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ON THE SIMULTANEOUS BEHAVIOR OF THE DEPENDENCE COEFFICIENTS ASSOCIATED WITH THREE MIXING CONDITIONS

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ABSTRACT. For strictly stationary random sequences, there are certain basic, elementary restrictions on the simultaneous behavior of the dependence coefficients associated with the strong mixing, ρ-mixing and "interlaced ρ-mixing" conditions. Here a class of strictly stationary random sequences is constructed in order to show that in a certain sense there are "almost" no other restrictions on the simultaneous behavior of these dependence coefficients.

1. Introduction. Suppose $(\Omega, \mathcal{F}, P)$ is a probability space. For any two σ-fields $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{F}$, define the following measures of dependence

$$\begin{aligned} \alpha(\mathcal{A},\mathcal{B}) &:= \sup |P(A\cap B) - P(A)P(B)|, \\ & \qquad A \in \mathcal{A},\ B \in \mathcal{B}; \\ \rho(\mathcal{A},\mathcal{B}) &:= \sup |\mathrm{Corr}\,(f,g)| \end{aligned} \tag{1.1}$$

where the latter supremum is taken over all pairs of square-integrable random variables f and g such that f is $\mathcal{A}$-measurable and g is $\mathcal{B}$-measurable. The quantity $\rho(\mathcal{A},\mathcal{B})$ is the "maximal correlation" [**13, 15**] between the σ-fields $\mathcal{A}$ and $\mathcal{B}$. The following inequalities are elementary and well known:

$$0 \le 4\alpha(\mathcal{A},\mathcal{B}) \le \rho(\mathcal{A},\mathcal{B}) \le 1. \tag{1.2}$$

For any family $(W_k, k \in S)$ of random variables on this probability space $(\Omega, \mathcal{F}, P)$, let $\sigma(W_k, k \in S)$ denote the σ-field of events generated by this family.

Suppose $X := (X_k, k \in \mathbf{Z})$ is a strictly stationary sequence of random variables (on the probability space $(\Omega, \mathcal{F}, P)$). For each positive integer n, define the following dependence coefficients

$$\begin{aligned} \alpha(n) = \alpha(X,n) &:= \alpha(\sigma(X_k, k\le 0), \sigma(X_k, k \ge n)); \\ \rho(n) = \rho(X,n) &:= \rho(\sigma(X_k, k\le 0), \sigma(X_k, k \ge n)); \\ \rho^*(n) = \rho^*(X,n) &:= \sup \rho(\sigma(X_k, k\in S), \sigma(X_k, k \in T)), \end{aligned} \tag{1.3}$$

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where this last supremum is taken over all pairs of nonempty disjoint sets S and $T \subset \mathbf{Z}$ such that

$$\text{dist}\,(S,T) := \min_{\substack{j\in S\\ k\in T}} |j-k| \geq n. \tag{1.4}$$

Obviously, by (1.2), one has that, for each positive integer n,

$$0 \leq 4\alpha(n) \leq \rho(n) \leq \rho^*(n) \leq 1. \tag{1.5}$$

Also, obviously, one has that

$$\begin{aligned}&\alpha(1) \geq \alpha(2) \geq \alpha(3) \geq \cdots ;\\ &\rho(1) \geq \rho(2) \geq \rho(3) \geq \cdots ;\end{aligned} \tag{1.6}$$

and

$$\rho^*(1) \geq \rho^*(2) \geq \rho^*(3) \geq \cdots .$$

The (strictly stationary) sequence X is "strongly mixing" [**25**] if $\alpha(n) \to 0$ as $n \to \infty$ and "ρ-mixing" [**18**] if $\rho(n) \to 0$ as $n \to \infty$. The origin of the mixing condition $\rho^*(n) \to 0$ seems hard to trace; that condition has been popular in the broader context of random fields, see, e.g., [**4, 9, 12, 19, 24, 26**].

The purpose of this note is to prove the following theorem:

Theorem 1.1. *Suppose $(a_1, a_2, a_3, \dots)$, $(b_1, b_2, b_3, \dots)$ and $(c_1, c_2, c_3, \dots)$ are each a nonincreasing sequence of numbers such that, for each positive integer n,*

$$0 < 4a_n \leq b_n \leq c_n \leq 1. \tag{1.7}$$

Suppose $(d_1, d_2, d_3, \dots)$ is a sequence of positive numbers. Then there exists (on some probability space) a strictly stationary sequence $(X_k, k \in \mathbf{Z})$ of random variables such that, for each positive integer n,

$$a_n \leq \alpha(n) \leq a_n + d_n; \tag{1.8}$$

$$\rho(n) = b_n; \tag{1.9}$$

and

$$\rho^*(n) = c_n. \tag{1.10}$$

In [**1**, Theorem 6], a similar result was proved which had Equations (1.8) and (1.9) but did not involve the dependence coefficients $\rho^*(n)$. That sequence will (indirectly) be a major component of the construction here, but most of it will have to be redone here (in a considerably simpler manner) in order to keep track of the dependence coefficients $\rho^*(n)$. Another major component of the construction here will be a piece (in a substantially extended form) of a construction in [**2**].

The construction for Theorem 1.1 will be given in Section 3; it will involve putting together pieces developed in Section 2. The rest of Section 1 here will be devoted to several remarks connected with Theorem 1.1.

Remark 1.2. This theorem shows (at least for strictly stationary random sequences that are not m-dependent) that "essentially" the only restrictions on the simultaneous behavior of the dependence coefficients $\alpha(n)$, $\rho(n)$ and $\rho^*(n)$ are the basic, elementary inequalities in (1.5) and (1.6). (See also Remark 1.10.)

Remark 1.3. In Theorem 1.1, for any positive integer n such that $4a_n = b_n$, (1.8) can be replaced by the equation $\alpha(n) = a_n$. This is a consequence of (1.5) (or (1.2)).

