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# ASYMPTOTIC CONSTANCY OF SOLUTIONS OF DELAY-DIFFERENTIAL EQUATIONS OF IMPLICIT TYPE

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ABSTRACT. We study delay-differential equations with time-state depending lag. We will prove that, under integrability conditions, any solution converges. Reciprocally, for any possible $\xi$, there exists a solution $x$ such that $x(t) \to \xi$.

**1. Introduction.** Let $b$ and $\tau$ be two positive reals, $I = [0, \infty)$ and $B_n[0, 2b]$ the closed ball centered at the origin with radius $b < \infty$, contained in $\mathbf{C}^n$.

Consider the functions $f$ and $r$ satisfying the following assumption $C_1$.

$C_1$) $f : I \times B_n[0, 2b] \to \mathbf{C}^n$ is a continuous function satisfying $|f(t, x)| \leq \mu(t)|x|$ where $\mu : I \to I$ is continuous and $r : I \times B_n[0, b] \to [0, \tau]$ is another continuous function.

We are interested in the existence and asymptotic behavior of solutions of delay-differential equations of the type

$$\dot{x}(t) = f(t, x(t - r(t, x(t))) - x(t)). \tag{1}$$

Define, for $t \geq 0$,

$$m_b(t) = \sup_{|x| \leq b} r(t, x) \quad \text{and} \quad \lambda_b(t) = \mu(t) \cdot \mu_t \cdot m_b(t),$$

where $\mu_t = \max_{s \in [t-\tau, t]} \hat{\mu}(s)$ and $\hat{\mu}(s) = \mu(s)$ for $s \geq 0$, $\hat{\mu}(s) = \mu(0)$ for $s < 0$.

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By a solution of (1) defined on $[t_0 - \tau, T)$, we mean a function $x : [t_0 - \tau, T) \to \mathbf{C}^n$ for some $t_0 \geq 0$ and $t_0 < T \leq \infty$, which is continuous on $[t_0 - \tau, T)$, differentiable on $[t_0, T)$, and satisfies $|x(t)| \leq b$ and equation (1) for $t \geq t_0$.

In many cases $f$ and $r$ are globally defined and $r$ does not need to be bounded, for which we will consider equation (1), assuming the following condition $C_2$.

$C_2$) $f$ is a continuous function defined in whole $I \times \mathbf{C}^n$, satisfying $|f(t, x)| \leq \mu(t)|x|$ where $\mu : I \to I$ is continuous and $r$ is a nonnegative, continuous function, defined on $I \times \mathbf{C}^n$ (and not necessarily bounded).

In this case, a function $x : [t_0 - \tau, \infty) \to \mathbf{C}^n$ is a solution of (1) defined on $[t_0 - \tau, \infty)$, if $\sup_{t \geq t_0} r(t, x(t)) \leq \tau$, $x$ is continuous on $[t_0 - \tau, \infty)$, differentiable on $[t_0, \infty)$ and satisfies (1) for $t \geq t_0$.

We will prove that if $\lambda_b \in L^1(I)$, then for any $\xi \in \mathbf{C}^n$ such that $|\xi| < b$, there exists $t_0$ big enough and a solution $x$ of (1) defined on $[t_0 - \tau, \infty)$ satisfying $|x(t)| \leq b$ for $t \geq t_0$ and

$$x(t) = \xi + 0\left( \int_t^\infty \lambda_b(s)\, ds \right), \quad t \to \infty. \tag{2}$$

Conversely, any solution $x$ of (1) defined on $[t_0 - \tau, \infty)$ such that $|x(t)| \leq b$, satisfies (2) for some $\xi \in \mathbf{C}^n$. Moreover, demanding the additional condition of smallness for $t$ large of

$$\gamma_b(t) = \int_{t - m_b(t)}^t \mu(s)\, ds,$$

this result is valid for solutions $x$ with arbitrary continuous initial condition. Furthermore, under uniqueness hypothesis any solution $x$ of Equation (1) initially small enough satisfies formula (2).

These results are also valid, in some cases, for several lags. So we can consider the equation

$$\dot{x}(t) = \sum_{i=1}^n f_i(t, x(t - r_i(t, x(t))) - x(t)) \tag{3}$$

if the lags $r_1, r_2, \ldots, r_n$ are bounded. We remark that we do not impose restrictions of order to the lags, see [**9, 14, 15, 16**].

Under the hypothesis $\lambda_b \in L^1(I)$, the same conclusions obtained for the Equation (1) are also obtained for a system

$$\dot{x}(t) = g(t, x(t)) - g(t, x(t - r(t, x(t)))) \tag{4}$$

satisfying $|g(t,x) - g(t,y)| \leq \mu(t)|x - y|$ for all $t, x, y$.

This equation appears extensively in models of various phenomena as growth of population, epidemiology, etc., see [**2, 3, 4**].

The existence of convergent solutions in delay differential equations has been studied for systems

$$\dot{x}(t) = f(t, x(g_1(t)), x(g_2(t)), \dots, x(g_n(t)))$$

with several delays under $L^1$ integrability of $f(\cdot, x)$, for $x$ fixed. See [**1, 6, 7, 8, 9, 12**–**16**]. However, Equation (1) is considered in [**3, 5, 6, 7**].

