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CONSTRUCTION OF INDECOMPOSABLE HERONIAN TRIANGLES

PAUL YIU

ABSTRACT. We give a simple characterization in terms of the tangents of the half-angles of a primitive Heronian triangle for the triangle to be decomposable into two Pythagorean triangles. This characterization leads to easy constructions of Heronian triangles not so decomposable.

1. Introduction. The ancient formula attributed to Heron of Alexandria on the area of a triangle in terms of the lengths of the sides a, b, c, and the semi-perimeter $s = (a+b+c)/2$,

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}, \tag{1}$$

naturally suggests the problem of constructing triangles with integer sides and integer areas. We shall call such triangles *Heronian*, and denote one such triangle with sides a, b, c and area Δ by $(a, b, c; \Delta)$. A common construction of Heronian triangles is to juxtapose two Pythagorean triangles (right triangles with integer sides) along a common leg. See, for example, Dickson [**5**, Chapter 5] and Thébault [**8**]. Indeed, a triangle is Heronian if and only if it is the juxtaposition of two Pythagorean triangles along a common leg, or a reduction of such a juxtaposition (Carlson [**2**], with correction in Singmaster [**7**]). Cheney [**4**] has given a construction of Heronian triangles in terms of positive rational numbers t_1, t_2, t_3, satisfying

$$t_1t_2 + t_2t_3 + t_3t_1 = 1. \tag{2}$$

Here t_i, $i = 1, 2, 3$, are the tangents of the half-angles of the triangles, and any two of them determine the third. Lehmer [**6**] had noted that a triangle with rational sides has rational area if and only if the tangents of its half-angles are all rational. Such triangles are called *rational*. The similarity class of a rational triangle contains triangles with integer sides. Such triangles are necessarily Heronian, see Proposition 1 below.

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Cheney observed that not every Heronian triangle can be decomposed into two Pythagorean triangles with a common side. In a footnote in [**4**], he gave the example (25, 34, 39; 420). The heights of this triangle are

$$\frac{168}{5}, \quad \frac{420}{17} \quad \text{and} \quad \frac{280}{13}.$$

Cheney obviously wrote the paper [**4**] with the problem of finding *indecomposable* Heronian triangles in mind. After making a list of Heronian triangles at the end of the paper, he noted another example of indecomposable Heronian triangles, namely, (39, 58, 95; 456). These examples, however, are not the smallest, which happens to be (5, 29, 30; 72), or (15, 34, 35; 252), if one insists on an acute triangle. In this paper we give a simple solution of Cheney's problem by studying the "triple of simplifying factors" for the tangents of the half-angles, Theorem 11. These "simplifying factors" are integers g_1, g_2, g_3 arising from the calculation of each of the numbers t_1, t_2, t_3 in terms of the other two.

2. Rational and Heronian triangles. We begin with the following basic result about rational triangles, a proof of which can be found in Bachman [**1**] or Cheney [**4**].

Proposition 1. *A rational triangle with integer sides in Heronian.*

Consider a triangle Γ with angles $\alpha_1, \alpha_2, \alpha_3$. Let $t_i := \tan(\alpha_i/2)$, $i = 1, 2, 3$. Note that these are positive. The relation (2) follows from the simple fact

$$\frac{\alpha_1}{2} + \frac{\alpha_2}{2} + \frac{\alpha_3}{2} = \frac{\pi}{2}.$$

Any two positive numbers t_1, t_2 determine a third positive number t_3 so that t_1, t_2, t_3 are the tangents of the half-angles of a triangle if and only if $t_1 t_2 < 1$.

The similarity class of Γ contains a Heronian triangle if and only if t_1, t_2, t_3 are all rational. Indeed, putting $t_i = n_i/d_i$, $i = 1, 2, 3$, with $\gcd(n_i, d_i) = 1$, we rewrite (2) in the form

$$n_1 n_2 d_3 + n_1 d_2 n_3 + d_1 n_2 n_3 = d_1 d_2 d_3. \tag{3}$$

Convention. Unless explicitly stated otherwise, whenever the three indices i, j, k appear altogether in an expression or an equation, they are taken as a *permutation* of the indices 1, 2, 3.