Remark 1.4. A careful examination of the random sequence (X_k) constructed (in Section 3) for Theorem 1.1 will show that it has the property that the "marginal" σ-field $\sigma(X_0)$ is purely nonatomic. Hence, for a given $\gamma \geq 0$, one can have in Theorem 1.1 the additional property that $E|X_0|^\gamma < \infty$ and $E|X_0|^{\gamma+\varepsilon} = \infty$ for all $\varepsilon > 0$. (To accomplish this, one can replace each X_k in Theorem 1.1 by $\phi(X_k)$ where $\phi : \mathbf{R} \to \mathbf{R}$ is a well chosen strictly increasing function.)

Remark 1.5. If a strictly stationary random sequence $(X_k, k \in \mathbf{Z})$ satisfies $EX_0 = 0$, $0 < EX_0^2 < \infty$, $\alpha(n) \to 0$ as $n \to \infty$, and $\rho(1) < 1$, then $E(X_1 + \cdots + X_n)^2 \to \infty$ as $n \to \infty$. This is a well-known consequence of a theorem of Leonov given in [**17**, Theorem 18.2.2]. (It can also be seen from [**6**, Theorem 2].) This will be important for the next remark.

Remark 1.6. Theorem 1.1 is motivated by two central limit theorems of Magda Peligrad, stated together here:

Theorem A (Peligrad). *Suppose $(X_k, k \in \mathbf{Z})$ is a strictly stationary sequence of random variables such that $EX_0 = 0$, $EX_0^2 < \infty$, $\sigma_n^2 := E(X_1 + \cdots + X_n)^2 \to \infty$ as $n \to \infty$, and $\alpha(n) \to 0$ as $n \to \infty$. Suppose also that at least one of the following two conditions holds*:

(1) *$\rho^*(n) < 1$ for some $n \geq 1$, or*

(2) *for some $\delta \in (0,1]$, $E|X_0|^{2+\delta} < \infty$ and $R := \lim_{n\to\infty} \rho(n)$ satisfies*

$$\frac{[2(1 + (\delta(1+\delta)/2)R^{2\delta/(2+\delta)} + (1+\delta)R^{2/(2+\delta)})]^{1/(2+\delta)}}{2^{1/2}(1-R)^{1/2}} < 1. \tag{1.11}$$

Then $(X_1 + \cdots + X_n)/\sigma_n \to N(0,1)$ in distribution as $n \to \infty$.

Theorem A with assumption (1) is taken from [**22**, Corollary 2.3] (and is extended in [**23**] to a weak invariance principle under the same hypothesis), and with assumption (2) it is taken from [**20**, Corollary 2.2] (also a weak invariance principle). In the broader context of random fields, Perera [**24**, Proposition 2.4] gave a result closely related to Theorem A with assumption (1).

By Theorem 1.1 (and Remark 1.4), there exists a strictly stationary random sequence (X_k) such that (say) $EX_0 = 0$, $0 < EX_0^2 < \infty$, $E|X_0|^{2+\delta} = \infty$ for all $\delta > 0$, $\alpha(n) \sim (\log \log n)^{-1}$ as $n \to \infty$, and $\rho(n) = \rho^*(n) = .97$ for all $n \geq 1$. For such a random sequence, the partial sums are asymptotically normally distributed. This fact can be derived from Theorem A(1) (and Remark 1.5), but it cannot be derived either from Theorem A(2) or from standard central limit theorems under strong mixing or ρ-mixing with a higher order moment assumption and/or a mixing rate assumption, see, e.g., [**11**, Theorem 1, **16**, Theorems 2.1 and 2.2, **17**, Theorems 18.5.3 and 18.5.4], or the result of M.I. Gordin [**14**] discussed in [**3**, Theorem 2.2].

By Theorem 1.1 (and Remark 1.4), for a given $\delta \in (0, 1]$ and a given $R \in (0, 1]$, there exists a strictly stationary random sequence (X_k) such that (say) $EX_0 = 0$, $E|X_0|^{2+\delta} < \infty$, $E|X_0|^\gamma = \infty$ for all $\gamma > 2 + \delta$, $\alpha(n) \sim (\log \log n)^{-1}$ as $n \to \infty$, $\rho(n) = R$ for all $n \geq 1$, and $\rho^*(n) = 1$ for all $n \geq 1$. Thus, Theorem A(2) covers some strictly stationary sequences that are not covered by Theorem A(1), by the standard CLT's in [**11, 14, 16, 17**] alluded to above, or by an earlier version of Theorem A(2) in [**1**, Theorem 5] in which (1.11) was replaced by a more stringent restriction on $R := \lim_{n\to\infty} \rho(n)$.

Remark 1.7. For a given strictly stationary sequence $X := (X_k, k \in \mathbf{Z})$, one might consider defining for each positive integer n the dependence coefficient

$$\alpha^*(n) := \sup \alpha(\sigma(X_k, k \in S), \sigma(X_k, k \in T))$$

where the supremum is taken over all pairs of nonempty disjoint sets S and $T \subset \mathbf{Z}$ such that (1.4) holds. However, if X is mixing in the ergodic-theoretic sense, in particular, if X is strongly mixing, then for all $n \geq 1$,

$$4\alpha^*(n) \leq \rho^*(n) \leq 2\pi\alpha^*(n) \tag{1.12}$$

by (1.2) and [**5**, Theorem 1 and Remark 3]. Consequently, there is not much motivation to keep track of both of the mixing coefficients $\alpha^*(n)$ and $\rho^*(n)$.

Remark 1.8. In connection with Remark 1.7, let us mention some information provided by Igor Zhurbenko [**28**]. In the 1980's, some probabilists (faculty and students) at Moscow State University became aware of a possible connection, such as in [**5**], between "strong mixing" and "ρ-mixing" types of conditions for random fields. Apparently they never published anything on that. According to Zhurbenko [**28**], there appeared to be a little uncertainty about the statements or proofs. (The work in [**5**] was done independently.)