Some results along this line were obtained by Gyori [**6**] and Pituk [**13**], studying equation (3) or (4) for $r(t,x) = r(t)$. As an example, if we consider the equation

$$\dot{x}(t) = P(t)[x(t) - x(t - r)], \quad r \text{ constant}, \tag{5}$$

the condition $\lambda_b \in L^1(I)$, implied by $\mu_t\mu \in L^1(I)$ is similar to condition $P \in L^2(I)$ obtained by Atkinson and Haddock [**1**] to study the asymptotic constancy of solutions of (5). In this paper the same conclusion is true for $r = r(t,x)$ under the same condition or even with one weaker.

In our work the point is to consider $r = r(t,x)$ sufficiently small, (the smallness condition is $\lambda_b \in L^1(I)$), so that the solutions of the system (1) behave as constants when $t$ tends to infinity. To our knowledge, the problem studied here and the results obtained have not appeared in the literature so far.

**2. Main results.** Through this section we assume that hypothesis $C_1$ holds. Let $E = C([t_0 - \tau, +\infty), \mathbf{C}^n)$, the topological vector space of the continuous functions on $[t_0 - \tau, +\infty)$ provided by the open-compact topology. Define $C_\mu \subseteq E$ by:

$x \in C_\mu$ if and only if $x$ is constant on $[t_0-\tau, t_0]$, $|x(t)| \le b$, for $t \ge t_0$ and $|x(t)-x(t')| \le 2b\mu_t(t-t')$, for $t, t' \ge t_0$, $0 \le t-t' \le \tau$.

$C_\mu$ is a closed and convex set in $E$. Given $\xi \in \mathbf{C}^n$, $|\xi| < b$, we define the function

$$\begin{aligned} \mathcal{N} : C_\mu &\longrightarrow E \\ x &\longrightarrow \mathcal{N}(x), \end{aligned}$$

where

$$\begin{aligned} &\mathcal{N}x(t) = \xi - \int_t^\infty f(s, x(s-r(s,x(s)))) - x(s))\, ds \\ &\qquad\qquad \text{for } t \ge t_0 \\ &\mathcal{N}x(t) = \mathcal{N}x(t_0) \quad \text{for } t_0-\tau \le t \le t_0. \end{aligned}$$

$\mathcal{N}$ is well defined, since for $x \in C_\mu$,

$$\begin{aligned} |f(s, x(s-r(s,x(s))) - x(s))| &\le \mu(s)|x(s-r(s,x(s))) - x(s)| \\ &\le 2b\mu(s)\mu_s r(s,x(s)) \\ &\le 2b\mu(s)\mu_s \cdot m_b(s) = 2b\lambda_b(s), \end{aligned} \tag{6}$$

from where the integral is finite and $\mathcal{N}x \in E$. Now we prove that

1. $\mathcal{N}(C_\mu) \subseteq C_\mu$.
2. $\mathcal{N}$ is continuous in $C_\mu$, with respect to the open-compact topology.
3. $C_\mu$ is a closed, convex and compact set.

Proving the above, the fixed point theorem of Tychonoff implies that there exists $x \in C_\mu$ satisfying $\mathcal{N}(x) = x$, i.e.,

$$\begin{aligned} &x(t) = \xi - \int_t^\infty f(s, x(s-r(s,x(s))) - x(s))\, ds \\ &\qquad\qquad \text{for } t \ge t_0 \\ &x(t) = x(t_0), \quad \text{for } t_0-\tau \le t \le t_0. \end{aligned}$$

Then

$$\dot{x}(t) = f(t, x(t-r(t,x(t))) - x(t)) \quad \text{for } t \ge t_0,$$

and by (6) we have $x(t) = \xi + 0(\int_t^\infty \lambda_b(s)\, ds)$, $t \to \infty$.

**Lemma 1.** *If* $\xi \in \mathbf{C}^n$ *and* $t_0 \geq \tau$ *satisfy*

$$|\xi| + 2b \int_{t_0}^{\infty} \lambda_b(s)\, ds \leq b,$$

*then* $\mathcal{N}(C_\mu) \subseteq C_\mu$.

*Proof.* Let $x \in C_\mu$ and $y = \mathcal{N}(x)$. By definition of $\mathcal{N}$, $y$ is constant on $[t_0 - \tau, t_0]$. Moreover, on account of (6),

$$\begin{aligned} |y(t)| &\leq |\xi| + \int_t^{\infty} |f(s, x(s - r(s, x(s))) - x(s))|\, ds \\ &\leq |\xi| + 2b \int_{t_0}^{\infty} \lambda_b(s)\, ds \leq b \end{aligned}$$

for any $t \geq t_0$. On the other hand,

$$\dot{y}(t) = f(t, x(t, r(t, x(t))) - x(t))$$

for $t \geq t_0$, and hence,

$$\begin{aligned} |\dot{y}(t)| &\leq \mu(t)|x(t - r(t, x(t))) - x(t)| \\ &\leq \mu(t)(|x(t - r(t, x(t)))| + |x(t)|) \\ &\leq 2b\mu(t). \qquad \square \end{aligned}$$

**Lemma 2.** *With respect to the open-compact topology, the operator* $\mathcal{N}$ *is continuous in* $C_\mu$.

*Proof.* Let $\{x_n\}_n$ be a sequence in $C_\mu$, and let $x$ in $C_\mu$. Assume that $x_n \to x$. We must prove that $\mathcal{N}(x_n) \to \mathcal{N}(x)$ in the open-compact topology.