In the process of expressing $t_i = n_i/d_i$ in terms of $t_j = n_j/d_j$ and $t_k = n_k/d_k$, we encounter certain "simplifying factors," namely

$$g_i := \gcd(d_jd_k - n_jn_k, n_jd_k + d_jn_k),$$

so that

$$\begin{aligned} g_in_i &= d_jd_k - n_jn_k, \\ g_id_i &= d_jn_k + n_jd_k. \end{aligned} \tag{4}$$

We shall call (g_1, g_2, g_3) the *triple of simplifying factors* for the numbers (t_1, t_2, t_3), or of the similarity class of triangles they define.

Examples. For the decomposable Heronian triangle (13, 14, 15; 84), we have $t_1 = 1/2$, $t_2 = 4/7$ and $t_3 = 2/3$. From

$$\frac{1 - t_2t_3}{t_2 + t_3} = \frac{7 \cdot 3 - 4 \cdot 2}{7 \cdot 2 + 4 \cdot 3} = \frac{13}{26} = \frac{1}{2},$$

it follows that $g_1 = 13$. Similarly, $g_2 = 1$ and $g_3 = 5$. On the other hand, for the indecomposable Heronian triangle (25, 34, 39; 420), we have $(t_1, t_2, t_3) = (5/14, 4/7, 6/7)$. The simplifying factors are $(g_1, g_2, g_3) = (5, 17, 13)$. Also, for the smallest indecomposable Heronian triangle (15, 34, 35; 252), the simplifying factors are $(g_1, g_2, g_3) = (5, 17, 5)$.

In Theorem 15 (and Proposition 12) below, it is established that a triple of positive integers (g_1, g_2, g_3) determines a finite number of similarity classes of Heronian triangles provided it satisfies some mild conditions. The "smallest" Heronian triangle in one such class is decomposable into two Pythagorean components if and only if each of g_1, g_2, g_3 contains an odd prime divisor (Theorem 13). This criterion leads to simple constructions of indecomposable Heronian triangles.

Here is a Heronian triangle with angles $\alpha_i = 2\arctan t_i$, $i = 1, 2, 3$:

$$a_i = n_i(d_jn_k + n_jd_k) = g_in_id_i. \tag{5}$$

We shall denote this triangle by $\Gamma = \Gamma(t_1, t_2, t_3)$. It is clear that the semi-perimeter is $s = n_1n_2d_3 + n_1d_2n_3 + d_1n_2n_3 = d_1d_2d_3$ by (3) and $s - a_i = d_in_jn_k$, so that the area is

$$\Delta = n_1d_1n_2d_2n_3d_3. \tag{6}$$

Dividing the sides of Γ by $g := \gcd(a_1, a_2, a_3)$, we obtain the *primitive* Heronian triangle $\Gamma_0 := \Gamma_0(t_1, t_2, t_3)$ in its similarity class. Every primitive Heronian triangle, i.e., one in which the sides do not admit common divisors, arises in this way.

3. Decomposition of Heronian triangles. A Heronian triangle $\Gamma := (a_1, a_2, a_3; \Delta)$ is said to be *decomposable* if there are (nondegenerate) Pythagorean triangles $\Gamma_1 := (x_1, y, a_1; \Delta_1)$, $\Gamma_2 := (x_2, y, a_2; \Delta_2)$ and $\varepsilon = \pm 1$ such that

$$a_3 = \varepsilon x_1 + x_2, \qquad \Delta = \varepsilon \Delta_1 + \Delta_2.$$

According to whether $\varepsilon = 1$ or -1, we shall say that Γ is obtained by juxtaposing Γ_1 and Γ_2, $\Gamma = \Gamma_1 \cup \Gamma_2$, or by excising Γ_1 from Γ_2, $\Gamma = \Gamma_2 \backslash \Gamma_1$.