Remark 1.9. It was shown in [**4**, Theorem 5] that if (say) a strictly stationary nondegenerate random sequence $(X_k, k \in \mathbf{Z})$ is such that

$EX_0^2 I(|X_0| \le c)$ is slowly varying as $c \to \infty$, $EX_0 = 0$, $\rho^*(1) < 1$ and $\rho^*(n) \to 0$ as $n \to \infty$, then $a_n := (\pi/2)^{1/2} E|X_1 + \cdots + X_n| \to \infty$ as $n \to \infty$, and $(X_1 + \cdots + X_n)/a_n \to N(0,1)$ in distribution as $n \to \infty$. After seeing a preprint of that work, Magda Peligrad [**21**] pointed out that in that result the assumption $\rho^*(n) \to 0$ can be replaced by $\alpha(n) \to 0$. Her comment (and also Theorem A(1) in Remark 1.6) were based partly on (a preprint of) the work of Bryc and Smolenski [**8**]. The result in [**4**, Theorem 5], and Peligrad's [**21**] comment on it, were in fact in the broader context of random fields $(X_k, k \in \mathbf{Z}^d)$, with the dependence coefficients defined as in, e.g., [**4, 5**]. (Also, in that context, in [**4**, Theorem 5] the assumption $\rho^*(1) < 1$ can also be replaced by a weaker assumption, $\rho'(1) < 1$ in the terminology of [**7**].) For related results on random fields, see also Miller [**19**] and Perera [**24**].

Remark 1.10. Theorem 1.1 (and Remark 1.3) can be extended to include a particular class of m-dependent sequences. That is, under the extra restriction that $b_n = c_n$ for all $n \ge 1$, Theorem 1.1 still holds with the first inequality in (1.7) replaced by $0 \le 4a_n$, thus allowing $b_n = c_n = 0$ for some n. The proof of this extra (m-dependent) case is simply part of the argument in Sections 2 and 3 (essentially the construction in [**1**, Theorem 6])–using only Lemmas 2.1–2.5, not Lemmas 2.6–2.7, parts (1) and (2), not part (3), of Lemmas 3.1–3.2 (with $r_n = 0$ allowed in Lemma 3.2 (1)(2)), and the random sequences U and V, not W, in the final argument. The details are left to the reader.

2. Preliminaries. This section consists of seven lemmas. Lemmas 2.1–2.3 give useful basic facts. Then Lemmas 2.4–2.5 give (in a simplified form) a class of building blocks from the construction in [**1**, Theorem 6]. Finally, Lemmas 2.6–2.7 give (in an extended form) a class of building blocks from the construction in [**2**]. In Section 3, these two classes of building blocks will be put together to construct the random sequence for Theorem 1.1.

Lemma 2.1. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are σ-fields. Then*

$$\rho(\mathcal{A}, \mathcal{B}) = \sup \frac{\|E(f|\mathcal{B})\|_2}{\|f\|_2}$$

where this supremum is taken over all $\mathcal{A}$-measurable random variables f such that $Ef = 0$ and $Ef^2 < \infty$.

Here 0/0 is interpreted to be 0. This lemma is well known. Its proof is an elementary exercise, left to the reader.

Lemma 2.2. *Suppose $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots$ and $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \ldots$ are σ-fields, and the σ-fields $(\mathcal{A}_n \vee \mathcal{B}_n)$, $n = 1, 2, 3, \ldots$ are independent. Then*

$$\alpha\Big(\bigvee_{n=1}^{\infty} \mathcal{A}_n, \bigvee_{n=1}^{\infty} \mathcal{B}_n \Big) \le \sum_{n=1}^{\infty} \alpha(\mathcal{A}_n, \mathcal{B}_n), \tag{2.1}$$

and

$$\rho\Big(\bigvee_{n=1}^{\infty} \mathcal{A}_n, \bigvee_{n=1}^{\infty} \mathcal{B}_n \Big) = \sup_{n \ge 1} \rho(\mathcal{A}_n, \mathcal{B}_n). \tag{2.2}$$

Equation (2.1) is taken from [**1**, Lemma 8]. Equation (2.2) is due to Csaki and Fischer [**10**]; a short proof of it is given by Witsenhausen [**27**, Theorem 1].

Lemma 2.3. *Suppose X and Y are random variables, each of which takes only two values. Then*

$$\alpha(\sigma(X), \sigma(Y)) = |P(X = x, Y = y) - P(X = x) \cdot P(Y = y)| \tag{2.3}$$

where x, respectively y, is either one of the two values taken by the random variable X, respectively Y; and

$$\rho(\sigma(X), \sigma(Y)) = |\mathrm{Corr}\,(X, Y)|. \tag{2.4}$$

Proof. To prove (2.3), note that in the definition of $\alpha(\mathcal{A}, \mathcal{B})$ in (1.1), one can restrict to events A and B whose probabilities are neither 0 nor 1. Also, the number $|P(A \cap B) - P(A)P(B)|$ does not change if either A or B is replaced by its complement. Equation (2.3) follows.

Equation (2.4) follows from the fact that any function of the random variable X, respectively Y, can be expressed as an affine function of X, respectively Y.

Lemma 2.4. *Suppose* $0 < q < 1$ *and* $0 \le r \le 1$. *Suppose* V *and* W *are random variables which take only the values* 0 *and* 1, *and their joint distribution is as follows*:

$$P(V = 0, W = 0) = (1 - r)(1 - q)^2 + r(1 - q),$$
$$P(V = 0, W = 1) = (1 - r)q(1 - q),$$
$$P(V = 1, W = 0) = (1 - r)q(1 - q),$$

and

$$P(V = 1, W = 1) = (1 - r)q^2 + rq.$$

Then $\alpha(\sigma(V), \sigma(W)) = rq(1 - q)$ *and* $\rho(\sigma(V), \sigma(W)) = r$.

