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ON A CLASS OF ADDITIVE GROUP ACTIONS ON AFFINE THREE-SPACE

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ABSTRACT. Every algebraic action of the additive group of complex numbers on complex affine space is obtained as the exponential of a locally nilpotent derivation on its coordinate ring. Moreover, a locally nilpotent derivation is equivalent to a polynomial vector field on affine space admitting a strictly polynomial flow. For three-dimensional affine space it is known that the group action is triangulable if and only if the centralizer of the corresponding vector field, in the Lie algebra of vector fields on affine space, contains a constant vector field. A class of additive group actions is investigated with this criterion, and the generic member is shown to be nontriangulable.

1. Introduction. In [**13**] Rentschler showed that, for any field K of characteristic zero, all algebraic actions of the additive group of K on the affine plane over K are triangulable. The corresponding assertion in higher dimensions is false, as first demonstrated by Bass in [**1**]. Since this example first appeared, several authors have found other nontriangulable actions of the additive group of complex numbers, henceforth denoted by G_a, or complex affine three space [**12, 3, 2, 9, 10**]. The interest in these examples stems from the attempts to understand the structure of the automorphism group of complex affine space, as an infinite dimensional algebraic group, by investigating the homomorphisms to it from well understood, finite dimensional, algebraic groups. Indeed, the Jung–van der Kulk theorem [**11**] can be viewed as accomplishing this for the complex plane, and the theorem of Rentschler can be viewed as a nonlinear Lie-Kolchin theorem. However, the nontriangulable G_a actions on complex three space indicate that no Lie-Kolchin theorem holds in dimension higher than two, and that the structure of the automorphism group is far more complicated in these dimensions.

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To each polynomial derivation $\delta = \sum_{i=1}^n p_i(\partial/\partial x_i)$, with $p_i \in \mathbf{C}[x_1, \dots, x_n] = \mathbf{C}[x]$, there corresponds the autonomous polynomial differential equation $\dot{x} = P(x)$ in $\mathbf{C}^n$, where $x = x(t)$ has its values in $\mathbf{C}^n$, and the vector field P is given by $P = (p_1, \dots, p_n)$. We will not distinguish between the notions of "differential equation" and "vector field" and use the terms synonymously.

It is well know that locally nilpotent derivations give rise to algebraic actions of the additive group of complex numbers as automorphisms of $\mathbf{C}[X]$, hence as polynomial automorphisms of $\mathbf{C}^n$. From the differential equations perspective, the consequence of local nilpotency is that the differential equation admits a general solution which is polynomial in t and the initial value. Under this condition we say that the differential equation has a strictly polynomial flow, see [**9**] for more details.

As the title suggests, our interest is in the case of complex three space, where it is known that the ring of invariants for any algebraic G_a action (which is equal to the ring of polynomial first integrals of a strictly polynomial flow vector field) is isomorphic to a polynomial ring in two variables, cf., Sugie [**16**]. It should be noted that an analogous result does not hold in dimension higher than three, e.g., there are G_a actions on complex four space where the ring of invariants is the coordinate ring of a singular hypersurface in four space [**7**], and an action on seven space where the ring of invariants is not finitely generated [**6**].

If f and $g \in \mathbf{C}[x_1, x_2, x_3]$ generate the ring of invariants for a G_a action induced by the vector field F, then $G = \text{grad}\,(f) \times \text{grad}\,(g)$ is a polynomial vector field with first integrals f and g. It is straightforward to show that $G = \phi F$ for some rational function ϕ. Moreover, it has been shown that, if the given G_a action is fixed point free, then $G = \lambda F$ for some complex number λ [Deveney and Finston, unpublished]. Until recently, there were only examples of G_a actions on $\mathbf{C}^3$ with the property that their ring of invariants contains a variable. Due to work of Freudenburg [**10**], it is now known that there exist other classes of such actions. The question of whether every fixed point free G_a action on $\mathbf{C}^3$ fixes a variable, and thus, according to Deveney and Finston [**5**], is conjugate to a translation, is still unanswered.

The class of G_a actions investigated here are generated by vector fields (or corresponding locally nilpotent derivations) of the form

$$\dot{x} = \begin{pmatrix} 0 \\ 0 \\ r(x_1) \end{pmatrix} + \sigma(\phi_1, \phi_2) \cdot \begin{pmatrix} 0 \\ p(x_1) \\ q(x_1, x_2) \end{pmatrix},$$

where $\phi_1(x) = x_1$ and $\phi_2(x) = p(x_1)x_3 - \int_0^{x_2} q(x_1, u)\, du$.

The vector field is conveniently represented by $\operatorname{grad}(x_1) \times \operatorname{grad} \gamma_2(x)$ where $\gamma_2(x) = r(x_1)x_2 - \tau(\phi_1(x), \phi_2(x))$, with $\tau(x_1, x_2) = \int_0^{x_2} \sigma(x_1, u)\, du$. (The origin of γ_2 will become apparent in the remarks preceding Lemma 2.3.)

This class of examples is included in a class investigated by Daigle and Freudenburg [**3**], who determined all reduced locally nilpotent derivations on $\mathbf{C}^3$ that annihilate a variable. Many known examples of nontriangulable G_a actions are members of the more special class under investigation in this paper, see [**10**] for an example not in this class. Bass's example [**1**] has $r(x_1) = 0$, $p(x_1) = x_1$, $q(x_1, x_2) = x_2$ and $\sigma(\phi_1, \phi_2) = \phi_2$. Those of Popov [**12**] have $r(x_1) = 0$ and $\sigma(\phi_1, \phi_2)$ chosen to define a surface in $\mathbf{C}^3$ with an isolated singularity. Popov also has given examples of this type in higher dimension. Daigle and Freudenburg [**3**] examined the case $\gamma_2(x) = x_1x_2 + (x_1x_3 + x_2^2)^2$ in detail. Other special cases of such actions were considered by Daigle in [**2**] and by the authors in [**9**].

The general differential equation on $\mathbf{C}^3$ corresponding to a locally nilpotent derivation that kills a variable can be transformed into

$$\dot{x} = \begin{pmatrix} 0 \\ 0 \\ r^*(x_1) \end{pmatrix} + \sigma^*(x) \begin{pmatrix} 0 \\ p^*(x) \\ q^*(x) \end{pmatrix}.$$

This follows from [**3**, Corollary 3.2], and perhaps an additional conjugation by a polynomial automorphism of $\mathbf{C}^3$ that fixes x_1 and induces linear automorphism of $\mathbf{C}(x_1)[x_2, x_3]$. The additional requirements we introduce are that σ^* is a first integral of the second vector field and that both vector fields commute. An equivalent characterization is that the associated locally nilpotent derivation of $\mathbf{C}(x_1)[x_2, x_3]$ can be transformed to $(\partial/\partial x_3)$ by a product of two triangular automorphism. (This can be seen from the computations in Section 2.)

Results of Smith [**14**] and subsequent work in [**17**] show that all the group actions in the class we consider are stably tame, and in fact that their canonical extensions to four-dimensional space are tame. In [**4**] it was shown that all these group actions are rationally triangulable.