In general a Heronian triangle is decomposable into two Pythagorean components if and only if it has at least one integer height.

Theorem 2. *A primitive Heronian triangle can be decomposed into two Pythagorean components in at most one way.*

We make this more precise in the following three propositions.

Proposition 3. *A primitive Pythagorean triangle is indecomposable.*

Proof. We prove this by contradiction. A Pythagorean triangle, if decomposable, is partitioned by the altitude on the hypotenuse into two similar but *smaller* Pythagorean triangles. None of these, however, can have all sides of integer length by the primitivity assumption on the original triangle. □

Proposition 4. *A primitive, isosceles, Heronian triangle is decomposable, the only decomposition being into two congruent Pythagorean triangles.*

Proof. The triangle being isosceles and Heronian, the perimeter and hence the base must be even by Proposition 1. Each half of the isosceles triangle is a (primitive) Pythagorean triangle, $(m^2-n^2, 2mn, m^2+n^2)$, with m, n relatively prime, and of different parity. The height on each slant side of the isosceles triangle is

$$\frac{2mn(m^2-n^2)}{m^2+n^2},$$

which clearly cannot be an integer. This shows that the only way of decomposing a primitive isosceles triangle is into two congruent Pythagorean triangles. □

Proposition 5. *If a non-Pythagorean Heronian triangle has two integer heights, then it cannot be primitive.*

Proof. Let $(a, b, c; \Delta)$ be a Heronian triangle, not containing any right angle. Suppose the heights on the sides b and c are integers.

Clearly, b and c cannot be relatively prime, for otherwise, the heights of the triangle on these sides are, respectively, ch and bh, for some integer h. This is impossible since the triangle not containing any right angle, the height on b must be less than c.

Suppose, therefore, $\gcd(b, c) = g > 1$. We write $b = b'g$ and $c = c'g$ for relatively prime integers b' and c'. If the height on c is h, then that on the side b is $ch/b = c'h/b'$. If this is also an integer, then h must be divisible by b'. Replacing h by $b'h$, we may now assume that the heights on b and c are respectively $c'h$ and $b'h$. The side c is divided into $b'k$ and $\pm(c - b'k) \neq 0$, where $g^2 = h^2 + k^2$. It follows that

$$\begin{aligned} a^2 &= (b'h)^2 + (c'g - b'k)^2 \\ &= b'^2(h^2+k^2) + c'^2g^2 - 2b'c'gk \\ &= g[g(b'^2+c'^2) - 2b'c'k]. \end{aligned}$$

From this, it follows that g divides a^2, and every prime divisor of g is a common divisor of a, b and c. The Heronian triangle cannot be primitive. □

4. Gaussian integers. We shall associate with each positive rational number $t = n/d$, n, d relatively prime, the primitive, positive Gaussian integer $z(t) := d + n\sqrt{-1} \in \mathbf{Z}[\sqrt{-1}]$. Here we say that a Gaussian integer $x + y\sqrt{-1}$ is

- *primitive* if x and y are relatively prime, and
- *positive* if both x and y are positive.

The norm of the Gaussian integer $z = x + y\sqrt{-1}$ is the integer $N(z) := x^2 + y^2$. The norm in $\mathbf{Z}[\sqrt{-1}]$ is *multiplicative*:

$$N(z_1 z_2) = N(z_1)N(z_2).$$

The *argument* of a Gaussian integer $z = x + y\sqrt{-1}$ is the unique real number $\phi = \phi(z) \in [0, 2\pi)$ defined by

$$\cos\phi = \frac{x}{\sqrt{x^2+y^2}}, \qquad \sin\phi = \frac{y}{\sqrt{x^2+y^2}}.$$

A Gaussian integer z is positive if and only if $0 < \theta(z) < \pi/2$. Each positive Gaussian integer $z = x + y\sqrt{-1}$ has a *complement*

$$z^* := y + x\sqrt{-1} = \sqrt{-1} \cdot \bar{z},$$

where $\bar{z} := x - y\sqrt{-1}$ is the conjugate of z. Note that $N(z^*) = N(z)$, and

$$\phi(z) + \phi(z^*) = \frac{\pi}{2}, \tag{7}$$

for each pair of complementary positive Gaussian integers.