This follows from Lemma 2.3 and simple arithmetic.

In the proofs of Lemmas 2.5 and 2.7 below, the following notation will be used. If S is a subset of $\mathbf{Z}$ and j is any integer, then $S - j := \{k \in \mathbf{Z} : j + k \in S\}$.

Lemma 2.5. *Suppose* $0 < q < 1$ *and* $0 \le r \le 1$. *Then there exists a strictly stationary* 1*-dependent sequence* $X := (X_k, k \in \mathbf{Z})$ *of random variables such that*

$$\alpha(X, 1) = rq(1 - q), \tag{2.5}$$
$$\rho(X, 1) = r, \tag{2.6}$$

and

$$\rho^*(X, 1) = r. \tag{2.7}$$

Proof. Let $((V_{k,0}, V_{k,1}), k \in \mathbf{Z})$ be a sequence of independent, identically distributed random vectors such that each $V_{k,l}$ takes only the values 0 and 1, and for each $k \in \mathbf{Z}$ the joint distribution of $(V_{k,0}, V_{k,1})$ is the same as that of the random vector (V, W) in Lemma 2.4 (with the same q and r).

Define the (strictly stationary) random sequence $X := (X_k, k \in \mathbf{Z})$ as follows. For all $k \in \mathbf{Z}$,

$$X_k := 2V_{k,0} + V_{k-1,1}.$$

It is easy to see that, for each $k \in \mathbf{Z}$, $\sigma(X_k) = \sigma(V_{k,0}, V_{k-1,1})$. For any nonempty set $S \subset \mathbf{Z}$, one therefore has that

$$\sigma(X_k, k \in S) = \bigvee_{j \in \mathbf{Z}} \sigma(V_{j,l}, l \in (S-j) \cap \{0,1\}) \tag{2.8}$$

where, for a given $j \in \mathbf{Z}$, $\sigma(V_{j,l}, l \in (S-j) \cap \{0,1\})$ is interpreted to be $\{\Omega, \phi\}$ if the set $(S-j) \cap \{0,1\}$ is empty.

The sequence X is clearly 1-dependent. We just need to verify (2.5), (2.6) and (2.7).

Suppose S and T are any two nonempty disjoint subsets of $\mathbf{Z}$. By (2.8) and Lemma 2.2, Equation (2.2),

$$\begin{aligned} &\rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \\ &\qquad = \sup_{j \in \mathbf{Z}} [\rho(\sigma(V_{j,l}, l \in (S-j) \cap \{0,1\}), \\ &\qquad\qquad \sigma(V_{j,l}, l \in (T-j) \cap \{0,1\}))]. \end{aligned} \tag{2.9}$$

For a given $j \in \mathbf{Z}$, the sets $S-j$ and $T-j$ are disjoint, and hence the term in the brackets in the righthand side of (2.9) is either 0 (if $(S-j) \cap \{0,1\}$ or $(T-j) \cap \{0,1\}$ is empty) or r (if $S-j$ and $T-j$ each have one of the elements $0, 1$) by Lemma 2.4. It follows that the two sides of (2.9) are 0 or r. Hence,

$$\rho^*(X, 1) \le r. \tag{2.10}$$

Now consider the choice of sets $S = \{\dots, -2, -1, 0\}$ and $T = \{1, 2, 3, \dots\}$. From (2.9) we have that

$$\rho(\sigma(X_k, k \le 0), \sigma(X_k, k \ge 1)) = \rho(\sigma(V_{0,0}), \sigma(V_{0,1})) = r \tag{2.11}$$

by Lemma 2.4 again. Hence, $\rho(X,1) = r$, which is (2.6). Now by (2.10) and (1.5), one has that (2.7) holds as well.

By an argument similar to that of (2.11) but using (say) Lemma 2.2 (Equation (2.1)) and Lemma 2.4, one has that

$$\alpha(\sigma(X_k, k \leq 0), \sigma(X_k, k \geq 1)) = \alpha(\sigma(V_{0,0}), \sigma(V_{0,1})) = rq(1-q).$$

That is, (2.5) holds. This completes the proof of Lemma 2.5. □

Lemma 2.6. *Suppose* $\varepsilon > 0$ *and* $0 \leq r \leq 1$. *Then there exists a random vector* (X, Y, Z) *with the following properties*:

$$\rho(\sigma(X,Y), \sigma(Z)) \leq \varepsilon, \tag{2.12}$$

$$\rho(\sigma(X), \sigma(Y,Z)) \leq \varepsilon, \tag{2.13}$$

and

$$\rho(\sigma(X,Z), \sigma(Y)) = r. \tag{2.14}$$

Proof. Without loss of generality, we assume $0 < \varepsilon < 1$.

Let $((V_k, W_k), k = 1, 2, 3, \dots)$ be a sequence of independent, identically distributed random vectors, with each V_k and each W_k taking only the values 1 and -1, such that

$$P(V_1 = W_1 = 1) = P(V_1 = W_1 = -1) = (1+\varepsilon)/4 \tag{2.15}$$

and

$$P(V_1 = 1, W_1 = -1) = P(V_1 = -1, W_1 = 1) = (1-\varepsilon)/4.$$

Let T, T^* and U be independent random variables, with the random vector (T, T^*, U) being independent of $(V_k, W_k, k \geq 1)$, with T and T^* taking only the values 1 and -1 and U taking only the values 0 and 1, such that

$$P(T = 1) = P(T = -1) = P(T^* = 1) = P(T^* = -1) = 1/2 \tag{2.16}$$

and

$$P(U = 0) = 1 - r \quad \text{and} \quad P(U = 1) = r.$$

Define the random sequences

$$V := (V_1T, V_2T, V_3T, \dots) \tag{2.17}$$

and

$$W := (W_1, W_2, W_3, \dots).$$

(Note the role of the random variable T in the definition of the sequence V.)