While finding new examples of additive group actions is of obvious interest in the quest to understand the automorphism group of affine space, it is equally important to address conjugacy problems. In this paper we obtain a fairly complete answer to the question of whether the actions under consideration are conjugate to a triangular action. (The case $r = 0$ has been discussed in [**9**] so we will assume $r \neq 0$ here.) Our main result is that generically the group actions induced by the above vector fields are not triangulable. It must be noted, however, that there are triangulable examples among these actions even with $r(x_1) \neq 0$ and

$$\sigma(\phi_1, \phi_2) \begin{pmatrix} 0 \\ p(x_1) \\ q(x_1, x_2) \end{pmatrix}$$

nontriangulable. The proofs are based on the results of [**9**], where it was shown that the question of triangulability in three-dimensional space can be completely answered by investigating the centralizer of the given vector field. We also discuss one vector field that is not reduced, to illustrate that our method works equally well for those cases. Finally, we suggest a strategy based on centralizers for investigating more general vector fields on $\mathbf{C}^3$ that kill a variable.

We will formulate our computations and results in terms of vector fields, which seems appropriate for our purpose. For more information, the reader may consult [**9**] where the terminology used here and the perhaps more familiar language of derivations were reconciled. A number of computations have been included for the reader's convenience, although they are quite straightforward.

2. Preliminaries and computations. In the following we will consider the differential equation

$$\dot{x} = F(x) := \begin{pmatrix} 0 \\ 0 \\ r(x_1) \end{pmatrix} + \sigma(\phi_1, \phi_2) \begin{pmatrix} 0 \\ p(x_1) \\ q(x_1, x_2) \end{pmatrix},$$

where p and q are relatively prime polynomials with a common zero, r is a nonzero polynomial, $\phi_1(x) := x_1$, $h(x_1, x_2) := \int_0^{x_2} q(x_1, u)\, du$, $\phi_2(x) := p(x_1)x_3 - h(x_1, x_2)$ and σ is a polynomial not depending only on the first variable. It is known, see [**9**], that the general solution of this equation is a polynomial in x_1, x_2, x_3 and t. We will therefore describe $\dot{x} = F(x)$ as having *strictly polynomial flow.*

In addition we will assume that F has a stationary point, and finally, we require F to be reduced, i.e., the entries of F are assumed to be relatively prime. In particular, σ and r are relatively prime, although this condition is not sufficient for reducedness.

(In other language, we are dealing with the locally nilpotent derivation

$$\sigma(\phi_1, \phi_2)p(x_1)\frac{\partial}{\partial x_2} + (r(x_1) + \sigma(\phi_1, \phi_2)q(x_1, x_2))\frac{\partial}{\partial x_3}$$

of the polynomial algebra $\mathbf{C}[x_1, x_2, x_3]$.)

The purpose of this paper is to investigate triangulability of $\dot{x} = F(x)$. The method is to utilize two known results which reduce the problem to elementary, although somewhat involved, computations. Recall that the Lie bracket of two vector fields F, G, is defined by $[G, F](x) := DF(x)G(x) - DG(x)F(x)$, which corresponds to the commutator of the associated derivations. The polynomial centralizer of F consists of all G for which $[G, F] = 0$.

Lemma 2.1. *The equation $\dot{x} = F(x)$ on $\mathbf{C}^3$ is triangulable if and only if the polynomial centralizer $\mathcal{C}_{\mathrm{pol}}(F)$ contains a vector field conjugate to a* (*nonzero*) *constant vector field.*

This was shown in [**9**, Theorem 2.7]. Incidentally, this criterion can be shown to be equivalent to the one given by Daigle [**2**, Corollary 3.4]. (The second variable Q mentioned there corresponds to a centralizer element of F which has no stationary point!) We proceed with investigating centralizer elements, since this will turn out to be quite straightforward.

The second result we need can be found in Daigle-Freudenburg [**3**].

Lemma 2.2. *If the differential equation*

$$\dot{x} = G(x) := \begin{pmatrix} 0 \\ g_2(x) \\ g_3(x) \end{pmatrix}$$

has strictly polynomial flow and no stationary point, then the vector field G is conjugate to a constant vector field.

This lemma explains some of the additional conditions imposed on F. If F has no stationary point, then F is conjugate to a constant vector field, hence triangulable. If p and q have no common zero, then $(0, p, q)^t$ gives rise to a free triangular action, hence is conjugate to a constant vector field, Snow [**15**], and triangulability of F follows easily. (Also, if σ depends only on the first variable, then F is obviously triangulable.)

Similar to the strategy pursued in [**9**], we start by determining the rational centralizer $\mathcal{C}_{\mathrm{rat}}(F)$. We first transform F to a constant vector field with the help of a birational map. (Finding such a birational map relies on the well-known technique of inverting an invariant in the image of the derivation to obtain a slice on an open subset of $\mathbf{C}^3$.)

Let $H(x) := (0, p(x_1), q(x_1, x_2))$ and note that $[H, F] = 0$. Define $\phi_3(x) := x_2/p(x_1)$. Then it is known, see [**9**], that $\Phi(x) := (\phi_1(x), \phi_2(x), \phi_3(x))^t$ transforms H to $(0, 0, 1)^t$, in other words, $(x_2/p(x_1))$ is a slice for the action on an open subset of $\mathbf{C}^3$), and an easy computation shows that

$$D\Phi(x)F(x) = \begin{pmatrix} 0 \\ p(x_1)r(x_1) \\ \sigma(x_1, \phi_2) \end{pmatrix} = F^*(\Phi(x)),$$

where

$$F^*(x) := \begin{pmatrix} 0 \\ p(x_1 r(x_1) \\ \sigma(x_1, x_2) \end{pmatrix}.$$

Thus, Φ transforms F to F^*, which is triangular, and it is straightforward to find a birational map that transforms F^* to a constant vector field. Let $\psi_1(x) := x_1$, $\tau(x_1, x_2) := \int_0^{x_2} \sigma(x_1, u)\, du$, $\psi_2(x) := p(x_1)r(x_1)x_3 - \tau(x_1, x_2)$, and $\psi_3(x) := x_2/(p(x_1)r(x_1))$. Then ψ_3 is a slice for F^* on an open subset of $\mathbf{C}^3$, hence

$$\Psi(x) := \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix}$$

satisfies

$$D\Psi(x)F^*(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and therefore the birational map $\Gamma := \Psi \circ \Phi$ straightens F to $(0,0,1)^t$. (The corresponding derivation is $(\partial/\partial x_3)$.) Explicitly,

$$\begin{aligned} \Gamma(x) &= \begin{pmatrix} x_1 \\ p(x_1)r(x_1)\phi_3 - \tau(x_1, \phi_2) \\ \phi_2/(p(x_1)r(x_1)) \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ r(x_1)x_2 - \tau(x_1, p(x_1)x_3 - h(x_1, x_2)) \\ (p(x_1)x_3 - h(x_1, x_2))/(p(x_1)r(x_1)) \end{pmatrix}. \end{aligned}$$

In particular, the polynomials $\gamma_1(x) := x_1$ and $\gamma_2(x) := r(x_1)x_2 - \tau(x_1, \phi_2(x))$ are first integrals of $\dot{x} = F(x)$ (and invariants for the induced group action).