Let $[t_0 - \tau, L]$ be fixed, and estimate $|\mathcal{N}(x_n)(t) - \mathcal{N}(x)(t)|$ for $t_0 - \tau \leq t \leq L$. Since $\mathcal{N}(x_n)$ and $\mathcal{N}(x)$ are constants in $[t_0 - \tau, t_0]$, we have, for $t \geq t_0 - \tau$,

$$\begin{aligned} (7) \quad |\mathcal{N}(x_n)(t) - \mathcal{N}(x)(t)| \leq \int_t^{\infty} &|f(s, x(s - r(s, x(s))) - x(s)) \\ &- f(s, x_n(s - r(s, x_n(s))) - x_n(s))|\, ds. \end{aligned}$$

Let

$$\Delta(x,s) = f(s, x(s - r(s, x(s))) - x(s)),$$
$$\Delta_n(x,s) = \Delta(x,s) - \Delta(x_n, s).$$

We will prove that, for $s \geq t_0 - \tau$,

$$\lim_{n\to\infty} \Delta_n(x,s) = 0. \tag{8}$$

We will use the uniform continuity of $f$ on $[t_0, L]\times B_n[0, 2b]$, of $x$ on $[t_0-\tau, L]$, of $r$ on $[t_0, L] \times B_n[0, b]$ and that $x_n \to x$ uniformly on compacts. Thus, there exists $\delta > 0$ so that $(s, x), (s, x') \in [t_0, L] \times B_n[0, 2b]$, $|(s, x) - (s, x')| < \delta$ imply $|f(s, x) - f(s, x')| < \varepsilon$. Moreover, there exists $\delta' > 0$ so that $|t-t'| < \delta'$, $t, t' \in [t_0, L]$ imply $|x(t)-x(t')| < \delta/3$. Finally, there exists $\delta'' > 0$ so that $(s, x), (s, x') \in [t_0, L] \times B_n[0, b]$, $|(s, x) - (s, x')| = |x - x'| < \delta''$ imply $|r(s, x) - r(s, x')| < \delta/3$ and there exists $N$ so that $n \geq N$ imply $|x_n(s) - x(s)| < \delta/3$ and $|x_n(s) - x(s)| < \delta''$ for any $s \in [t_0, L]$.

Then, $n \geq N$ implies $|(s, x_n(s)) - (s, x(s))| = |x_n(s) - x(s)| < \delta''$ on $[t_0, L]$ and $|r(s, x_n(s)) - r(s, x(s))| < \delta'$ or

$$|(s - r(s, x_n(s))) - (s - r(s, x(s)))| < \delta'$$

on $[t_0, L]$. This implies

$$|x_n(s - r(s, x_n(s))) - x(s - r(s, x(s)))| < \delta/3$$

since $s - r(s, x_n(s))$ and $s - r(s, x(s))$ belong to $[t_0 - \tau, L]$.

Now, using the above, we estimate $\Delta_n(x, s)$. If $n \geq N$ and $s \in [t_0, L]$, then

$$\begin{aligned}
|(x(s - r(s, x(s))) - x(s)) - (x_n(s - r(s, x_n(s))) - x_n(s))|&\\
\leq |x(s - r(s, x(s))) - x(s - r(s, x_n(s)))|&\\
+ |x(s - r(s, x_n(s))) - x_n(s - r(s, x_n(s)))|&\\
+ |x_n(s) - x(s)|&\\
< \delta/3 + \delta/3 + \delta/3 = \delta.&
\end{aligned}$$

So $(s, x(s-r(s, x(s)))-x(s))$ and $(s, x_n(s-r(s, x_n(s)))-x_n(s))$ belong to $[t_0, L] \times B_n[0, 2b]$ and hence $\Delta_n(x, s) < \varepsilon$. As $[t_0, L]$ is arbitrary, (8) follows. Then, by the Lebesgue theorem on dominated convergence, we have

$$\lim_{n\to\infty}\int_{t_0}^{\infty}\Delta_n(x,s)\,ds = 0. \tag{9}$$

From (7) and (9), we get

$$\begin{aligned}|\mathcal{N}(x_n)(t)-\mathcal{N}(x)(t)| &\le \int_t^{\infty}\Delta_n(x,s)\,ds\\ &\le \int_{t_0}^{\infty}\Delta_n(x,s)\,ds \longrightarrow 0,\\ &\text{as } n\to\infty,\end{aligned}$$

i.e., $\mathcal{N}(x_n)\to\mathcal{N}(x)$ uniformly on $[t_0,+\infty)$. □

**Lemma 3.** *$C = C_\mu$ is a compact set in $E$.*

*Proof.* Any $x\in C$ is a Lipschitz function on any interval $[t_0,L]$. In fact, for $t,t'\in[t_0,L]$, $t\ge t'$, there exists $n\in\mathbf{N}$ such that