Claim 1. *The random ordered quintuple* (V, W, T, T^*, U) *has the same distribution* (*on* $\{-1,1\}^{\mathbf{N}} \times \{-1,1\}^{\mathbf{N}} \times \{-1,1\} \times \{-1,1\} \times \{0,1\}$) *as the random ordered quintuple* (W, V, T, T^*, U).

Proof. The random vectors (V, W, T) and (T^*, U) are independent. Hence, it suffices to show that the random ordered triplets (V, W, T) and (W, V, T) have the same distribution.

From (2.15), it is easy to see that the distribution of the random vector (V_1, W_1) is "symmetric," in the sense that it is the same as the distribution of the random vector (W_1, V_1). In the same sense one also has that the distribution of the random vector $(-V_1, W_1)$ is symmetric. Hence, for each $t \in \{-1, 1\}$, conditional on the event $\{T = t\}$, one has that the random vectors (V_kT, W_k), $k = 1, 2, 3, \dots$ are independent, identically distributed and symmetric. Hence, for each $t \in \{-1, 1\}$, conditional on the event $\{T = t\}$, the random ordered pairs (V, W) and (W, V) have the same distribution on $\{-1,1\}^{\mathbf{N}} \times \{-1,1\}^{\mathbf{N}}$. Hence the random ordered triplets (V, W, T) and (W, V, T) have the same distribution. This completes the proof of Claim 1. □

Claim 2. *One has that*

$$\rho(\sigma(V), \sigma(W, T, T^*, U)) = \rho(\sigma(W), \sigma(V, T, T^*, U)) \le \varepsilon.$$

Proof. The first equality here is an elementary consequence of Claim 1. To prove the latter inequality ($\ldots \leq \varepsilon$), first note that, for each $k \geq 1$,

$$\rho(\sigma(V_k), \sigma(W_k)) = |\mathrm{Corr}\,(V_k, W_k)| = \varepsilon,$$

by Lemma 2.3 and a simple calculation, and then apply Lemma 2.2 (Equation (2.2)) to the pairs of σ-fields $(\sigma(W_k), \sigma(V_k))$, $k = 1, 2, 3, \ldots$, $(\{\Omega, \phi\}, \sigma(T, T^*, U))$. Thus, Claim 2 holds. □

Next, define the random variable Y by

$$Y := T \cdot I(U = 1) + T^* \cdot I(U = 0). \tag{2.18}$$

Claim 3. *$\rho(\sigma(V, W), \sigma(Y)) = r$.*

Proof. By (2.16) and simple arithmetic, the random variable Y takes just the two values 1 and -1, with probability 1/2 each. In particular, $EY = 0$. If f is a $\sigma(Y)$-measurable random variable with mean 0, then, by a trivial argument, $f = cY$ for some constant c. Hence, by Lemma 2.1,

$$\rho(\sigma(V, W), \sigma(Y)) = \frac{\|E(Y|\sigma(V, W))\|_2}{\|Y\|_2}.$$

Clearly $\|Y\|_2 = 1$. Hence, to prove Claim 3, it suffices to show that

$$\|E(Y|\sigma(V, W))\|_2 = r. \tag{2.19}$$

By a simple calculation, $EV_1W_1 = \varepsilon$. By the strong law of large numbers, $n^{-1}\sum_{k=1}^{n} V_kW_k \to \varepsilon$ almost s. as $n \to \infty$. Hence, $n^{-1}\sum_{k=1}^{n}(V_kT)W_k \to \varepsilon T$ almost s. as $n \to \infty$. Since $\varepsilon > 0$, it follows that T is equal almost s. to a random variable which is $\sigma(V, W)$-measurable. Since (T^*, U) is independent of (V, W, T), we have that

$$E(Y|\sigma(V, W)) = rT \quad \text{a.s.}$$

by (2.18) and a simple calculation. Since $\|T\|_2 = 1$, by (2.16), Equation (2.19) follows. This completes the proof of Claim 3. □

Now let us complete the proof of Lemma 2.6. Let $\phi : \{-1,1\}^{\mathbf{N}} \to \mathbf{R}$ be a bimeasurable isomorphism. (The existence of such a function ϕ is well known.) Define the random variables X and Z by $X := \phi(V)$ and $Z := \phi(W)$. Equations (2.12) and (2.13) hold by (2.18) and Claim 2. Equation (2.14) holds by Claim 3. This completes the proof of Lemma 2.6. □

Lemma 2.7. *Suppose* $0 < \varepsilon \le r \le 1$. *Then there exists a strictly stationary* 2*-dependent sequence* $X := (X_k, k \in \mathbf{Z})$ *of random variables such that*

$$\rho(X,1) \le \varepsilon, \tag{2.20}$$

$$\rho^*(X,1) = r, \tag{2.21}$$

and

$$\rho^*(X,2) \le \varepsilon. \tag{2.22}$$

Proof. Applying Lemma 2.6, let $((W_{k,0}, W_{k,1}, W_{k,2}), k \in \mathbf{Z})$ be a sequence of independent, identically distributed random vectors such that, for each $k \in \mathbf{Z}$,

$$\rho(\sigma(W_{k,0}, W_{k,1}), \sigma(W_{k,2})) \le \varepsilon, \tag{2.23}$$

$$\rho(\sigma(W_{k,0}), \sigma(W_{k,1}, W_{k,2})) \le \varepsilon, \tag{2.24}$$

and

$$\rho(\sigma(W_{k,0}, W_{k,2}), \sigma(W_{k,1})) = r. \tag{2.25}$$

Let $\lambda : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a bimeasurable isomorphism. Define the random sequence $X := (X_k, k \in \mathbf{Z})$ as follows. For all $k \in \mathbf{Z}$,

$$X_k := \lambda(W_{k,0}, W_{k-1,1}, W_{k-2,2}). \tag{2.26}$$

Clearly the random sequence X is strictly stationary and 2-dependent. Our task is to verify (2.20), (2.21) and (2.22).