Recall that the units of $\mathbf{Z}[\sqrt{-1}]$ are precisely ± 1 and $\pm\sqrt{-1}$. An odd (rational) prime number p ramifies into two nonassociate primes $\pi(p)$ and $\overline{\pi(p)}$ in $\mathbf{Z}[\sqrt{-1}]$, namely, $p = \pi(p)\overline{\pi(p)}$, if and only if $p \equiv 1 \pmod 4$. For applications in the present paper, we formulate the unique factorization theorem in $\mathbf{Z}[\sqrt{-1}]$ as follows.

Theorem 6. *Let $g > 1$ be an odd number. There is a primitive Gaussian integer θ satisfying $N(\theta) = g$ if and only if each prime divisor of g is congruent to* $1 \pmod 4$.

5. Heronian triangles and Gaussian integers. Consider the Heronian triangle $\Gamma := \Gamma(t_1, t_2, t_3)$ with sides given by (5). In terms of the Gaussian integers $z_i := z(t_i) = d_i + n_i\sqrt{-1}$, the relations (4) can be rewritten as

$$g_i z_i = \sqrt{-1} \cdot \overline{z_j z_k} = (z_j z_k)^*. \tag{8}$$

Lemma 7. $N(z_i) = g_j g_k$.

Proof. From the relation (8), we have

$$g_i^2 N(z_i) = N(z_j) N(z_k),$$

and the result follows easily. □

Proposition 8. (a) g_i *is a common divisor of* $N(z_j)$ *and* $N(z_k)$.

(b) *At least two of* g_i, g_j, g_k *exceed* 1.

(c) g_i *is even if and only if all* n_j, d_j, n_k *and* d_k *are odd.*

(d) *At most one of* g_i, g_j, g_k *is even, and none of them is divisible by* 4.

(e) g_i *is prime to each of* n_j, d_j, n_k *and* d_k.

(f) *Each odd prime divisor of* g_i, $i = 1, 2, 3$, *is congruent to* 1 (mod 4).

Proof. (a) follows easily from Lemma 7.

(b) Suppose $g_1 = g_2 = 1$. Then $N(z_3) = 1$, which is clearly impossible.

(c) is clear from the relation (4).

(d) Suppose g_i is even. Then n_j, d_j, n_k, d_k are all odd. This means that g_i, being a divisor of $N(z_j) = d_j^2 + n_j^2 \equiv 2 \pmod 4$, is not divisible by 4. Also, $d_j d_k - n_j n_k$ and $n_j d_k + d_j n_k$ are both even, and

$$\begin{aligned}(d_j d_k - n_j n_k) + (n_j d_k + d_j n_k) &= (d_j + n_j)(d_k + n_k) - 2n_j n_k \\ &\equiv 2 \pmod 4,\end{aligned}$$

it follows that one of them is divisible by 4, and the other is 2 (mod 4). After cancelling the common divisor 2, we see that exactly one of n_i and d_i is odd. This means, by (c), that g_j and g_k cannot be odd.

(e) If g_i and n_j admit a common prime divisor p, then p divides both n_j and $n_j^2+d_j^2$, and hence d_j as well, contradicting the assumption that $d_j + n_j\sqrt{-1}$ be primitive.

(f) is a consequence of Theorem 6. □

Proposition 9. $\gcd(g_1, g_2, g_3) = 1$.

