) \le 0 \quad \textit{and} \quad u(x,y) \ge -a\psi(r) - B\psi(r), \quad x^2+y^2 < R^2,$$

it follows that

$$u(x,y) \le a\psi(y) + (B+b)\psi(r), \quad x^2+y^2 < r^2,$$

provided that $0 < a < A$ *and* $0 < r < R/H$.

Proof. Assume the lemma is false. Then we can find positive constants A, B and b, sequences of numbers a_n, R_n and r_n with $0 < a_n < A$ and $R_n/r_n \ge n$, and a sequence of harmonic functions u_n such that

$$(3.1) \qquad u_n(x,0) \le 0 \quad \text{and} \quad u_n(x,y) \ge -a_n\psi(r_n) - B\psi(r_n),$$

$x^2+y^2 < Rn^2$,

and

$$(3.2) \qquad u_n(x_n,y_n) > a_n\psi(y_n) + (B+b)\psi(r_n),$$

for some $(x_n, y_n) \in R^2$ with $x_n^2 + y_n^2 < r_n^2$.

We now set

$$v_n(x,y) = u_n(r_n x, r_n y)/\psi(r_n), \qquad x_n^* = x_n/r_n, \qquad y_n^* = y_n/r_n.$$

Since $R_n/r_n \geq n$, it follows from (3.1) and (3.2) that

$$\text{(3.3)} \quad v_n(x,0) \leq 0 \quad \text{and} \quad v_n(x,y) \geq -a_n \frac{\psi(r_n y)}{\psi(r_n)} - B, \quad x^2 + y^2 < n^2,$$

and

$$\text{(3.4)} \qquad v_n(x_n^*, y_n^*) > a_n \frac{\psi(r_n y_n^*)}{\psi(r_n)} + B + b.$$

By assumption, ψ is regularly varying at zero and at infinity with positive indices of variation. Therefore, given any $h > 0$, the functions $\psi(r_n y)/\psi(r_n)$ are uniformly bounded for $|y| \leq h$ by Corollary 1. Accordingly, from (3.3) and Harnack's inequality, it follows that the sequence $\{v_n\}$ is uniformly bounded on every compact set in $\mathbf{R}^2$. Applying the "compactness theorem" for harmonic functions, we can select a subsequence $\{v_n'\}$ of $\{v_n\}$ which converges to a harmonic function v uniformly on every compact subset of $\mathbf{R}^2$. Since $(x_n^*)^2 + (y_n^*)^2 \leq 1$, $0 < a_n < A$ and $0 < r_n < \infty$, we can choose the subsequence $\{v_n'\}$ of $\{v_n\}$ so that the sequences $\{x_n^{*\prime}\}, \{y_n^{*\prime}\}$, $\{r_n'\}$ and $\{a_n'\}$ have limits x_0, y_0, r_0 and a_0, respectively, where r_0 may be ∞. Then, by Theorems 1 and 2, Remark 2 and Corollary 2, $\psi(r_n' y)/\psi(r_n')$ converges uniformly on every interval $|y| \leq h$ to a function $L(y)$ such that

$$0 \leq L(y) \leq C(1 + |y|^p), \quad y \in \mathbf{R},$$

for some constants C and p. In particular, $\psi(r_n' y_n^*)/\psi(r_n')$ converges to $L(y_0)$.

Thus, we have

$$\text{(3.6)} \qquad v(x,y) \leq 0, \quad v(x,y) \geq -a_0 L(y) - B, \quad (x,y) \in \mathbf{R}^2,$$

and

$$\text{(3.7)} \qquad v(x_0, y_0) \geq a_0 L(y_0) + B + b.$$

We apply Harnack's inequality again and infer from (3.5) and (3.6) that $(1+|y|^p)^{-1}v(x,y)$ is bounded on $\mathbf{R}^2$. Hence, v is a harmonic polynomial that does not depend on x, and so it must be a linear function of y. Suppose that $v(x,y) = cy + d$. Then $d \leq 0$, by the first inequality in (3.6) and, from (3.6) and (3.7), it follows that

$$a_0 L(y) + cy + B + d \geq 0 \tag{3.8}$$

and

$$a_0 L(y_0) - cy_0 + B + b - d \leq 0. \tag{3.9}$$

In particular, setting $y = -y_0$ in (3.8), we obtain

$$a_0 L(y_0) - cy_0 + B + d \geq 0,$$

which contradicts the inequality (3.9) since $b > 0$ and $d \leq 0$. Thus, the lemma is proved. □

4. Application to the hypoellipticity problem in $\mathcal{K}'_M$. As stated in Section 1, every distribution $S \in \mathcal{O}'_C(\mathcal{K}'_M : \mathcal{K}'_M)$ can be characterized in its Fourier transform $\hat{S}$ by the Paley-Wiener type condition (PW). If S is hypoelliptic in $\mathcal{K}'_M$, then $\hat{S}$ also satisfies conditions (H_r) and (H_c). Note that the function N appearing in conditions (H_r) and (PW) is the dual to M in the sense of Young. Therefore, N is continuous and increasing on $[0,\infty)$ with $N(0) = 0$. We now prove our main result.