First, note that, for any nonempty set $S \subset \mathbf{Z}$, one has that

$$\sigma(X_k, k \in S) = \bigvee_{j \in \mathbf{Z}} \sigma(W_{j,l}, l \in (S-j) \cap \{0,1,2\}),$$

where $\sigma(W_{j,l}, l \in (S-j)\cap\{0,1,2\}) := \{\Omega, \phi\}$ if the set $(S-j)\cap\{0,1,2\}$ is empty.

Hence, for any two nonempty disjoint sets $S, T \subset \mathbf{Z}$, by Lemma 2.2 one has that

$$\begin{aligned} \rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \\ = \sup_{j \in \mathbf{Z}} \rho(\sigma(W_{j,l}, l \in (S-j) \cap \{0,1,2\}), \\ \sigma(W_{j,l}, l \in (T-j) \cap \{0,1,2\})). \end{aligned} \tag{2.27}$$

Proof of (2.20). Let $S := \{\dots, -2, -1, 0\}$ and $T := \{1, 2, 3, \dots\}$. Then (2.27) reduces to the equation

$$\begin{aligned} \rho(X,1) = \max\{\rho(\sigma(W_{-1,0}, W_{-1,1}), \sigma(W_{-1,2})), \\ \rho(\sigma(W_{0,0}), \sigma(W_{0,1}, W_{0,2}))\}. \end{aligned}$$

The righthand side is $\le \varepsilon$ by (2.23) and (2.24). Thus, (2.20) holds. □

Proof of (2.21). First note that

$$\begin{aligned} \rho^*(X,1) &\ge \rho(\sigma(X_{-1}, X_1), \sigma(X_0)) \\ &\ge \rho(\sigma(W_{-1,0}, W_{-1,2}), \sigma(W_{-1,1})) \\ &= r, \end{aligned}$$

by (2.25). Now we need to prove that $\rho^*(X,1) \le r$. Let S and T be arbitrary fixed nonempty disjoint subsets of $\mathbf{Z}$. To complete the proof of (2.21), it suffices to prove that $\rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \le r$. By (2.27), it suffices to prove that, for each $j \in \mathbf{Z}$,

$$\rho(\sigma(W_{j,l}, l \in (S-j) \cap \{0,1,2\}), \sigma(W_{j,l}, l \in (T-j) \cap \{0,1,2\})) \le r. \tag{2.28}$$

Suppose $j \in \mathbf{Z}$. The sets $(S-j) \cap \{0,1,2\}$ and $(T-J) \cap \{0,1,2\}$ are (possibly empty) disjoint subsets of $\{0,1,2\}$. Hence, by (2.23), (2.24), (2.25) and the hypothesis $\varepsilon \le r$ (in the statement of Lemma 2.7), Equation (2.28) holds. This completes the proof of (2.21). □

Proof of (2.22). Let S and T be any two nonempty disjoint subsets of $\mathbf{Z}$ such that $\operatorname{dist}(S,T) \ge 2$. It suffices to prove that $\rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \le \varepsilon$. By (2.27), it suffices to prove that, for each $j \in \mathbf{Z}$,

(2.29)
$$\rho(\sigma(W_{j,l}, l \in (S-j) \cap \{0,1,2\}), \sigma(W_{j,l}, l \in (T-j) \cap \{0,1,2\})) \le \varepsilon.$$

Suppose $j \in \mathbf{Z}$. Now $\operatorname{dist}(S-j, T-j) \ge 2$. Hence, either one of the two sets $(S-j) \cap \{0,1,2\}$ or $(T-j) \cap \{0,1,2\}$ is empty; or else these two sets are (in either order) $\{0\}$ and $\{2\}$, and (2.29) reduces to the equation $\rho(\sigma(W_{j,0}), \sigma(W_{j,2})) \le \varepsilon$. This last equation holds by (2.23) (or (2.24)). This completes the proof of (2.22), and of Lemma 2.7. □

3. Proof of Theorem 1.1. Starting with the random sequences described in Lemmas 2.5 and 2.7 as building blocks, we shall proceed to the construction of the random sequence for Theorem 1.1. This construction will require two further intermediate stages, given in Lemmas 3.1 and 3.2 below.

Lemma 3.1. *Suppose n is a positive integer. Then the following three statements hold.*

(1) *For any $r \in (0,1]$, there exists a strictly stationary n-dependent random sequence $X := (X_k, k \in \mathbf{Z})$ such that*

$$\rho^*(X,1) = \rho(X,n) = 4\alpha(X,n) = r. \tag{3.1}$$

(2) *For any $r \in (0,1]$ and any $\varepsilon > 0$, there exists a strictly stationary n-dependent random sequence $X := (X_k, k \in \mathbf{Z})$ such that*

$$\rho^*(X,1) = \rho(X,n) = r \tag{3.2}$$

and

$$\alpha(X,1) \le \varepsilon. \tag{3.3}$$

(3) *For any* $r \in (0,1]$ *and any* $\varepsilon > 0$, *there exists a strictly stationary* $(2n)$*-dependent random sequence* $X := (X_k, k \in \mathbf{Z})$ *such that*

$$\rho^*(X,1) = \rho^*(X,n) = r, \tag{3.4}$$

$$\rho^*(X,n+1) \le \varepsilon, \tag{3.5}$$

and

$$\rho(X,1) \le \varepsilon. \tag{3.6}$$

Proof. The proofs of the three statements are similar. We shall first give the proof of statement (3), and then indicate the changes for the proofs of (2) and (1), in that order. □

Proof of (3). Applying Lemma 2.7, let $W := (W_k, k \in \mathbf{Z})$ be a strictly stationary 2-dependent random sequence such that

$$\rho^*(W,1) = r, \tag{3.7}$$

$$\rho^*(W,2) \le \varepsilon, \tag{3.8}$$

and

$$\rho(W,1) \le \varepsilon. \tag{3.9}$$

Let $X := (X_k, k \in \mathbf{Z})$ be a (strictly stationary) random sequence with the following two properties: (a) For each $l = 0, 1, \dots, n-1$, the random sequence $X^{(l)} := (X_{l+nk}, k \in \mathbf{Z})$ has the same distribution (on $\mathbf{R}^{\mathbf{Z}}$) as the sequence W. (b) These sequences $X^{(0)}, X^{(1)}, \dots, X^{(n-1)}$ are independent of each other. (Of course, (b) is vacuous if $n = 1$.)

