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ON RAMANUJAN'S TAU FUNCTION

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ABSTRACT. A formula for Ramanujan's function τ, defined by the expansion

$$x\prod_1^\infty (1-x^n)^{24} = \sum_1^\infty \tau(n)x^n, \quad |x|<1,$$

is presented.

1. A formula for τ. In this paper we present a formula for the arithmetical function τ defined by the expansion

$$x\prod_1^\infty (1-x^n)^{24} = \sum_1^\infty \tau(n)x^n, \tag{1.1}$$

which is valid for each complex number x such that $|x| < 1$. As intimated in the above title, S. Ramanujan [**6**, p. 151] was the first mathematician to consider this function. Since the formula involves several additional functions, we collect these in the following definition.

Definition 1.1. For $\mathbf{N} := \{0,1,2,\dots\}$, put $\mathbf{P} := N - \{0\}$. Then, for each $k \in \mathbf{P}$ and each $n \in \mathbf{N}$,

$$r_k(n) := |\{(x_1, x_2, \dots, x_k) \in \mathbf{Z}^k | n = x_1^2 + x_2^2 + \cdots + x_k^2\}|.$$

(Of course, $\mathbf{Z} := \{0, \pm 1, \pm 2, \dots\}$.)

For each $n \in \mathbf{P}$, $b(n)$ is the exponent of the exact power of 2 dividing n, and then $\mathrm{Od}(n) := n2^{-b(n)}$ is the odd part of n.

For each $k \in \mathbf{N}$ and each $n \in \mathbf{P}$, $\sigma_k(n)$ is the sum of the kth powers of all of the positive divisors of n. For simplicity, $\sigma(n) := \sigma_1(n)$.

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We further require the identities:

$$\prod_1^\infty (1-x^{2n})(1+tx^{2n-1})(1+t^{-1}x^{2n-1}) = \sum_{-\infty}^{\infty} x^{n^2} t^n \tag{1.2}$$

and

$$\begin{aligned}\prod_1^\infty (1-x^{2n})^2 &(1+abx^{2n-1})(1+a^{-1}b^{-1}x^{2n-1}) \\ &\cdot (1+ab^{-1}x^{2n-1})(1+a^{-1}bx^{2n-1}) \\ &= \sum_{-\infty}^{\infty} x^{2m^2} a^{2m} \sum_{-\infty}^{\infty} x^{2n^2} b^{2n} \\ &+ x \sum_{-\infty}^{\infty} x^{2m(m+1)} a^{2m+1} \sum_{-\infty}^{\infty} x^{2n(n+1)} b^{2n+1},\end{aligned} \tag{1.3}$$

which are respectively valid for each pair t, x and each triple a, b, x of complex numbers such that $t \neq 0$, $a \neq 0$, $b \neq 0$ and $|x| < 1$. The first of these two identities, the triple-product identity,is a celebrated result, and elementary proofs of it abound. For example, see [**5**, pp. 282–283]. The second identity, due to the author, is not so widely known. But there is an accessible proof of it in [**3**, pp. 1287–1293].

We are now prepared to establish our main result.

Theorem 1.2. *For each* $n \in \mathbf{P}$*,*

$$\begin{aligned}\tau(n) = \sum_{k=1}^{n} &(-1)^{n-k} r_8(n-k) \\ &\cdot \sum_{j=0}^{k-1} \sigma_3(2(k-j)-1)\sigma_3(2j+1) \\ &- 64 \sum_{k=2}^{n} (-1)^{n-k} r_8(n-k) \\ &\cdot \sum_{j=1}^{k-1} 2^{3b(k-j)+3b(j)} \sigma_3(\mathrm{Od}(k-j))\sigma_3(\mathrm{Od}(j)).\end{aligned}$$

Proof. To prove this theorem we first appeal to identity (1.2) to express each series on the right side of identity (1.3) as an infinite product.