$$t'+(n-1)\tau < t \quad\text{and}\quad t'+n\tau\ge t.$$

Define, for $i=1,2,\dots,n-1$; $t_i' = t'+i\tau$. So

$$\begin{aligned}|x(t)-x(t')| &= |x(t)-x(t_{n-1}')| + |x(t_{n-1}')-x(t_{n-2}')|\\ &\quad + \cdots + |x(t_1')-x(t')|\\ &\le 2b\mu_t(t-t_{n-1}') + 2b\mu_{t_{n-1}'}(t_{n-1}'-t_{n-2}')\\ &\quad + \cdots + 2b\mu_{t_1'}(t_1'-t')\\ &\le 2b\bar{u}(t-t_{n-1}') + 2b\bar{\mu}(t_{n-1}'-t_{n-2}')\\ &\quad + \cdots + 2b\bar{\mu}(t_1'-t')\\ &\le 2b\bar{\mu}(t-t'),\end{aligned}$$

where $\bar{\mu} = \max_{s\in[t_0-\tau,L]}|\hat{\mu}(s)|$.

Since the Lipschitz constant is the same for any $x\in C$, we deduce that $C$ is equicontinuous on $[t_0,L]$. Moreover, for any $x$ in $C$, $|x(t)|\le b$, for $t\ge t_0$, in particular on $[t_0,L]$.

So we can apply the Arzela-Ascoli theorem on $[t_0, t_0 + 1]$ to a sequence $\{x_n\}_n$ in $C$ to obtain a uniformly convergent subsequence on $[t_0, t_0+1]$. Next, to this sequence obtained, we again apply Arzela-Ascoli theorem on the interval $[t_0, t_0 + 2]$ and we obtain a uniformly convergent subsequence on $[t_0, t_0 + 2]$. Continuing this process, we finally get a subsequence $\{x_n\}_n$, uniformly convergent on compacts, to a function $x \in C$. Since $E$ is metrizable, this proves that $C$ is compact. $\square$

We are ready to present the main result in this section.

**Theorem 1.** *Let $\lambda_b \in L^1(I)$ and assume that hypothesis $C_1$ holds. Then*

i) *For any $\xi \in \mathbf{C}^n$, $|\xi| < b$ and $t_0 \geq 0$ such that $|\xi| + 2b\int_{t_0}^{\infty} \lambda_b(s)\,ds \leq b$, there exists a solution $x$ of equation* (1) *defined on $[t_0 - \tau, +\infty)$ satisfying $|x(t)| \leq b$ for $t \geq t_0 - \tau$ and formula* (2) *holds.*

ii) *Moreover, for any solution $x$ of* (1) *defined for $t \geq t_0 - \tau$ such that $|x(t)| \leq b$, for $t \geq t_0 - \tau$ there exists $\xi \in \mathbf{C}^n$, $|\xi| \leq b$, satisfying* (2).

*Proof.* Part i). It follows from Lemmas 1–3 and the Tychonoff fixed point theorem. To prove part ii), let $x$ be a solution of (1) defined on $[t_0 - \tau, +\infty)$, $t_0 \geq 0$, such that $|x(t)| \leq b$ for $t \geq t_0 - \tau$. We remark that $t \geq t' \geq t_0$ and $0 \leq t - t' \leq \tau$ imply

$$|x(t) - x(t')| \leq 2b\mu_t(t - t'). \tag{10}$$

In fact,

$$x(t) - x(t') = \int_{t'}^{t} \dot{x}(s)\,ds = \int_{t'}^{t} f(s, x(s - r(s, x(s))) - x(s))\,ds,$$

and

$$\begin{aligned}|x(t) - x(t')| &\leq \int_{t'}^{t} |f(s, x(s - r(s, x(s))) - x(s))|\,ds \\ &\leq \int_{t'}^{t} \mu(s)(|x(s - r(s, x(s)))| + |x(s)|)\,ds \\ &\leq \int_{t'}^{t} 2b\mu(s)\,ds \leq 2b\mu_t(t - t').\end{aligned}$$

Thus, (10) implies (6), and we can write:

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} f(s, x(s - r(s, x(s))) - x(s))\, ds \\ &= \xi + 0\bigg( \int_t^{\infty} \lambda_b(s)\, ds \bigg), \quad t \to \infty, \end{aligned}$$

where $\xi = x(t_0) + \int_{t_0}^{\infty} f(s, x(s - r(s, x(s)))\, ds$, which satisfies $|\xi| \le b$, since $x(t) \to \xi$ as $t \to \infty$ and $|x(t)| \le b$ for all $t \ge t_0 - \tau$. Then the proof of Theorem 1 is complete. □

For $b > 0$, define $\rho_b = \sup\{r(t, x) : (t, x) \in I \times B_n[0, b]\}$. Now we can establish the following corollary.