If S and T are any two nonempty disjoint subsets of $\mathbf{Z}$, then, by (3.7) and Lemma 2.2,

$$\begin{aligned} &\rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \\ &\qquad = \max_{0 \le l \le n-1} \rho(\sigma(X_k, k \in S, k \equiv l \bmod n), \\ &\qquad\qquad \sigma(X_k, k \in T, k \equiv l \bmod n)) \\ &\qquad \le \rho^*(W,1) = r. \end{aligned} \tag{3.10}$$

Hence $\rho^*(X,1) \le r$. On the other hand, $\rho^*(X,n) \ge \rho^*(W,1) = r$ (by (3.7) again). From these two facts, (3.4) follows.

By similar arguments, using (3.8) and (3.9) and Lemma 2.2, one has that $\rho^*(X,n+1) = \rho^*(W,2) \le \varepsilon$ and $\rho(X,1) = \rho(W,1) \le \varepsilon$. Thus (3.5) and (3.6) hold, and the proof of statement (3) is complete. □

Proof of (2). Proceed as in the proof of statement (3), but with $W := (W_k, k \in \mathbf{Z})$ chosen via Lemma 2.5 (with q very small there) to be a strictly stationary 1-dependent random sequence satisfying

$$\rho^*(W,1) = \rho(W,1) = r \tag{3.11}$$

and

$$\alpha(W,1) \le \varepsilon/n \tag{3.12}$$

(instead of (3.7), (3.8) and (3.9)). Using (3.11) and arguing as in (3.10), we get $\rho^*(X,1) \le r$; and, on the other hand, $\rho(X,n) \ge \rho(W,1) = r$. Equation (3.2) follows. Also, by (3.12) and Lemma 2.2, $\alpha(X,1) \le n \cdot \alpha(W,1) \le \varepsilon$. Thus, (3.3) holds, and statement (2) is proved. □

Proof of (1). Proceed as in the proof of statement (3), but with $W := (W_k, k \in \mathbf{Z})$ chosen via Lemma 2.5 (with $q = 1/2$ there) to be a strictly stationary 1-dependent random sequence satisfying

$$\rho^*(W,1) = \rho(W,1) = 4\alpha(W,1) = r. \tag{3.13}$$

Then, as in the proof of statement (2), one has that $\rho^*(X,1) = \rho(X,n) = r$. Also, $\alpha(X,n) \ge \alpha(W,1) = r/4$ by (3.13). Hence (3.1) holds by (1.5) (or (1.2)). Thus, statement (1) holds, and the proof of Lemma 3.1 is complete. □

Lemma 3.2. *Suppose $r_1, r_2, r_3, \ldots$ is a nonincreasing sequence of numbers in $(0,1]$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ is a sequence of positive numbers. Then the following three statements hold:*

(1) *There exists a strictly stationary random sequence $X := (X_k, k \in \mathbf{Z})$ such that, for each $n \ge 1$,*

$$\rho^*(X,n) = \rho(X,n) = 4\alpha(X,n) = r_n. \tag{3.14}$$

(2) *There exists a strictly stationary random sequence* $X := (X_k, k \in \mathbf{Z})$ *such that, for each* $n \geq 1$,

$$\rho^*(X,n) = \rho(X,n) = r_n \tag{3.15}$$

and

$$\alpha(X,n) \leq \varepsilon_n. \tag{3.16}$$

(3) *There exists a strictly stationary random sequence* $X := (X_k, k \in \mathbf{Z})$ *such that, for each* $n \geq 1$,

$$\rho^*(X,n) = r_n \tag{3.17}$$

and

$$\rho(X,n) \leq \varepsilon_n. \tag{3.18}$$

Proof. Again, the proofs of the three statements are similar. We shall first give the proof of statement (3) and then indicate the changes for the proofs of (1) and (2). □

Proof of (3). Applying Lemma 3.1(3), for each positive integer j, let $X^{(j)} := (X_k^{(j)}, k \in \mathbf{Z})$ be a strictly stationary $(2j)$-dependent random sequence such that

$$\rho^*(X^{(j)},1) = \rho^*(X^{(j)},j) = r_j, \tag{3.19}$$

$$\rho^*(X^{(j)},j+1) \leq r_{2j}, \tag{3.20}$$

and

$$\rho(X^{(j)},1) \leq \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2j}\}. \tag{3.21}$$

Further, let these sequences $X^{(1)}, X^{(2)}, X^{(3)}, \ldots$ be independent of each other.

Let $\phi : \mathbf{R}^{\mathbf{N}} \to \mathbf{R}$ be a bimeasurable isomorphism. Define the (strictly stationary) random sequence $X := (X_k, k \in \mathbf{Z})$ as follows. For all $k \in \mathbf{Z}$,

$$X_k := \phi(X_k^{(1)}, X_k^{(2)}, X_k^{(3)}, \dots).$$

Of course, for each $k \in \mathbf{Z}$, $\sigma(X_k) = \sigma(X_k^{(1)}, X_k^{(2)}, X_k^{(3)}, \dots)$.