$$\begin{aligned}\prod_1^\infty &(1-x^{2n})^2(1+abx^{2n-1})(1+a^{-1}b^{-1}x^{2n-1})\\ &\cdot(1+ab^{-1}x^{2n-1})(1+a^{-1}bx^{2n-1})\\ &=\prod_1^\infty(1-x^{4n})^2(1+a^2x^{4n-2})(1+a^{-2}x^{4n-2})\\ &\qquad\cdot(1+b^2x^{4n-2})(1+b^{-2}x^{4n-2})\\ &+x(a+a^{-1})(b+b^{-1})\prod_1^\infty(1-x^{4n})^2(1+a^2x^{4n})\\ &\qquad\cdot(1+a^{-2}x^{4n})(1+b^2x^{4n})(1+b^{-2}x^{4n}).\end{aligned}$$

Next, in the foregoing identity, we let $a=b$, and then let $a\to ia$ to get

$$\begin{aligned}\prod_1^\infty &(1-x^{2n})^2(1+x^{2n-1})^2(1-a^2x^{2n-1})(1-a^{-2}x^{2n-1})\\ &=\prod_1^\infty(1-x^{4n})^2(1-a^2x^{4n-2})^2(1-a^{-2}x^{4n-2})^2\\ &-(a-a^{-1})^2x\prod_1^\infty(1-x^{4n})^2(1-a^2x^{4n})^2(1-a^{-2}x^{4n})^2.\end{aligned}$$

In this identity we sequentially

(i) let $x\to -x$,

(ii) multiply the resulting identity and the immediately preceding identity, and

(iii) in the product of the two identities let $x \to x^{1/2}$ to get

$$(1.4)\quad (a-a^{-1})^4 x \prod_1^\infty (1-x^{2n})^4(1-a^2x^{2n})^4(1-a^{-2}x^{2n})^4$$
$$= \prod_1^\infty (1-x^{2n})^4(1-a^2x^{2n-1})^4(1-a^{-2}x^{2n-1})^4$$
$$- \prod_1^\infty (1-x^{2n})^4(1-x^{2n-1})^6$$
$$\cdot (1-a^4x^{2n-1})(1-a^{-4}x^{2n-1}).$$

Now, in identity (1.4), let $x \to -x$. Then multiply the last identity and (1.4) to get

$$(1.5)\quad (a-a^{-1})^8 F(a,x) = -G(a,x) + H(a,x) + H(a,-x) - I(a,x),$$

where

$$F(a,x) := x^2 \prod_1^\infty (1-x^{2n})^8(1-a^2x^{2n})^8(1-a^{-2}x^{2n})^8,$$

$$G(a,x) := \prod_1^\infty (1-x^{2n})^8(1-a^4x^{4n-2})^4(1-a^{-4}x^{4n-2})^4,$$

$$H(a,x) := \prod_1^\infty (1-x^{2n})^8(1+x^{2n-1})^6$$
$$\cdot (1-a^2x^{2n-1})^4(1-a^{-2}x^{2n-1})^4$$
$$\cdot (1+a^4x^{2n-1})(1+a^{-4}x^{2n-1}),$$

$$I(a,x) := \prod_1^\infty (1-x^{2n})^8(1-x^{4n-2})^6$$
$$\cdot (1-a^8x^{4n-2})(1-a^{-8}x^{4n-2}).$$

With D_a denoting derivation with respect to a, put $\theta_a := aD_a$. Our immediate goal is to operate on both sides of identity (1.5) with θ_a^8 and thereafter let $a = 1$. To this end we easily establish $\theta_a^8\{(a-a^{-1})^8F(a,x)\}|_{a=1} = 2^8 \cdot 8!F(1,x)$. Explicit evaluation of the four terms

$$-\theta_a^8 G(a,x)|_{a=1}, \theta_a^8 H(a,x)|_{a=1}, \theta_a^8 H(a,-x)|_{a=1}, -\theta_a^8 I(a,x)|_{a=1}$$

is somewhat tedious. So we illustrate how to evaluate $\theta_a^8 G(a,x)|_{a=1}$. Put $v_k(z) := z^k/(1-z^{2k})$, z a complex number such that $|z|<1$ and $k \in \mathbf{P}$. Then,

$$\begin{aligned}\theta_a G(a,x) &= G(a,x)\Big\{-16\sum_1^\infty v_k(x^2)(a^{4k}-a^{-4k})\Big\}\\ &:= G(a,x)\cdot\alpha(a,x),\end{aligned}$$

say. Since $\theta_a^j \alpha(a,x)|_{a=1} = 0$, whenever j is even, we have

$$\begin{aligned}\theta_a^8 G(a,x)|_{a=1} = \sum_{j=0}^{3}\binom{7}{2j+1}&\{\theta_a^{7-2j-1}G(a,x)|_{a=1}\}\\ &\cdot\{\theta_a^{2j+1}\alpha(a,x)|_{a=1}\}.\end{aligned}$$