**Corollary 1.** *Assuming condition $C_2$ and that $\rho_b < \infty$ and $\lambda_b \in L^1(I)$ for any (respectively some) $b > 0$, then*

i) *for any $\xi \in \mathbf{C}^n$, respectively $|\xi| < b$, there exists a bounded solution $x$ of (1), respectively with $|x(t)| \le b$, defined on $[t_0 - \tau, \infty)$, for $t_0$ big enough and an adequate $\tau$, respectively $\tau = \rho_b$, satisfying the asymptotic formula (2).*

ii) *Any bounded solution $x$, respectively $|x(t) \le b$ of (1), defined on $[t_0 - \tau, \infty)$ satisfies (2) for some $\xi \in \mathbf{C}^n$, respectively $|\xi| < b$.*

*Proof.* i) Let $b > 0$ such that $|\xi| < b$, and we consider $f$ and $r$ restricted to domain $I \times B_n[0, 2b]$ and $I \times B_n[0, b]$, respectively. In this domain, as $\rho_b < \infty$, the delay $r$ is bounded by $\tau = \rho_b$ so that $r : I \times B_n[0, b] \to [0, \tau]$.

As $\lambda_b \in L^1(I)$, Theorem 1(i) can be applied, and for $t_0$ sufficiently large, we obtain a solution $x$ of (1) defined on $[t_0 - \tau, \infty)$ satisfying $|x(t)| \le b$ and the asymptotic formulae (2), with $\xi$ given above.

ii) Let $x$ be a solution of (1) defined on $[t_0 - \tau, \infty)$ and bounded by $b > 0$. Proceeding as in i), the conclusion ii) follows from Theorem 1 (ii). □

**3. Complete solution to the problem.** Now, with an additional condition, we are able to obtain the above results for any continuous initial function $\phi$.

To state the next theorem, denote by $C_{t_0,b}$ the class formed by the continuous functions $\phi : [t_0 - \tau, t_0] \to \mathbf{C}^n$ such that $|\phi(t)| \leq b$. Let $\tau_1$ be the first real $\geq t_0$ such that

$$\tau_1 - m_b(\tau_1) = t_0 \quad \text{and} \quad s - m_b(s) > t_0 \quad \text{for } s > \tau_1. \tag{11}$$

Let $\gamma_b(t) = \int_{t-m_b(t)}^{t} \mu(s)\, ds$ defined for $t \geq \tau_1$.

Take $c \in \mathbf{C}^n$ and $t_0 \geq \tau$ satisfying

$$|c| + 2b\gamma_b(\tau_1) + 2b \int_{\tau_1}^{\infty} \lambda_b(s)\, ds \leq b, \tag{12}$$

and $E = C([t_0 - \tau, +\infty), \mathbf{C}^n)$ as in Theorem 1. For $\phi \in C_{t_0,b}$ such that $\phi(t_0) = c$, define the set $S$ formed by $x \in E$ such that $x|_{[t_0-\tau,t_0]} = \phi$, $|x(t)| \leq b$ for $t \geq t_0$ and

$$|x(t) - x(t')| \leq 2b\mu_t(t - t')$$

for $t, t' \geq t_0$, $0 \leq t - t' \leq \tau$. Now define

$$\begin{aligned} T : S &\longrightarrow E \\ x &\longrightarrow T(x) = y, \end{aligned}$$

where

$$\begin{aligned} &y(t) = c + \int_{t_0}^{t} f(s, x(s - r(s, x(s))) - x(s))\, ds, \quad t \geq t_0 \\ &y(t) = \phi(t), \quad t_0 - \tau \leq t \leq t_0. \end{aligned}$$

As in Theorem 1, we will prove that 1) $T(S) \subseteq S$, 2) $T$ is continuous in $S$ in the open-compact topology and 3) $S$ is closed, convex and compact. Once this is proved, by the Tychonoff fixed point theorem there exists $x \in S$ such that $T(x) = x$. Then $x$ satisfies Equations (1) and (6). So, as $\lambda_b \in L_1(I)$, we get

$$x(t) = \xi + 0\bigg( \int_{t}^{\infty} \lambda_b(s)\, ds \bigg), \quad t \to \infty.$$

That is, $x$ satisfies formula (2), where

$$\xi = c + \int_{t_0}^{\infty} f(s, x(s - r(s, x(s))) - x(s))\, ds,$$

and $|\xi| \le b$, according to (12).

**Lemma 4.** *$T(S) \subseteq S$.*

*Proof.* Let $x \in S$ and $y = T(x)$. First, obviously $y|_{[t_0-\tau,t_0]} = \phi$, by definition of $T$, We consider only $t \ge \tau_1$. The case $t_0 \le t \le \tau_1$ will be included. We have

$$\begin{aligned} |y(t)| &\le |c| + \int_{t_0}^{\tau_1} |f(s, x(s - r(s, x(s))) - x(s))|\, ds \\ &\quad + \int_{\tau_1}^{t} |f(s, x(s - r(s, x(s))) - x(s))|\, ds \\ &\le |c| + 2b \int_{t_0}^{\tau_1} \mu(s)\, ds + 2b \int_{\tau_1}^{\infty} \mu(s)\mu_s m_b(s)\, ds \\ &\le |c| + 2b\gamma_b(\tau_1) + 2b \int_{\tau_1}^{\infty} \lambda_b(s)\, ds \le b. \end{aligned}$$

This proves that $|y(t)| \le b$ for $t \ge t_0$. The rest of the proof follows as in Lemma 1. □

**Lemma 5.** *$T$ is continuous on $S$, in the open-compact topology.*

The proof uses the same techniques employed in Lemma 2, and the dominated convergence theorem of Lebesgue is not needed. We omit the proof.