Since the rational centralizer of $(0,0,1)^t$ is the set of all $(g_1(x_1, x_2), g_2(x_1, x_2), g_3(x_1, x_2))^t$, with rational functions g_i, the rational centralizer of F is the set of all

$$D\Gamma(x)^{-1} \begin{pmatrix} g_1(x_1, \gamma_2(x)) \\ g_2(x_1, \gamma_2(x)) \\ g_3(x_1, \gamma_2(x)) \end{pmatrix}, \quad \text{with rational } g_i.$$

A computation yields

$$D\Gamma(x) = \begin{pmatrix} 1 & 0 & 0 \\ u(x) & r(x_1) + \sigma(x_1, \phi_2(x))q(x_1, x_2) & -\sigma(x_1, \phi_2(x))p(x_1) \\ v(x) & -q(x_1, x_2)/(p(x_1)r(x_1)) & 1/r(x_1) \end{pmatrix},$$

with

$$u(x) := r'(x_1)x_2 - \frac{\partial \tau}{\partial x_1}(x_1, \phi_2(x)) - \sigma(x_1, \phi_2(x))x_3 p'(x_1),$$

and

$$v(x) := -\frac{x_3 r'(x_1)}{r(x_1)^2} - \frac{p(x_1)r(x_1)(\partial h/\partial x_1)(x_1, x_2) - (p(x_1)r(x_1))' h(x_1, x_2)}{(p(x_1)r(x_1))^2}.$$

The inverse is

$$D\Gamma(x)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1/r(x_1) & \sigma(x_1, \phi_2(x))p(x_1) \\ * & q(x_1, x_2)/(p(x_1)r(x_2)) & r(x_1) + \sigma(x_1, \phi_2(x))q(x_1, x_2) \end{pmatrix},$$

where the terms abbreviated by $*$ are not recorded explicitly here. It is useful to observe that

$$(q, -p) \cdot \begin{pmatrix} 1/r & \sigma p \\ q/pr & r + \sigma q \end{pmatrix} = (0, -pr),$$

and, as a consequence,

$$(*) \qquad (0, q, -p) \cdot D\Gamma(x)^{-1} = \left(\frac{x_3 pr'}{r} + \frac{\partial h}{\partial x_1} - \frac{(pr)'}{pr} \cdot h, 0, pr \right).$$

The last preliminary result we need is about the algebra of invariants of the group action induced by F.

Lemma 2.3. *The algebra of invariants of* $\dot{x} = F(x)$ *is* $\mathbf{C}[\gamma_1, \gamma_2]$.

Proof. It is known from a result of Daigle and Freudenburg [**3**, Corollary 3.2] that there is a polynomial μ_2 such that the algebra of invariants is $\mathbf{C}[x_1, \mu_2]$ and furthermore that

$$F(x) = \operatorname{grad}(x_1) \times \operatorname{grad}(\mu_2)$$

holds, where $\times$ denotes the vector product.

It is sufficient to show that the degrees of γ_2 and μ_2 are equal, since then their difference will be at worst a polynomial in x_1 only, so that $\mathbf{C}[x_1, \mu_2] = \mathbf{C}[x_1, \gamma_2]$. But this follows from the fact that the degree of F equals $\deg \gamma_2 - 1$, and is at most equal to $\deg \mu_2 - 1$.

It should be noted that Lemma 2.3 can also be proved with the help of van den Essen's algorithm, cf. [**8**].

3. On the polynomial centralizer. The results in Section 2 are sufficient to determine $\mathcal{C}_{\mathrm{rat}}(F)$. In particular, it follows that every element $\tilde{G} \in \mathcal{C}_{\mathrm{pol}}(F)$ is of the type

$$\tilde{G} = \begin{pmatrix} \tilde{g}_1 \\ \tilde{g}_2 \\ \tilde{g}_3 \end{pmatrix} = D\Gamma(x)^{-1} \cdot \begin{pmatrix} g_1(\gamma_1, \gamma_2) \\ g_2(\gamma_1, \gamma_2) \\ g_3(\gamma_1, \gamma_2) \end{pmatrix},$$

with the g_i still a priori rational functions. From the explicit form of $D\Gamma(x)^{-1}$ it follows that $\tilde{g}_1 = g_1(\gamma_1, \gamma_2)$, and Lemma 2.3 implies that g_1 is a polynomial, since $\tilde{g}_1$ is a polynomial and invariant. From $(*)$ as in Section 2, we get

Lemma 3.1. *If $\tilde{G}$ is a polynomial, then*

$$\left(px_3\frac{r'}{r} - \frac{(pr)'}{pr}h\right) \cdot g_1(\gamma_1, \gamma_2) + prg_3(\gamma_1, \gamma_2)$$

is a polynomial.

Proof. If $\tilde{G}$ is a polynomial, then so is $(0, q, -p) \cdot \tilde{G}$. □

The next result is crucial for finding the elements with strictly polynomial flow in $\mathcal{C}_{\mathrm{pol}}(F)$, whenever F satisfies a weak additional condition.

Theorem 3.2. *Suppose that there is a root α of p such that $\sigma(\alpha, y)$ is not constant, or that there is a root β of r such that $\sigma(\beta, y)$ is not constant (where y is an indeterminate). Then $x_1 - \alpha$ divides g_1*

respectively, $x_1-\beta$ *divides* g_1 *and* $x_1-\alpha$ *respectively* $x_1-\beta$ *is a semi-invariant of* $\tilde{G}$.

In particular, if $\tilde{G}$ *has strictly polynomial flow, then* $g_1=0$.

Proof. The last assertion follows from [**9**, Remark 2.1].

In the proof of the first assertion, we will repeatedly use the following standard result from algebra. Let K be any field, y transcendental over K and $K(y)$ the field of rational functions in y. Then, for every nonconstant polynomial f, the degree of the field extension $[K(y):K(f)]$ is equal to the degree of f. Thus, if g is another nonconstant polynomial, then $K(f)\subseteq K(g)$ implies that $\deg g\le \deg f$.

From Lemma 3.1, it follows that

$$(\dagger)\quad (p^2x_3r'-(pr)'h)g_1(\gamma_1,\gamma_2)+g_3^*(\gamma_1,\gamma_2)=pr\cdot\ \text{(some polynomial)},$$

with $g_3^*(\gamma_1,\gamma_2):=(pr)^2g_3(\gamma_1,\gamma_2)$ necessarily a polynomial.

(i) Assume that $p(\alpha)=0$ and that $\sigma(\alpha,y)$ is not constant. The assumption $(x_1-\alpha)\nmid g_1$ will lead to a contradiction.

Thus, assume that $x_1-\alpha$ does not divide g_1, and divide by highest possible powers of $x_1-\alpha$ to get $p=(x_1-\alpha)^n\hat{p}$, $r=(x_1-\alpha)^m\hat{r}$ and $g_3^*=(x_1-\alpha)^d\hat{g}_3$ with $n>0$ and $m,d\ge 0$.