Proof. We shall derive a contradiction by assuming a common rational prime divisor $p \equiv 1 \pmod 4$ of g_i, g_j, g_k with *positive* exponents r_i, r_j, r_k in their prime factorizations. By the relation (8), the product $z_j z_k$ is divisible by the *rational* prime power p^{r_i}. This means that the primitive Gaussian integers z_j and z_k should contain in their prime factorizations powers of the distinct primes $\pi(p)$ and $\overline{\pi(p)}$. The same reasoning also applies to each of the pairs (z_k, z_i) and (z_i, z_j) so that z_k and z_i, respectively z_i and z_j, each contains one of the nonassociate Gaussian primes $\pi(p)$ and $\overline{\pi(p)}$ in their factorizations. But then this means that z_j and z_k are divisible by the *same* Gaussian prime, a contradiction. □

Corollary 10. $\gcd(a_1, a_2, a_3) = \gcd(n_1d_1, n_2d_2, n_3d_3)$.

Proof. This follows from the expressions (5): $a_i = g_i n_i d_i$ for $i = 1, 2, 3$ and Proposition 9. □

6. Simplifying factors of Pythagorean triangles. Recall that primitive Pythagorean triangles are indecomposable by Proposition 2 above.

Proposition 11. *A Heronian triangle is Pythagorean if and only if its triple of simplifying factors is of the form $(1, 2, g)$, for an odd number g whose prime divisors are all of the form $4m + 1$.*

Proof. If the Heronian triangle contains a right angle, we may take $t_3 = \tan(\pi/4) = 1$ so that $g_1 g_2 = N(1+\sqrt{-1}) = 2$. From this, the numbers g_1 and g_2 must be 1 and 2 in some order.

Conversely, for an odd number g satisfying the given condition, $g = u^2 + v^2$ for unique positive integers $u < v$. The Heronian triangle has $z_3 = 1 + \sqrt{-1}$ and $\alpha_3 = 2\arctan 1 = \pi/2$. The other two angles are $2\arctan(u/v)$ and $2\arctan((v-u)/(v+u))$. $\square$

7. Orthocentric quadrangles. Now we consider a rational triangle which does not contain a right angle. The vertices and the orthocenter form an orthocentric quadrangle, i.e., each of these four points is the orthocenter of the triangle with vertices at the remaining three points. See, for example, Coxeter [**3**]. If any of the four triangles is rational, then so are the remaining three. The convex hull of these four points is an acute-angle triangle Γ. We label the vertices A_1, A_2, A_3, and the orthocenter in the interior by A. Denote by Γ_i the (obtuse-angled) triangle with vertices A_j, A_k and A.

Let t_1, t_2, t_3 be the tangents of the half angles of Γ, z_1, z_2, z_3 the associated Gaussian integers, and (g_1, g_2, g_3) the corresponding simplifying factors. Then the tangents of the half angles of Γ_k are

$$\frac{1-t_i}{1+t_i}, \quad \frac{1-t_j}{1+t_j} \quad \text{and} \quad \frac{1}{t_k}.$$

We first assume that g_1, g_2, g_3 are all odd, so that for $i = 1, 2, 3$, d_i and n_i are of different parity, Proposition 8(c). The triangle Γ_k has associated primitive Gaussian integers

$$\begin{aligned} z_i' &= (d_i + n_i) + (d_i - n_i)\sqrt{-1} = (1+\sqrt{-1})\overline{z_i}, \\ z_j' &= (d_j + n_j) + (d_j - n_j)\sqrt{-1} = (1+\sqrt{-1})\overline{z_j}, \\ z_k' &= n_k + d_k\sqrt{-1} = \sqrt{-1}\cdot\overline{z_k}. \end{aligned} \tag{9}$$

From these,

$$\begin{aligned} z_j' z_k' &= (1+\sqrt{-1})\sqrt{-1}\cdot\overline{z_j z_k} = g_i(1+\sqrt{-1})z_i \\ &= g_i\sqrt{-1}\cdot\overline{z_j'} \\ z_i' z_k' &= (1+\sqrt{-1})\sqrt{-1}\cdot\overline{z_i z_k} = g_j(1+\sqrt{-1})z_j \\ &= g_j\sqrt{-1}\cdot\overline{z_j'}, \\ z_i' z_j' &= 2\sqrt{-1}\cdot\overline{z_i z_j} = 2g_k z_k = 2g_k\sqrt{-1}\cdot\overline{z_k'}. \end{aligned}$$

Thus the triangle Γ_k has simplifying factors $(g_i, g_j, 2g_k)$.