Suppose n is a positive integer. By Lemma 2.2,

$$\rho^*(X, n) = \sup_{j \geq 1} \rho^*(X^{(j)}, n). \tag{3.22}$$

For any j such that $1 \leq j < n/2$, $\rho^*(X^{(j)}, n) = 0$, since $X^{(j)}$ is $(2j)$-dependent. For any j such that $n/2 \leq j < n$,

$$\rho^*(X^{(j)}, n) \leq \rho^*(X^{(j)}, j+1) \leq r_{2j} \leq r_n$$

by (3.20) (and the assumption that the sequence $r_1, r_2, r_3, \dots$ is non-increasing). Also, by (3.19), $\rho^*(X^{(n)}, n) = r_n$; and, for each $j \geq n+1$, $\rho^*(X^{(j)}, n) = r_j \leq r_n$. Hence, by (3.22), Equation (3.17) holds.

Similarly, by Lemma 2.2,

$$\rho(X, n) = \sup_{j \geq 1} \rho(X^{(j)}, n). \tag{3.23}$$

For any j such that $1 \leq j < n/2$, $\rho(X^{(j)}, n) = 0$. For any $j \geq n/2$, $\rho(X^{(j)}, n) \leq \varepsilon_n$ by (3.21). Hence, (3.18) holds by (3.23). This completes the proof of statement (3). □

Proof of (1). Proceed as in the proof of statement (3), but (applying Lemma 3.1(1)) with $X^{(j)} := (X_k^{(j)}, k \in \mathbf{Z})$ being a strictly stationary j-dependent random sequence such that

$$\rho^*(X^{(j)}, 1) = \rho(X^{(j)}, j) = 4\alpha(X^{(j)}, j) = r_j$$

(instead of (3.19), (3.20) and (3.21)). Then, by (1.5) and an argument somewhat similar to the proof of statement (3), one obtains, for each $n \geq 1$,

$$4\alpha(X, n) \leq \rho(X, n) = \rho^*(X, n) = r_n$$

as well as $\alpha(X, n) \geq \alpha(X^{(n)}, n) = r_n/4$. Equation (3.14) follows, and thus statement (1) holds. □

Proof of (2). Proceed as in the proof of statement (3), but (applying Lemma 3.1(2)) with $X^{(j)} := (X_k^{(j)}, k \in \mathbf{Z})$ being a strictly stationary j-dependent random sequence such that

$$\rho^*(X^{(j)}, 1) = \rho(X^{(j)}, j) = r_j$$

and

$$\alpha(X^{(j)}, 1) \le 2^{-j} \cdot \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_j\}$$

(instead of (3.19), (3.20) and (3.21)). As in the proof of statement (1), one obtains Equation (3.15) for each $n \ge 1$. Also, by Lemma 2.2, for each $n \ge 1$,

$$\begin{aligned}\alpha(X, n) &\le \sum_{j=1}^{\infty} \alpha(X^{(j)}, n) = 0 + \sum_{j=n}^{\infty} \alpha(X^{(j)}, n)\\ &\le \sum_{j=n}^{\infty} 2^{-j}\varepsilon_n\\ &\le \varepsilon_n.\end{aligned}$$

Thus, (3.16) holds. This completes the proof of statement (2), and of Lemma 3.2. □

Proof of Theorem 1.1. Let the sequences $(a_n), (b_n), (c_n)$ and (d_n) of numbers be as in the statement of Theorem 1.1. Without loss of generality, assume that, for each $n \ge 1$,

$$d_n \le a_n. \tag{3.24}$$

Applying Lemma 3.2(1), let $U := (U_k, k \in \mathbf{Z})$ be a strictly stationary random sequence such that, for each $n \ge 1$,

$$\rho^*(U, n) = \rho(U, n) = 4\alpha(U, n) = 4a_n. \tag{3.25}$$

Applying Lemma 3.2(2), let $V := (V_k, k \in \mathbf{Z})$ be a strictly stationary random sequence such that, for each $n \ge 1$,

$$\rho^*(V, n) = \rho(V, n) = b_n \tag{3.26}$$

and

$$\alpha(V,n) \le d_n/2. \tag{3.27}$$

Applying Lemma 3.2(3), let $W := (W_k, k \in \mathbf{Z})$ be a strictly stationary random sequence such that, for each $n \ge 1$,

$$\rho^*(W,n) = c_n \tag{3.28}$$

and

$$\rho(W,n) \le d_n. \tag{3.29}$$

Further, let these random sequences U, V and W be independent of each other.

Let $\phi : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a bimeasurable isomorphism. Define the (strictly stationary) random sequence $X := (X_k, k \in \mathbf{Z})$ as follows. For all $k \in \mathbf{Z}$,

$$X_k = \phi(U_k, V_k, W_k).$$

Of course, for each $k \in \mathbf{Z}$, $\sigma(X_k) = \sigma(U_k, V_k, W_k)$.

By (3.25), Lemma 2.2, (3.27), (3.29) and (1.5), for each $n \ge 1$,

$$\begin{aligned} a_n = \alpha(U,n) &\le \alpha(X,n) \\ &\le \alpha(U,n) + \alpha(V,n) + \alpha(W,n) \\ &\le a_n + d_n/2 + d_n/4 \\ &< a_n + d_n. \end{aligned}$$

Thus, (1.8) holds.

Similarly, by Lemma 2.2, (3.24), (3.25), (3.26), (3.28), (3.29) and (1.7), for each $n \ge 1$,

$$\rho(X,n) = \max\{\rho(U,n), \rho(V,n), \rho(W,n)\} = b_n$$

and

$$\rho^*(X,n) = \max\{\rho^*(U,n), \rho^*(V,n), \rho^*(W,n)\} = c_n.$$

Thus, (1.9) and (1.10) hold. This completes the proof of Theorem 1.1. □

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