Since $(x_1-\alpha)\nmid g_1$ and $(x_1-\alpha)\nmid h$ (otherwise $(x_1-\alpha)|q$, a contradiction), the highest $(x_1-\alpha)$-powers dividing the terms on the lefthand side of $(\dagger)$ are $(x_1-\alpha)^{2n+m-1}$, $(x_1-\alpha)^{n+m-1}$ and $(x_1-\alpha)^d$, respectively, while $(x_1-\alpha)^{n+m}$ divides the righthand side. Therefore, $d=n+m-1$, and division of $(\dagger)$ by $(x_1-\alpha)^{n+m-1}$ yields $((x_1-\alpha)^n\cdot(\dots)+a(x_1)h(x_1,x_2))g_1(x_1,\gamma_2)+\hat{g}_3(x_1,\gamma_2)=(x_1-\alpha)\cdot$ (some polynomial), with $a(x_1)$ a polynomial such that $a(\alpha)\neq 0$. Upon setting $x_1\to\alpha$, $x_3\to 0$, we find

$$(\dagger\dagger)\quad a(\alpha)\cdot h(\alpha,x_2)\cdot g_1(\alpha,r(\alpha)x_2-\tau(\alpha,-h(\alpha,x_2))) + \hat{g}_3(\alpha,r(\alpha)x_2-\tau(\alpha,-h(\alpha,x_2)))=0.$$

In this equation $h(\alpha,x_2)$ is not constant; otherwise, $q(\alpha,x_2)=0$ and $(x_1-\alpha)|q$, a contradiction. Since $\sigma(\alpha,y)$ is not constant, the degree of $\tau(\alpha,y)$ with respect to y is greater than 1, whence $r(\alpha)x_2-\tau(\alpha,-h(\alpha,x_2))$ has greater degree in x_2 than $h(\alpha,x_2)$.

On the other hand, the relation (††) (with $a(\alpha) \neq 0$ and $g_1(\alpha, *) \neq 0$) implies that $h(\alpha, x_2) \in \mathbf{C}(r(\alpha)x_2 - \tau(\alpha, -h(\alpha, x_2)))$, a contradiction. Therefore, $(x_1 - \alpha)|g_1$, according to the remark at the beginning of this proof.

(ii) Now suppose that $r(\beta) = 0$, that $\sigma(\beta, y)$ is not constant and that $x_1 - \beta$ does not divide q.

Again, assume that $(x_1 - \beta) \nmid g_1$, and let $r = (x_1 - \beta)^m \hat{r}$, $p = (x_1 - \beta)^n \hat{p}$ and $g_3^* = (x_1 - \beta)^d \hat{g}_3$, with highest possible powers divided out, and $m > 0$. Since $(x_1 - \beta) \nmid q$, we have that $(x_1 - \beta) \nmid h$, and therefore the highest powers of $x_1 - \beta$ dividing the terms on the lefthand side of (†) are $(x_1 - \beta)^{m+2n-1}$, $(x_1 - \beta)^{m+n-1}$ and $(x_1 - \beta)^d$, respectively, while $(x_1 - \beta)^{m+n}$ divides the righthand side.

Comparing degrees shows that $d \geq m + n - 1$ (we may have "$>$" in case $n = 0$) and division by $(x_1 - \beta)^{m+n-1}$ and setting $x_1 \to \beta$, $x_3 \to 0$ yields

$$a(\beta) \cdot h(\beta, x_2) \cdot g_1(\beta, -\tau(\beta, -h(\beta, x_2))) + \tilde{g}_3(\beta, -\tau(\beta, -h(\beta, x_2))) = 0,$$

with a polynomial a such that $a(\beta) \neq 0$ and $\tilde{g}_3 := (x_1 - \beta)^{d+1-m-n} \hat{g}_3$.

Since $h(\beta, x_2)$ is not constant and $\tau(\beta, y)$ has degree greater than 1 in y, this relation implies $h(\beta, x_2) \in \mathbf{C}(-\tau(\beta, -h(\beta, x_2)))$ (whether $\tilde{g}_3(\beta, *) = 0$ or not), which is a contradiction as in (i). Therefore, $(x_1 - \beta)|g_1$.

(iii) Finally let $r(\beta) = 0$, and suppose that $\sigma(\beta, y)$ is not constant and that $x_1 - \beta$ divides q. Then $(x_1 - \beta) \nmid p$, since p and q are relatively prime, and $(x_1 - \beta)|h$. Let $h = (x_1 - \beta)^s \hat{h}$, with a polynomial $\hat{h}$ and s maximal. Furthermore, let $r = (x_1 - \beta)^m \hat{r}$, and $g_3^* = (x_1 - \beta)^d \hat{g}_3$, with highest powers factored out.

Once more, assume that $(x_1 - \beta) \nmid g_1$. Then $(x_1 - \beta)^{m-1}$, $(x_1 - \beta)^{m+s-1}$ and $(x_1 - \beta)^d$, respectively, are the highest powers dividing each term on the lefthand side of (†), while $(x_1 - \beta)^m$ divides the righthand side. It follows that $d = m - 1$, since $s > 0$. Divide (†) by $(x_1 - \beta)^{m-1}$ and set $x_1 \to \beta$ to obtain

$$(\dagger\dagger\dagger) \qquad ax_3 \cdot g_1(\beta, \gamma_2(\beta, x_2, x_3)) + \hat{g}_3(\beta, \gamma_2(\beta, x_2, x_3)) = 0,$$

with some $a \in \mathbf{C}^*$.

Now $h(\beta, x_2) = 0$ and $r(\beta) = 0$ imply $\phi_2(\beta, x_2, x_3) = p(\beta)x_3$ and $\gamma_2(\beta, x_2, x_3) = -\tau(\beta, p(\beta)x_3)$.

With $p(\beta) \neq 0$, the relation $(\dagger\dagger\dagger)$ implies $x_3 \in \mathbf{C}(\tau(\beta, p(\beta)x_3))$, a contradiction, since $\tau(\beta, y)$ has degree > 1 in y. Therefore, $(x_1 - \beta)|g_1$, and the proof is finished. □

The partial computation of $D\Gamma(x)^{-1}$ in Section 2 immediately yields

Corollary 3.3. *Under the assumptions of the theorem, an element of $\mathcal{C}_{\mathrm{pol}}(F)$ with strictly polynomial flow has the form*

$$g_2^*(x_1, \gamma_2) \cdot \begin{pmatrix} 0 \\ p \\ q \end{pmatrix} + g_3^*(x_1, \gamma_2) \cdot F.$$

It should be emphasized, however, that g_2^* and g_3^* are not necessarily polynomials. From

$$\det \begin{pmatrix} p & \sigma p \\ q & r + \sigma q \end{pmatrix} = pr$$

it follows that prg_2^* and prg_3^* must be polynomials.

Now it is easy to determine all the elements with strictly polynomial flow in $\mathcal{C}_{\mathrm{pol}}(F)$.