Theorem. *Suppose that the function N is regularly varying at zero and infinity with positive indices of variation. Then conditions (H_r), (H_c) and* (PW) *imply condition* (H).

Proof. Let $\zeta = \xi + i\eta \in \mathbf{C}$ be such that $0 < N(\eta) < m \log|\zeta|$. We define an analytic function of one complex variable z by

$$F_\zeta(z) = \hat{S}\left(\xi + z\frac{\eta}{|\eta|}\right),$$

for all z with $N(|z|) < m_0 \log|\zeta|$, where m_0 is a constant to be determined later. If $|\zeta|$ is sufficiently large, then $F_\zeta(z) \neq 0$, in view of (H_c). Applying condition (H_r) and the fact that $N(x)/x \to \infty$ as $x \to \infty$, for $x \in \mathbf{R}$, we obtain

$$F_\zeta(x) \geq (2|\zeta|)^{-B_1}, \qquad \text{if } x \in \mathbf{R} \quad \text{and} \quad N(x) < m_0 \log|\zeta|, \tag{4.1}$$

provided that $|\zeta|$ is sufficiently large.

Also, by condition (PW), for a given $\varepsilon > 0$, we have

$$|F_\zeta(z)| \leq (2|\zeta|)^{B_2} e^{\varepsilon N(y)}, \qquad \text{if } z \in \mathbf{C} \quad \text{and} \quad N(|z|) < m_0 \log|\zeta| \tag{4.2}$$

when $|\zeta|$ is sufficiently large.

We consider the function

$$u_\zeta(x,y) = \log\{(2|\zeta|)^{-B_1}|F_\zeta(z)|^{-1}\}, \qquad z = x+iy,$$

which is harmonic when $N(|z|) < m_0 \log|\zeta|$ and when $|\zeta|$ is sufficiently large. Moreover, from (4.1) and (4.2), it follows that

$$u_\zeta(x,0) \leq 0 \qquad \text{if } N(x) < m_0 \log|\zeta|$$

and

$$\begin{aligned} u_\zeta(x,y) &\geq -\varepsilon N(y) - (B_1+B_2)\log(2|\zeta|) \\ &\geq -\varepsilon N(y) - (B_1+B_2+1)\log(|\zeta|), \end{aligned}$$

if $N(|z|) < m_0 \log|\zeta|$. In both inequalities, we assume that $|\zeta|$ is sufficiently large.

We now apply the basic lemma with $A = 1+\varepsilon$, $B = (B_1+B_2+1)/(m+1)$, $b = 1/(m+1)$, $r = N^{-1}[(m+1)\log|\zeta|]$ and $\psi = N$. Let H be the constant in that lemma. Since N is regularly varying at zero and at infinity, there is a constant $h > 0$ such that $N(Hx) < hN(x)$, $0 \leq x < \infty$. Hence $N^{-1}(x) < N^{-1}(hx)/H$, $0 \leq x < \infty$. If we set $m_0 = h(m+1)$, then from the lemma it follows that

$$u_\zeta(x,y) \leq \varepsilon N(y) - (B_1+B_2+2)\log(|\zeta|) \tag{4.3}$$

if $N(|z|) < (m+1)\log|\zeta|$.

Since $N(\eta) < m \log|\zeta| < (m+1)\log|\zeta|$, we may substitute $z = i\eta$ in (4.3), which yields

$$\log\left\{\frac{(2|\zeta|)^{-B_1}}{|\hat{S}(\zeta)|}\right\} \leq \varepsilon N(\eta) + (B_1 + B_2 + 2)\log|\zeta|.$$

Hence, we conclude that

$$\frac{1}{|\hat{S}(\zeta)|} \leq |\zeta|^{(B_1+B_2+3)} e^{\varepsilon N(\eta)}, \qquad \text{if } N(\eta) < m\log|\zeta|,$$

when $|\zeta|$ is sufficiently large. This completes the proof of the theorem. □

Corollary. *If N is regularly varying at zero and at infinity with positive indices of variation, the condition* (H) *is necessary for the hypoellipticity of S in $\mathcal{K}'_M$.*

REFERENCES

1. J. Karamata, *Sur un mode de croissance reguliere des fonctions*, Mathematica (Cluj) **4** (1930), 38–53.

2. D.H. Pahk, *Hypoelliptic convolution operators in the spaces $\mathcal{K}'_M$*, Dissertation, SUNY at Buffalo, 1981.

3. G. Sampson and Z. Zielezny, *Hypoelliptic convolution equations in $\mathcal{K}'_p$, $p > 1$*, Trans. Amer. Math. Soc. **223** (1976), 133–154.

4. E. Seneta, *Regularly varying functions*, Lecture Notes in Math. **508**, 1976.

Tatung Institute of Technology, Department of Applied Mathematics, 40 Chungshan North Road, Sec. 3, Taipei, Taiwan

Merrimack College, Department of Mathematics and Computer Science, Turnpike St., N. Andover, MA 01845

SUNY at Buffalo, Department of Mathematics, 106 Diefendorf Hall, Buffalo, NY 14124
E-mail address: ZIELEZNY@ACSU.BUFFALO.EDU