Computing the derivatives and subsequently simplifying, we find

$$\begin{aligned}-\theta_a^8 G(a,x)|_{a=1} = G(1,x)\Big\{&2^{19}\sum_1^\infty 2^{7b(n)}\sigma_7(\mathrm{Od}(n))x^{2n}\\ &-(2^{24}\times 7)\sum_1^\infty 2^{b(n)}\sigma(\mathrm{Od}(n))x^{2n}\\ &\cdot\sum_1^\infty 2^{5b(n)}\sigma_5(\mathrm{Od}(n))x^{2n}\\ &+(2^{26}\times 105)\Big[\sum_1^\infty 2^{b(n)}\sigma(\mathrm{Od}(n))x^{2n}\Big]^2\\ &\cdot\sum_1^\infty 2^{3b(n)}\sigma_3(\mathrm{Od}(n))x^{2n}\\ &+(2^{22}\times 35)\Big[\sum_1^\infty 2^{3b(n)}\sigma_3(\mathrm{Od}(n))x^{2n}\Big]^2\\ &-(2^{28}\times 105)\Big[\sum_1^\infty 2^{b(n)}\sigma(\mathrm{Od}(n))x^{2n}\Big]^4\Big\}.\end{aligned}$$

Similarly, we compute $\theta_a^8 H(a,x)|_{a=1}$, $\theta_a^8 H(a,-x)|_{a=1}$, $-\theta_a^8 I(a,x)|_{a=1}$, and find

$$\begin{aligned}
2^8\cdot 8!F(1,x) &= -\theta_a^8 G(a,x)|_{a=1} + \theta_a^8 H(a,x)|_{a=1} \\
&\quad + \theta_a^8 H(a,-x)|_{a=1} - \theta_a^8 I(a,x)|_{a=1} \\
&= 0\cdot H(1,x)\sum_1^\infty 2^{7b(n)}\sigma_7(\mathrm{Od}(n))x^{2n} \\
&\quad + 0\cdot H(1,x)\sum_1^\infty 2^{b(n)}\sigma(\mathrm{Od}(n))x^{2n}\sum_1^\infty 2^{5b(n)}\sigma_5(\mathrm{Od}(n))x^{2n} \\
&\quad - (2^{21}\times 315)H(1,x)\biggl[\sum_1^\infty 2^{3b(n)}\sigma_3(\mathrm{Od}(n))x^{2n}\biggr]^2 \\
&\quad + (2^{15}\times 315)H(1,x)\biggl[\sum_0^\infty \sigma_3(2n+1)x^{2n+1}\biggr]^2 \\
&\quad + 0\cdot H(1,x)\biggl[\sum_1^\infty 2^{b(n)}\sigma(\mathrm{Od}(n))x^{2n}\biggr]^2\sum_1^\infty 2^{b(n)}\sigma_3(\mathrm{Od}(n))x^{2n} \\
&\quad + 0\cdot H(1,x)\biggl[\sum_1^\infty 2^{b(n)}\sigma(\mathrm{Od}(n))x^{2n}\biggr]^4.
\end{aligned}$$

Note. We easily verify that $G(1,x) = H(1,x) = H(1,-x) = I(1,x)$. Hence, observing that $2^8\cdot 8! = 2^{15}\times 315$, we divide both sides of the foregoing identity by $2^{15}\times 315$ to get

$$\begin{aligned}
F(1,x) = H(1,x)&\biggl[\sum_0^\infty \sigma_3(2n+1)x^{2n+1}\biggr]^2 \\
&- 64H(1,x)\biggl[\sum_1^\infty 2^{3b(n)}\sigma_3(\mathrm{Od}(n))x^{2n}\biggr]^2,
\end{aligned}$$

where

$$\begin{aligned}
H(1,x) &= \prod_1^\infty (1-x^{4n})^8(1-x^{4n-2})^{16} = \biggl\{\sum_{-\infty}^\infty (-1)^n x^{2n^2}\biggr\}^8 \\
&= \sum_0^\infty (-1)^n r_8(n)x^{2n}.
\end{aligned}$$