Proposition 3.4. *The polynomial vector field*

$$\tilde{G} = g_2^*(x_1, \gamma_2) \cdot \begin{pmatrix} 0 \\ p \\ q \end{pmatrix} + g_3^*(x_1, \gamma_2) \cdot F$$

has strictly polynomial flow if and only if g_2^* is a function of x_1 alone.

Proof. To see that the condition is necessary, consider the map

$$\Delta : \mathbf{C}^3 \longrightarrow \mathbf{C}^2, \quad x \longmapsto \begin{pmatrix} \gamma_1(x) \\ \gamma_2(x) \end{pmatrix}.$$

Then $D\Delta(x)\tilde{G}(x) = \tilde{\tilde{G}}(D(x))$, with $\tilde{\tilde{G}}(x) := (0, p(x_1)r(x_1)g_2^*(x_1, x_2))^t$, as the following computation shows:

$$\begin{aligned}
\gamma_2(x) &= r(x_1)x_2 - \tau(x_1, \phi_2(x)); \qquad \phi_2(x) = p(x_1)x_3 - h(x_1, x_2);\\
\frac{\partial \gamma_2}{\partial x_2} &= r(x_1) - \frac{\partial \tau}{\partial x_2}(x_1, \phi_2) \cdot \frac{\partial \phi_2}{\partial x_2}\\
&= r(x_1) + \sigma(x_1, \sigma_2) \cdot q(x_1, x_2);\\
\frac{\partial \gamma_2}{\partial x_3} &= -\frac{\partial \tau}{\partial x_2}(x_1, \phi_2) \cdot \frac{\partial \phi_2}{\partial x_3} = -\sigma(x_1, \phi_2) \cdot p(x_1);
\end{aligned}$$

therefore, $D\gamma_2(x)\tilde{G}(x) = p(x_1)r(x_1)g_2^*(x_1, \gamma_2)$.

Thus Δ maps solutions of $\dot{x} = \tilde{G}(x)$ to solutions of $\dot{x} = \tilde{\tilde{G}}(x)$ in $\mathbf{C}^2$, and the latter will have strictly polynomial flow if the former has. But any vector field of the form $(0, u(x_1, x_2))^t$ has strictly polynomial flow only if u is a polynomial in x_1 alone.

To prove sufficiency, denote the derivation corresponding to $\tilde{G}$ by δ, and let B be the subalgebra of $\mathbf{C}[x_1, x_2, x_3]$ consisting of all polynomials annihilated by some power of δ.

Then $x_1 \in B$ is obvious, and the above computation yields $\delta(\gamma_2) = p(x_1)r(x_1)g_2^*(x_1) \in B$, so $\gamma_2 \in B$. Furthermore,

$$\delta(\phi_2) = g_3^*(x_1, \gamma_2)r(x_1)(\partial\phi_2/\partial x_3) = p(x_1)r(x_1)g_3^*(x_1, \gamma_2) \in B,$$

hence $\phi_2 \in B$.

Finally, $\delta(x_2) \in \mathbf{C}[x_1, \gamma_2, \phi_2] \subseteq B$, hence $x_2 \in B$, and $\delta(x_3) \in \mathbf{C}[x_1, x_2, \gamma_2, \phi_2] \subseteq B$, whence $x_3 \in B$. This shows $B = \mathbf{C}[x_1, x_2, x_3]$ and δ is locally nilpotent. □

It is worth emphasizing that, without any additional assumptions on F, every element $\tilde{G}$ as given in Proposition 3.4 lies in the centralizer of F and has strictly polynomial flow. The restriction on F in Theorem 3.2 is a technicality making sure that the given proof works. Indeed, it can be shown to follow from Daigle [**2**, Theorem 2.5] that the conclusion of Theorem 3.2 holds without any further restriction on F. On the other hand, our proof of Theorem 3.2 requires only elementary methods, and the restrictive condition forced by our proof has the benefit of identifying triangulable vector fields:

Proposition 3.5. *For every polynomial m in one variable and every polynomial σ^* in two variables, the vector field*

$$F = \begin{pmatrix} 0 \\ 0 \\ r(x_1) \end{pmatrix} + (m(x_1) + p(x_1)r(x_1)\sigma^*(x_1, \phi_2)) \cdot \begin{pmatrix} 0 \\ p(x_1) \\ q(x_1, x_2) \end{pmatrix}$$

is triangulable.

Proof. According to Proposition 3.4, the vector field

$$F - m \cdot \begin{pmatrix} 0 \\ p \\ q \end{pmatrix} = r \cdot \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + p\sigma^* \begin{pmatrix} 0 \\ p \\ q \end{pmatrix} \right)$$

is contained in the centralizer of F and has strictly polynomial flow. But then the same is true for

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + p\sigma^* \begin{pmatrix} 0 \\ p \\ q \end{pmatrix},$$

which has no stationary point. From Lemmas 2.1 and 2.2, it follows that F is triangulable. □

Note that the vector fields of Proposition 3.5 do not satisfy the hypothesis of Theorem 3.2, and that they are the only ones with this property when p and r are given as relatively prime polynomials with simple roots. Incidentally, the centralizer of such a vector field also satisfies the conclusion of Theorem 3.2, as was proven in [**9**] for all triangular vector fields having a stationary point. It seems likely that all the vector fields F that violate the hypothesis of Theorem 3.2 are triangulable. The following example lends additional support to this conjecture.

Example 3.6. The vector field

$$F = \begin{pmatrix} 0 \\ 0 \\ x_1 \end{pmatrix} + (1 + 2x_1\phi_2) \cdot \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix}$$

is triangulable.

To see this, consider the polynomial centralizer element

$$\frac{1}{x_1}\left(F - \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix}\right) + 2\gamma_2 F,$$

for which it is elementary to verify that there is no stationary point. Triangulability follows as in Proposition 3.5.

Note that, once more, the hypothesis of Theorem 3.2 is not satisfied, yet this is not a special case of Proposition 3.5.

4. Nontriangulability in the generic case. This section is devoted to the investigation of polynomial vector fields of the type

$$\tilde{G}(x) = g_2^*(x_1)\begin{pmatrix} 0 \\ p(x_1) \\ q(x_1,x_2) \end{pmatrix} + g_3^*(x_1,\gamma_2)\begin{pmatrix} 0 \\ \sigma(x_1,\phi_2)p(x_1) \\ r(x_1) + \sigma(x_1,\phi_2)q(x_1,x_2) \end{pmatrix}.$$

From Section 3 we know that every such vector field is contained in the centralizer of F and has strictly polynomial flow. If, in addition, the hypothesis of Theorem 3.2 is satisfied, then these are the only vector fields of this kind. Our first result is that F is not triangulable in general.

Theorem 4.1. *If p has a root α such that $\sigma(\alpha, y)$ and $q(\alpha, y)$ are not constant (where y is some indeterminate), then $\tilde{G}$ has a stationary point. In particular, $\dot{x} = F(x)$ is not triangulable.*

Proof. Lemma 2.1 shows that it is sufficient to prove the first assertion.