The proof of the compactness of $S$ is the same as that in Lemma 3. So, by the Tychonoff theorem, we have

**Theorem 2.** *Let $\lambda_b \in L_1(I)$, and assume that, for $t$ sufficiently large, $\gamma_b(t)$ is sufficiently small for which $c \in \mathbf{C}^n$ and $t_0 \ge \tau$ satisfy* (12)*. Then, for any $\phi \in C_{t_0,b}$ with $\phi(t_0) = c$, the equation* (1) *has a solution $x$ satisfying $x|_{[t_0-\tau,t_0]} = \phi$, $|x(t)| \le b$ for $t \ge t_0 - \tau$ and the asymptotic formulae* (2) *for some $\xi \in \mathbf{C}^n$, $|\xi| \le b$.*

If we consider now the Equation (1), under the condition $C_2$, and we assume that, for some $b > 0$, $\rho_b < \infty$, we can establish the following corollary.

**Corollary 2.** *Suppose that condition $C_2$ holds and that, for some $b > 0$, $\rho_b < \infty$ and with $\tau = \rho_b$, $\lambda_b \in L^1(I)$. Assume that, for $t$ sufficiently large, $\gamma_b(t)$ is sufficiently small for which $c \in \mathbf{C}^n$ and $t_0 \geq \tau$* (12)*. Then, for any $\phi \in C_{t_0,b}$ with $\phi(t_0) = c$, the Equation* (1) *has a solution $x$ defined on $[t_0 - \tau, \infty)$ satisfying $x|[t_0 - \tau, t_0] = \phi$, $|x(t)| \leq b$ for $t \geq 0$ and the asymptotic formula* (2)*, for some $\xi \in \mathbf{C}^n$, $|\xi| \leq b$.*

*Proof.* If we consider $r$ restricted to the domain $I \times B_n[0, b]$, then $r$ is bounded by $\tau = \rho_b$. That is, $r : I \times B_n[0, b] \to [0, \tau]$, and thus $\lambda_b$ (which depends on $\tau$) is in $L^1(I)$. Restricting now $f$ to $I \times B_n[0, 2b]$, Theorem 2 can be applied. □

**4. Assuming uniqueness of solutions.** We consider once more Equation (1) and the problem of the existence of solutions $x$, subject to an initial condition $x = \phi$ over the interval $[t_0 - \tau, t_0]$, being $t_0 \geq 0$ a real arbitrary.

We suppose that, for any $t_0$ and any $\phi \in C_{t_0,b}$, there exists a unique solution of (1) such that $x|_{[t_0-\tau,t_0]} = \phi$. Under these hypotheses, and assuming that $\gamma_b(t)$ is sufficiently small for $t$ large, we will prove that any solution $x$ of Equation (1) satisfies (2).

**Theorem 3.** *Suppose that $\lambda_b \in L^1(I)$ and that Equation* (1) *has a unique solution for any $t_0 \geq 0$ and $\phi \in C_{t_0,b}$. Assume that $\gamma_b(t)$ is small enough for $t$ sufficiently large. Then*

i) *Every solution $u$ of Equation* (1)*, defined on $[s_0 - \tau, +\infty)$, $s_0 \geq 0$, and small enough on $[s_0 - \tau, s_0]$, satisfies* (2) *for some $|\xi| \leq b$.*

ii) *For any $\xi \in \mathbf{C}^n$, $|\xi| < b$, any solution $u$ of Equation* (1) *such that $u(t) \to \xi$ as $t \to \infty$, satisfies* (2).

*Proof.* i) Let $u$ be a solution of Equation (1), defined on $[s_0 - \tau, +\infty)$, $(s_0 \geq 0)$. Let $t_0 \geq s_0$ be such that $\int_{t_0}^{\infty} \lambda_b(s)\, ds \leq 1/4$ and $\gamma_b(t) \leq 1/8$ for $t \geq t_0$. By Lemma 7, there exists $\delta > 0$ such that $\|u_{s_0}\| \leq \delta$ implies

$|u(t)| \leq b/4$ on $[s_0 - \tau, t_0]$. In this way, we have

$$|u(t_0)| + 2b\gamma_b(\tau_1) + 2b \int_{\tau_1}^{\infty} \lambda_b(s)\, ds \leq \frac{b}{4} + \frac{b}{4} + \frac{b}{2} = b,$$

and, moreover, as $u|_{[t_0-\tau,t_0]} \in C_{t_0,b}$, Theorem 2 implies that there exists a solution $x$ of Equation (1), with initial function $\phi = u|_{[t_0-\tau,t_0]}$, satisfying (2). Otherwise, the uniqueness of the solutions of (1), subject to condition $x = \phi$ over the interval $[t_0 - \tau, t_0]$, implies $u = x$ on $[t_0, +\infty)$, and thus $u$ satisfies (2).