Hence,

$$\begin{aligned}\sum_1^\infty \tau(n)x^{2n} &= F(1,x)\\ &= \sum_0^\infty (-1)^n r_8(n)x^{2n}\left[\sum_0^\infty \sigma_3(2n+1)x^{2n+1}\right]^2\\ &\quad - 64\sum_0^\infty (-1)^n r_8(n)x^{2n}\left[\sum_1^\infty 2^{3b(n)}\sigma_3(\mathrm{Od}(n))x^{2n}\right]^2.\end{aligned}$$

Finally, expanding the right side of the foregoing identity, and subsequently equating coefficients of like powers of x, we prove our theorem.

2. Congruence properties of τ.

Corollary 2.1. *For each $n \in \mathbf{P}$,*

$$\begin{aligned}\tau(n) \equiv &\sum_{j=0}^{n-1}\sigma_3(2n-2j-1)\sigma_3(2j+1)\\ &+\sum_{k=1}^{n-1}(-1)^{n-k}r_8(n-k)\\ &\cdot\sum_{j=0}^{k-1}\sigma_3(2k-2j-1)\sigma_3(2j+1) \pmod{64},\end{aligned} \tag{2.1}$$

and

$$\tau(n) \equiv \sum_{j=0}^{n-1}\sigma_3(2n-2j-1)\sigma_3(2j+1) \pmod{16}. \tag{2.2}$$

Proof. The first part (clause) of our statement is an obvious consequence of Theorem 1.2. To see the second part, we recall the formula

$$r_8(n) = 16(-1)^n\sum_{d\mid n}(-1)^d d^3,$$

valid for each $n \in \mathbf{P}$. For example, see [**5**, p. 314]. $\square$

Corollary 2.2. *For each $n \in \mathbf{P}$, if n is not an odd square, then*

$$\sum_{j=0}^{n-1} \sigma_3(2n-j-1)\sigma_3(2j+1) \equiv 0, \qquad (\mathrm{mod}\ 2). \tag{2.3}$$

Proof. Let $n \in \mathbf{P}$. Then it is well known that $\tau(n)$ is odd if and only if n is an odd square. For example, see [**2**, p. 39]. Hence, if n is not an odd square, then $\tau(n) \equiv 0 \pmod 2$, whence the desired conclusion (owing to (2.2)). $\square$

3. Concluding remarks. Embarking on a long discussion of the function τ, G.H. Hardy [**4**, p. 161] wrote: "We may seem to be straying into one of the backwaters of mathematics, but the genesis of $\tau(n)$ as a coefficient in so fundamental a function compels us to treat it with respect." Hardy then proceeded to demonstrate the importance and centrality of this function in the theory of modular functions. Our present discussion should certainly help to amplify this point of view. For, our formula representation of τ is a direct consequence of identity (1.3), which is itself an easy and straightforward consequence of the classical Gauss-Jacobi triple-product identity (1.2). And the importance of the triple-product identity in both enumerative combinatorics and additive number theory is beyond question.

In [**2**, p. 37] the author has derived another formula for τ, viz.,

$$\tau(n) = \sum_{j=1}^{n} (-1)^{n-j} r_{16}(n-j) 2^{3b(j)} \sigma_3(\mathrm{Od}(j)),$$

for each $n \in \mathbf{P}$. This formula has a simpler form than that of Theorem 1.2. However, r_{16} is certainly expressible as a convolution of r_8. So the "simpler form" is somewhat illusory.

It is perhaps instructive to compare the modular relation (2.2) with a similar result of Bambah [**1**, pp. 91–93].

If n is odd, then $\tau(n) \equiv \sigma_3(n) \pmod{32}$. However, in [**2**, p. 39] the following result, which contains Bambah's result, is presented.

For each $n \in \mathbf{P}$,

$$\tau(n) \equiv 2^{3b(n)}\sigma_3(\mathrm{Od}(n)), \qquad (\mathrm{mod}\ 32).$$

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