(i) Recall that $g_2 = prg_2^*$ and $g_3 = prg_3^*$ are polynomials. Since $\tilde{G}$ is a polynomial, it follows that

$$g_2 q + g_3(r + \sigma q) = pr \cdot (\text{some polynomial}).$$

Let $p = (x_1-\alpha)^n \cdot \hat{p}$ and $r = (x_1-\alpha)^m \cdot \hat{r}$, with highest powers factored out. According to the hypothesis, we have $n > 0$. Since $(x_1 - \alpha)$ does not divide q (as p and q are relatively prime), and $(x_1 - \alpha)$ does not divide $(r+\sigma q)$ (as F is reduced), the assumption that $g_2 = (x_1-\alpha)^d \cdot \hat{g}_2$, with $d < n+m$ and $\hat{g}_2(\alpha) \neq 0$, implies that $g_3 = (x_1-\alpha)^d \cdot \hat{g}_3$, with $\hat{g}_3(\alpha, *) \neq 0$. Then division by $(x_1-\alpha)^d$ and substitution $x_1 \to \alpha$ yields $\hat{g}_2(\alpha)\cdot q(\alpha, x_2)+\hat{g}_3(\alpha, r(\alpha)x_2-\tau(\alpha, -h(\alpha, x_2)))\cdot(r(\alpha)+\sigma(\alpha, -h(\alpha, x_2))\cdot q(\alpha, x_2)) = 0$.

Note that the second summand is not zero and has higher degree in x_2 than $q(\alpha, x_2)$. (This follows from the facts that $\sigma(\alpha, y)$ has degree ≥ 1 in y and $h(\alpha, x_2)$ has degree ≥ 1 in x_2.) But this leads to a contradiction; hence, we have found that $(x_1 - \alpha)^{m+n}$ divides g_2 and g_3.

Conclusion. Both $g_2^*(\alpha)$ and $g_3^*(\alpha, y)$ are defined.

(ii) Since $g_2^*(\alpha)$ and $g_3^*(\alpha, *)$ are defined, substituting $x_1 \to \alpha$ will annihilate $(g_2^* + \sigma g_3^*) \cdot p$. It remains to be shown that the polynomial $g_2^* q + g_3^*(r + \sigma q)$ will not become constant upon substituting $x_1 \to \alpha$, unless it becomes identically zero.

The substitution results in the polynomial

$$g_2^*(\alpha)q(\alpha, x_2) + g_3^*(\alpha, r(\alpha)x_2 - \tau(\alpha, -h(\alpha, x_2))) \\ \cdot (r(\alpha) + \sigma(\alpha, -h(\alpha, x_2)) \cdot q(\alpha, x_2)).$$

Now if $g_3^*(\alpha, y)$ is not identically zero, then the second summand is nonzero and has higher degree in x_2 than the first, as the argument in part (i) shows. Therefore, we have a nonconstant polynomial in x_2 in this case. If $g_3^*(\alpha, y) = 0$ and $g_2^*(\alpha) \neq 0$, then the first term is a nonconstant polynomial in x_2. The remaining case yields the zero polynomial. As a result, $\tilde{G}$ always has a stationary point. □

It remains to clarify that the hypothesis of Theorem 4.1 describes a "generic" situation. The collection of all quadruples (p, q, r, σ) of polynomials p, q, r and σ, whose degree does not exceed a given integer d, can be identified with an affine space of a certain dimension. It is easily verified that the subcollection of those quadruples satisfying the hypothesis of Theorem 4.1 contain a nonempty Zariski open subset of this affine space.

It is natural to ask whether there is a similar criterion involving the roots of r. The answer is as follows, with the proof completely analogous to that of Theorem 4.1.

Remark 4.2. Let β be a root of r such that $\sigma(\beta, y)$ is nonconstant and $q(\beta, y)$ is nonzero. Then both $g_2^*(\beta)$ and $g_3^*(\beta, y)$ are defined, and in case $g_3^*(\beta, y) \neq 0$ there is a stationary point of $\tilde{G}$ with first coordinate β.

Note that triangulability cannot be excluded in this case. For the remainder of this section, a few more triangulable examples will be exhibited. The following observation turns out to be useful for this purpose.

Lemma 4.3. *If y is a stationary point of $\tilde{G}$ that is not a common zero of $g_2^*(x_1)$ and $g_3^*(x_1, \gamma_2)$, then $p(y_1) \cdot r(y_1) = 0$.*

Proof. We know that each prg_i^* is a polynomial. If g_2^* or g_3^* is not defined at y, then pr must have a zero at y. Otherwise, the vectors

$$\begin{pmatrix} 0 \\ p(y_1) \\ q(y_1, y_2) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \sigma(y_1, \phi_2(y))p(y_1) \\ r(y_1) + \sigma(y_1, \phi_2(y))q(y_1, y_2) \end{pmatrix}$$

are linearly dependent in $\mathbf{C}^3$.

The following two examples satisfy the hypothesis of Theorem 3.2, but are triangulable. Thus, there are triangulable vector fields which do not belong to the class discussed in Proposition 3.5 and Example 3.6.

Example 4.4. The vector field

$$\begin{pmatrix} 0 \\ 0 \\ x_1 - 1 \end{pmatrix} + 2x_1\phi_2 \cdot \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix} =: F(x)$$

is triangulable.

Here we have $\phi_2(x) = x_1x_3 - x_2^2/2$ and $\gamma_2(x) = (x_1 - 1)x_2 - x_1\phi_2^2$. (Note that F does have stationary points!) Since $\sigma(1, *)$ is not constant, Theorem 3.2 shows that every centralizer element with strictly polynomial flow is of the type

$$g_2^*(x_1)\cdot\begin{pmatrix}0\\x_1\\x_2\end{pmatrix}+g_3^*(x_1,\gamma_2)\cdot\begin{pmatrix}0\\2x_1^2\phi_2\\(x_1-1)+2x_1x_2\phi_2\end{pmatrix},$$

with $g_i := x_1(x_1-1)g_i^*$ polynomials.

A consequence is that $g_2(x_1)+g_3(x_1,\gamma_2)\cdot 2x_1\phi_2 = (x_1-1)\cdot$ (some polynomial).

The substitution $x_1\to 1$, $x_3\to 0$ implies $\phi_2\to -x_2^2/2$ and $\gamma_2\to -x_2^4/4$, and therefore $g_2(1)+g_3(1,-x_2^4/4)\cdot(-x_2^2)=0$.

Since this is possible only for $g_2(1)=0$ and $g_3(1,*)=0$, it follows that the $\hat{g}_i := x_1g_i^*$ are already polynomials. Now the third entry of the centralizer element above shows that $\hat{g}_2(x_1)x_2+\hat{g}_3(x_1,\gamma_2)\cdot((x_1-1)+2x_1x_2\phi_2) = x_1\cdot$ (some polynomial), and $x_1\to 0$ yields $\phi_2\to -x_2^2/2$, $\gamma_2\to -x_2$, hence $\hat{g}_2(0)x_2+\hat{g}_3(0,-x_2)\cdot(-1)=0$.