ii) Let $u$ be a solution of Equation (1), defined on $[s_0 - \tau, +\infty)$, $(s_0 \geq 0)$, such that $u(t) \to \xi$, $t \to \infty$, where $|\xi| < b$. Since $u(t) \to \xi$ as $t \to \infty$, there exist $t_0^*$ such that $|u(t)| < b$ for all $t \geq t_0^* - \tau$. Let $t_0 \geq t_0^*$ big enough, such that (12) holds with $c = u(t_0)$. Thus, Theorem 2 can be applied with $\phi = u|_{[t_0-\tau,t_0]}$, obtaining that there exists a solution $x$ defined on $[t_0 - \tau, \infty)$ satisfying the asymptotic formula (2). On the other hand, as we are assuming uniqueness of solutions subject to an initial condition in $C_{t_0,b}$, we have that $x = u$ on $[t_0 - t, \infty)$ and thus $u$ satisfies (2). □

**5. Several lags.** The extension of the above results to Equation (3)

$$\dot{x}(t) = \sum_{i=1}^{m} f_i(t, x(t - r_i(t,x)) - x(t))$$

is simple.

Let $f_i : I \times B_n[0,b] \to \mathbf{C}^n$, $1 \leq i \leq m$, be continuous functions such that $|f_i(t,x)| \leq \mu_i(t)|x|$ for $\mu_i : I \to I$, $1 \leq i \leq m$, continuous functions. For some constant $\tau > 0$, the lags $r_i : I \times B_n[0,b] \to [0,\tau]$ are continuous functions, $1 \leq i \leq m$.

Define, for $1 \leq i \leq m$ and $t \geq 0$,

$$m_b^i(t) = \sup_{|z| \leq b} r_i(t,z) \quad \text{and} \quad \mu_{it}(t) = \sup_{s \in [t-\tau,t]} \hat{\mu}_i(s).$$

Moreover, let $\mu_t = \sum_{i=1}^{m} \mu_{it}$, $\lambda_b^i(t) = \mu_i(t)\mu_{it} m_b^i(t)$ and $\lambda_b = \sum_{i=1}^{m} \lambda_b^i(t)$. From this, we can establish the same result proved for Equation (1).

**Theorem 4.** *Let $\lambda_b \in L^1(I)$. Then*

i) *For any $\xi \in \mathbf{C}^n$, $|\xi| < b$ and $t_0 \geq 0$ such that $|\xi| + 2b\int_{t_0}^{\infty} \lambda_b(s)\, ds \leq b$, there exists a solution $x$ of* (3) *defined on $[t_0 - \tau, \infty)$ satisfying $|x(t)| \leq b$, for $t \geq t_0$ and the asymptotic formulae* (2).

ii) *Moreover, for any solution $x$ of* (3) *defined for $t \geq t_0 - \tau$, such that $|x(t)| \leq b$, for $t \geq t_0 - \tau$ there exists $\xi \in \mathbf{C}^n$, $|\xi| \leq b$, satisfying* (2).

*Proof.* let $E = C([t_0 - \tau, \infty), \mathbf{C}^n)$ and $C_\mu \subset E$ be the same space defined in Section 2, now with $\mu_t = \sum_{i=1}^{m} \mu_{it}$. The proof is similar to Theorem 1. □

Results for Equation (3) corresponding to Theorems 2 and 3 can be obtained in the same way.

**6. Examples.**

1. Consider the equation

$$\dot{x}(t) = A(t)[x(t - r(t, x(t))) - x(t)] \tag{13}$$

where $A(t)$ is bounded and $m_b \in L^1(I)$. Here $\lambda_b(t) = 0(m_b(t)) \in L^1(I)$, and we can use our results.

Applying Theorem 1, we have

i) For any $\xi \in \mathbf{C}^n$, $|\xi| < b$, there exists a solution $x$ of (13), defined on $[t_0 - \tau, \infty)$ for $t_0$ big enough, satisfying $|x(t)| \leq b$ for $t \geq t_0 - \tau$ and the asymptotic formula (2).

ii) Any solution $x$ of (13) defined on $[t_0 - \tau, \infty)$ such that $|x(t)| \leq b$ for $t \geq t_0$ satisfies the asymptotic formula (2) for some $|\xi| \leq b$.

The same conclusion is obtained if $A(t)|A_t| \in L^1(I)$ (with $A$ not necessarily bounded).

2. Consider the equation

$$\dot{x}(t) = A(t)[x(t - r(t, x(t))) - x(t)]$$

where $|A(t)|$ is nonincreasing, $\int^{\infty} |A(t)|^2\, dt < \infty$ and $r : I \times [-b, b] \to [0, \tau]$, a continuous function. Then $\lambda_b(t) \leq |A(t - \tau)|^2 \cdot m_b(t)$, i.e., $\lambda_b \in L^1(I)$. So our results are applicable.