With $\hat{g}_2=-1$ and $\hat{g}_3=x_2$, we obtain the centralizer element

$$\begin{aligned}Q(x) &:= -\frac{1}{x_1}\begin{pmatrix}0\\x_1\\x_2\end{pmatrix}+\frac{\gamma_2}{x_1}F(x)\\ &= \begin{pmatrix}0\\-1+2x_1\phi_2\gamma_2\\x_1x_2-2x_2-(x_1-1)\phi_2^2+2x_2\phi_2\gamma_2\end{pmatrix},\end{aligned}$$

and another centralizer element

$$Q-\gamma_2F=\begin{pmatrix}0\\-1+2x_1\phi_2\gamma_2-2x_1^2\phi_2\gamma_2\\ *\end{pmatrix}$$

with strictly polynomial flow. Now Lemma 4.3 implies that every stationary point y of this latter vector field satisfies $y_1=0$ or $y_1=1$. But both choices yield -1 in the second entry, showing that $Q-\gamma_2F$ has no stationary point. It follows from Lemmas 2.1 and 2.2 that F is triangulable.

Example 4.5. The vector field

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 2(x_1 - 1)\phi_2 \cdot \begin{pmatrix} 0 \\ x_1(x_1-1) \\ 1 + x_1 x_2 \end{pmatrix} =: F(x)$$

is triangulable.

In this example $\phi_2 = x_1(x_1 - 1)x_3 - (x_2 + x_1 x_2^2/2)$ and $\gamma_2 = x_2 - (x_1 - 1)\phi_2^2$. The strictly polynomial flow elements in the centralizer of F are the

$$g_2^*(x_1) \cdot \begin{pmatrix} 0 \\ x_1(x_1-1) \\ 1 + x_1 x_2 \end{pmatrix} + g_3^*(x_1, \gamma_2) \cdot F(x),$$

according to Theorem 3.2, with $\sigma(0, *)$ not constant. Again, the $g_i := x_1(x_1 - 1)g_i^*$ are polynomials. One such centralizer element is

$$\begin{aligned} Q(x) &:= \frac{1}{(x_1 - 1)} \left((1 + \gamma_2(x)) \cdot F(x) - \begin{pmatrix} 0 \\ x_1(x_1-1) \\ 1 + x_1 x_2 \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 \\ -x_1 + 2x_1(x_1-1)(1+\gamma_2)\phi_2 \\ -x_2 - \phi_2^2 + 2\phi_2(1+\gamma_2)(1+x_1x_2) \end{pmatrix}. \end{aligned}$$

It is not hard to verify that every strictly polynomial flow element in the centralizer of F can be written as

$$\tilde{G}(x) = \mu \cdot Q(x) + \tilde{g}_2(x_1) \cdot \begin{pmatrix} 0 \\ x_1(x_1-1) \\ 1 + x_1 x_2 \end{pmatrix} + \tilde{g}_3(x_1, \gamma_2) \cdot F(x),$$

with $\mu \in \mathbf{C}$ and polynomials $\tilde{g}_i$. Set $\mu = 1$ in the following. Then $\tilde{G}$ has no stationary point y with $y_1 = 1$, since the second entry of $\tilde{G}$ always yields -1. The search for stationary points with first entry 0 yields 0 in the second entry and the polynomial $1 - (1 + 2x_2) \cdot (1 + x_2 + x_2^2) + \tilde{g}_2(0) + \tilde{g}_3(0, x_2 + x_2^2) \cdot (1 + 2x_2)$ in the last entry.

The choice $\tilde{g}_2 = 0$ and $\tilde{g}_3(x_1, x_2) = 1 + x_2$ yields the constant polynomial 1, and therefore,

$$Q(x) + (1 + \gamma_2(x)) \cdot F(x)$$

has no stationary point with first component 0. Lemma 4.3 shows that this vector field has no stationary point, and we can conclude that F is triangulable.

It should be emphasized that it is actually possible to systematically investigate the polynomial centralizer of any given F, thus it is not a matter of trial and error to find strictly polynomial flow centralizer elements with no stationary points. We will sketch the strategy in the following.

In general, one uses Lemma 4.3 and exploits the arguments in the proof of (4.1) in greater detail. Suppose that $p(\alpha) = 0$ and that the hypothesis of (4.1) does not hold for α. One first has to obtain more precise information about the polynomial elements $\tilde{G}$ in the centralizer. (The notation is as in the proof of (4.1).) If $q(\alpha, y)$ is constant (and necessarily nonzero), then part (i) still shows that $g_2^*(\alpha)$ and $g_3^*(\alpha, y)$ are defined.

Now suppose that $\sigma(\alpha, y) = \sigma_0$ is constant, so $\tau(x_1, x_2) = \sigma_0 x_2$, while $q(\alpha, y)$ is nonconstant. Assume that $d < n + m$ in part (i) of the proof. If $\sigma_0 = 0$, then $r(\alpha) \neq 0$ (thanks to reducedness), and we have $\hat{g}_2(\alpha) q(\alpha, x_2) + \hat{g}_3(\alpha, r(\alpha) x_2) r(\alpha) = 0$. This determines $\hat{g}_3(\alpha, y)$ for every given $\hat{g}_2(\alpha)$.

If $\sigma_0 \neq 0$ then $\hat{g}_3(\alpha, y)$ is constant (as $h(\alpha, x_2)$ has degree ≥ 2) and nonzero. This is possible only for $r(\alpha) = 0$, and then again $\hat{g}_3(\alpha, y)$ is uniquely determined by $\hat{g}_2(\alpha)$.

Finally, assume that $q(\alpha, x_2) = \eta_0$ is constant. Then reducedness implies that $r(\alpha) + \sigma_0 \eta_0 \neq 0$, and one has $\hat{g}_2(\alpha) \eta_0 + \hat{g}_3(\alpha, (r(\alpha) + \sigma_0 \eta_0) x_2)(r(\alpha) + \sigma_0 \eta_0) = 0$. Once more, this shows that $\hat{g}_2(\alpha)$ determines $\hat{g}_3(\alpha, y)$ (which is constant).

In case $d < n + m - 1$, set $\hat{g}_2(x_1) = \hat{g}_2(\alpha) + (x_1 - \alpha) \cdot \hat{\hat{g}}_2(x_1)$, and similarly for $\hat{g}_3$, and obtain conditions for $\hat{\hat{g}}_2(\alpha)$ and so on. Eventually, this yields a representation

$$\tilde{G} = \mu_1 \cdot Q_1 + \cdots + \mu_m \cdot Q_m + \tilde{g}_2(x_1) \cdot \begin{pmatrix} 0 \\ p \\ q \end{pmatrix} + \tilde{g}_3(x_1, \gamma_2) \cdot \begin{pmatrix} 0 \\ \sigma p \\ r + \sigma q \end{pmatrix},$$

where the μ_i are complex numbers, and $\tilde{g}_2(\alpha)$ and $\tilde{g}_3(\alpha, y)$ are defined.

In the next step, one has to determine the conditions under which there is no stationary point $(\alpha, *, *)^t$ for $\tilde{G}$. We will discuss this only for the case that $q(\alpha, y)$ is nonconstant and $\sigma(\alpha, y) = 0$. (The other cases can be handled similarly.)

Upon setting $x_1 \to \alpha$, the third entry of $\tilde{G}$ becomes $s(\alpha, x_2) + \tilde{g}_2(\alpha)q(\alpha, x_2) + \tilde{g}_3(\alpha, r(\alpha)x_2)r(\alpha)$, where s comes from the sum of the $\mu_i Q_i$.

Now, for any given $\mu_1, \ldots, \mu_m$ (which determine s), and for any given $\tilde{g}_2(\alpha)$, one can determine all $\tilde{g}_3$ such that the above polynomial is a nonzero constant. (It is clear how to choose $\tilde{g}_3(\alpha, y)$, and then one may add an arbitrary multiple of $x_1 - \alpha$.) Moreover, there may be certain combinations of the μ_i such that the second entry of $\tilde{G}$ is a nonzero constant.

In this way one obtains all strictly polynomial flow elements in the centralizer of F that have no stationary point with first coordinate α. Going through all the roots of p, and similarly through all the roots of r, one ends up with centralizer elements having no stationary points, provided that such elements exist at all.

We were not able to find nontriangulable vector fields that violate the hypothesis of Theorem 4.1. Thus, it may very well be the case that the condition of Theorem 4.1 is also necessary for nontriangulability.

5. A nonreduced example, and final remarks. One class of nonreduced vector fields F consists of those where r and σ have a nontrivial common factor. A more interesting example is given by

$$F(x) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + (1 + 2x_1\phi_2) \cdot \begin{pmatrix} 0 \\ x_1(x_1+1) \\ 1 - 2x_1x_2 \end{pmatrix}.$$

Note that F is indeed not reduced, as one has $F(x) = x_1 \cdot \tilde{F}(x)$, where

$$\tilde{F}(x) = \begin{pmatrix} 0 \\ (1 + 2x_1\phi_2)(x_1+1) \\ 2\phi_2 - 2x_2(1 + 2x_1\phi_2) \end{pmatrix}.$$

With the usual definitions we have $\phi_2 = x_1(x_1+1)x_3 - x_2 + x_1x_2^2$ and $\gamma_2 = -x_2 - \phi_2 - x_1\phi_2^2 = (-x_1) \cdot \mu(x)$, with $\mu(x) := (x_1+1)x_3 + x_2^2 + \phi_2^2$.

Since a birational straightening function for F is known from Section 2, one easily gets the birational straightening function

$$\tilde{\Gamma}(x) = \begin{pmatrix} x_1 \\ \gamma_2(x) \\ -x_1x_3 + (x_2 + x_2^2)/(x_1 + 1) \end{pmatrix}.$$

Furthermore, the same proof as in Lemma 2.3 yields that $\mathbf{C}[x_1, \mu]$ is the algebra of polynomial invariants of the corresponding group action.

Using these ingredients, a straightforward imitation of the proofs of Theorem 3.2 and Proposition 3.4 yields that every strictly polynomial flow vector field in the centralizer of $\tilde{F}$ has the form

$$g_2^*(x_1) \cdot \begin{pmatrix} 0 \\ x_1(x_1 + 1) \\ 1 - 2x_1x_2 \end{pmatrix} + g_3^*(x_1, \mu) \cdot \tilde{F}(x),$$

and the g_i^* actually turn out to be polynomials. All these vector fields, however, have a stationary point (with first component -1) as is readily verified. The conclusion is that $\tilde{F}$ (and F) is not triangulable.

As this example indicates, there is no principal difficulty in the investigation of any given nonreduced vector field.

Finally it is worth noting that, in the reduced case, the centralizer elements of F described in Corollary 3.3 and Proposition 3.4 have a remarkable property. The reduced ones among these can be transformed over $\mathbf{C}(x_1)$ to $(\partial/\partial x_3)$ by a product of *three* triangular automorphisms, and conversely it can be shown that every reduced vector field G with this latter property occurs in the centralizer of some vector field F of the type under consideration here. Moreover, the process does not end at this stage. It can be shown that iterated computations of centralizer elements, i.e., centralizers of G, centralizers of centralizers of G,..., will eventually produce every vector field on $\mathbf{C}^3$ that corresponds to a locally nilpotent derivation killing a variable. It is clear that, for any *given* locally nilpotent derivation, the question of triangulability can be decided using the strategy employed in this paper. Whether a general picture can be obtained remains to be seen.

REFERENCES

1. H. Bass, *A nontriangular action of* G_a *on* A^3, J. Pure Appl. Algebra **33** (1984), 1–5.

2. D. Daigle, *A necessary and sufficient condition for triangulability of derivations of* $k[X, Y, Z]$, preprint, 1995.

3. D. Daigle and G. Freudenburg, *Locally nilpotent derivations over a UFD and an application to rank two locally nilpotent derivations of* $k[X_1, \dots, X_n]$, preprint, 1994.

4. J.K. Deveney and D.R. Finston, *Rationally triangulable automorphisms*, J. Pure Appl. Algebra **72** (1991), 1–4.

5. ———, *Algebraic aspects of additive group actions on complex affine space*, in *Automorphisms of affine spaces* (A. van den Essen, ed.), Kluwer, Norwell, 1995.

6. ———, G_a *actions on* $\mathbf{C}^3$ *and* $\mathbf{C}^7$, Comm. Algebra **22** (1994), 6295–6302.

7. ———, G_a *actions on* $\mathbf{C}^n$, Comm. Algebra **22** (1994), 4977–4988.

8. A. van den Essen, *An algorithm to compute the invariant ring of a* G_a *action on an affine variety*, J. Symbolic Comput. **16** (1993), 531–555.

9. D.R. Finston and S. Walcher, *Centralizers of locally nilpotent derivations*, J. Pure Appl. Algebra, to appear.

10. G. Freudenburg, *Actions of* G_a *on* A^3 *defined by homogeneous derivations*, J. Pure Appl. Algebra, to appear.

11. H.W.E. Jung, *Über ganze birationale Transformationen der Ebene*, J. Reine Ange. Math. **184** (1942), 161–174.

12. V. Popov, *On actions of* G_a *on* A^n, in *Algebraic groups*, Lecture Notes in Math. **1271** (1986), 237–242.

13. R. Rentschler, *Operations du groupe additif sur le plan affine*, C.R. Acad. Sci. Paris Sér. I Math. **267** (1968), 384–387.

14. M.K. Smith, *Stably tame automorphisms*, J. Pure Appl. Algebra **58** (1989), 209–212.

15. D.M. Snow, *Unipotent actions on affine space*, in *Topological methods in algebraic transformation groups* (H. Kraft, T. Petrie, G.W. Schwarz, eds.), Birkhäuser, Boston, 1989.

16. T. Sugie, *Algebraic characterization of the affine plane and the affine* 3*-space*, in *Topological methods in algebraic transformation groups* (H. Kraft, T. Petrie, G.W. Schwarz, eds.), Birkhäuser, Boston, 1989.

17. S. Walcher, *On sums of vector fields*, Resultate Math., to appear.

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