3. In the equation

$$\dot{x}(t) = -t\left[x\left(t - \frac{x^2(t)+1}{t^4+1}\right) - x(t)\right], \tag{14}$$

fixed $b > 0$, it is clear that $r(t,x) \le K r_1(t)$ for all $(t,x) \in I \times B_1[0,b]$ where $K = b^2+1$ and $r_1(t) = 1/(t^4+1)$. Moreover, since $\lambda_b(t) \in L^1(I)$, Corollary 1 can be applied, and we obtain

i) For any $\xi \in \mathbf{C}$ taking $t_0 \ge 0$ big enough, there exists a bounded solution $x$ of (14) defined on $[t_0 - \tau, \infty)$, with an adequate $\tau$, satisfying the asymptotic formula (2).

ii) Any bounded solution $x$ of (14), defined on $[t_0 - \tau, \infty)$ satisfies the asymptotic formula (2) for some $\xi \in \mathbf{C}$.

4. Let the equation

$$\dot{x}(t) = \left[x\left(t - \frac{|x(t)|}{1+t^2}\right) - x(t)\right]^2. \tag{15}$$

For $b > 0$ and $|x| \le b$, we have that $\lambda_b(t) \le b/(1+t^2) \in L^1(I)$. Then Corollary 1 can be applied and we obtain

i) For any $\xi \in \mathbf{C}$ and $t_0 \ge 0$ big enough, there exists a bounded solution $x$ of (15), defined on $[t_0 - \tau, \infty)$, satisfying the asymptotic formula $x(t) = \xi + O(\int_t^\infty ds/(1+s^2)$ as $t \to \infty$.

ii) Any bounded solution $x$ of (15), defined on $[t_0 - \tau, \infty)$ satisfies the asymptotic formula $x(t) = \xi + O(\int_t^\infty ds/(1+s^2)$ as $t \to \infty$ for some $\xi \in \mathbf{C}$.

5. Consider the equation

$$\dot{x}(t) = \frac{1}{t+1}[x(t-1) - x(t)].$$

Here $\lambda_b(t) = \mu(t)\mu_t m_b(t) = (1/(t+1))(1/t)$ for $t \ge 1$. We have $\lambda_b \in L^1[0,+\infty]$ and

$$\gamma_b(t) = \int_{t-m_b(t)}^{t} \mu(s)\,ds = \int_{t-1}^{t} \frac{ds}{s+1} = \log\left(\frac{t+1}{t}\right)$$

satisfies $\gamma_b(t) \to 0$ as $t \to \infty$. So Theorem 2 is applicable.

Given $c \in \mathbf{C}$ and $t_0 \geq 1$ such that

$$|c| + 2b\gamma_b(\tau_1) + 2b \int_{t_0+1}^{\infty} \lambda_b(s)\, ds \leq b, \tag{16}$$

then, for any $\phi \in C_{t_0,b}$ with $\phi(t_0) = c$, the problem

$$\begin{aligned} \dot{x}(t) &= \frac{1}{t+1}[x(t-1) - x(t)], \quad t \geq t_0 \\ x(t) &= \phi(t), \quad t_0 - 1 \leq t \leq t_0, \end{aligned} \tag{17}$$

has a solution $x$, continuous on $[t_0 - 1, +\infty)$, satisfying $|x(t)| \leq b$ for $t \geq t_0$ and, for some $\xi$,

$$x(t) = \xi + 0\bigg(\log \frac{(t+1)}{t}\bigg), \quad t \to \infty.$$

In this particular case, condition (16) becomes $|c| + 4b \log((t_0+2)/(t_0+1)) \leq b$ showing explicitly the condition satisfied by $t_0$ for which problem (17) has a solution on $[t_0 - 1, +\infty)$.

6. Let

$$\begin{aligned} \dot{x}(t) = -t\bigg[x\bigg(t - \frac{x^2(t)+1}{t^4+1}\bigg) - x(t)\bigg] \\ + \frac{1}{t+1}[x(t-1) - x(t)] \end{aligned} \tag{18}$$

be an equation with two lags, where $r_1 : I \times [-b, b] \to \mathbf{C}$ is defined by $r_1(t, x) = (x^2 + 1)/(t^2 + 1)$ and $r_2 : I \times [-b, b] \to \mathbf{C}$ by $r_2(t, x) = 1$.

Here $\lambda_b(t) = \lambda_b^1(t) + \lambda_b^2(t)$ and $\lambda_b^1(t), \lambda_b^2(t)$ are as in Examples 3 and 5. Thus, $\lambda_b^1(t) \leq K/(t^2 + 1)$ and $\lambda_b^2(t) = 1/(t(t + 1))$ for $t \geq 1$. It follows that $\lambda_b \in L^1(I)$ and so Theorem 4 can be applied:

i) For any $\xi \in \mathbf{C}$, $|\xi| < b$ and $t_0$ big enough, there exists a solution $x$ of (18), defined on $[t_0 - \tau, +\infty)$, $(\tau = K)$, satisfying $|x(t)| \leq b$ and

$$x(t) = \xi + O\bigg(\int_t^{\infty} \frac{ds}{1+s^2}\bigg), \quad t \to \infty.$$

(We remark that $\lambda_b(t) = O(1/(1+t^2))$.)

ii) Any solution $x$ of (18) such that $|x(t)| \le b$, satisfies

$$x(t) = \xi + O\bigg(\int_t^\infty \frac{ds}{1+s^2}\bigg), \quad t \to \infty,$$

for some $|\xi| \le b$.

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