Suppose now that one of the simplifying factors of Γ, say g_k, is even. Then n_i, d_i, n_j, d_j are all odd, and n_k, d_k have different parity. A similar calculation shows that the simplifying factors for the triangles Γ_i, Γ_j and Γ_k are $(2g_i, g_j, g_k/2)$, $(g_i, 2g_j, g_k/2)$ and $(g_i, g_j, g_k/2)$, respectively.

We summarize these in the following proposition.

Proposition 12. *The simplifying factors for the four (rational) triangles in an orthocentric quadrangle are of the form* (g_1, g_2, g_3), $(2g_1, g_2, g_3)$, $(g_1, 2g_2, g_3)$ *and* $(g_1, g_2, 2g_3)$ *with* g_1, g_2, g_3 *odd integers.*

8. Indecomposable primitive Heronian triangles. A routine computer search gives the following indecomposable, primitive Heronian triangles with sides ≤ 100, excluding Pythagorean triangles:

(5, 29, 30; 72)	(10, 35, 39; 168)	(15, 34, 35; 252)	(13, 40, 45; 252)	(17, 40, 41; 336)
(25, 34, 39; 420)	(5, 51, 52; 126)	(15, 52, 61; 336)	(20, 53, 55; 528)	(37, 39, 52; 720)
(17, 55, 60; 462)	(26, 51, 73; 420)	(17, 65, 80; 288)	(29, 65, 68; 936)	(34, 55, 87; 396)
(39, 55, 82; 924)	(41, 50, 89; 420)	(35, 65, 82; 1092)	(26, 75, 91; 840)	(39, 58, 95; 456)
(17, 89, 90; 756)	(26, 73, 97; 420)	(41, 60, 95; 798)	(51, 52, 97; 840)	

We study the condition under which the primitive Heronian triangle $\Gamma_0 = \Gamma_0(t_1, t_2, t_3)$ constructed in Section 2 is *indecomposable*. Clearly, $\Gamma_0 = \Gamma(t_1, t_2, t_3)$ is indecomposable if this is so for the triangle Γ defined by (5). More remarkable is the validity of the converse.

Theorem 13. *A non-Pythagorean, primitive Heronian triangle* $\Gamma_0 = \Gamma_0(t_1, t_2, t_3)$ *is indecomposable if and only if each of the simplifying factors* g_i, $i = 1, 2, 3$, *contains an odd prime divisor.*

Proof. We first prove the theorem for the triangle $\Gamma := \Gamma(t_1, t_2, t_3)$ defined by (5).

Since Γ has area $\Delta = n_1 d_1 n_2 d_2 n_3 d_3$, the height on the side $a_i =$

$g_i n_i d_i$ is given by

$$h_i = \frac{2n_j d_j n_k d_k}{g_i}.$$

Since the triangle does not contain a right angle, it is indecomposable if and only if none of the heights h_i, $i = 1, 2, 3$, is an integer. By Proposition 8(d), this is the case if and only if each of g_1, g_2, g_3 contains an odd prime divisor.

To complete the proof, note that the sides (and hence also the heights) of Γ_0 are $1/g$ times those of Γ. Here $g := \gcd(a_1, a_2, a_3) = \gcd(n_1d_1, n_2d_2, n_3d_3)$ by Corollary 10. The heights of Γ_0 are therefore

$$h_i' = \frac{2n_j d_j n_k d_k}{g_i \cdot g} = \frac{2}{g_i} \cdot \frac{n_j d_j n_k d_k}{\gcd(n_1d_1, n_2d_2, n_3d_3)}.$$

Note that $(n_j d_j n_k d_k)/\gcd(n_1d_1, n_2d_2, n_3d_3)$ is an *integer* prime to g_i. If h_i' is not an integer, then g_i must contain an odd prime divisor, by Proposition 8(d) again. □

Corollary 14. *Let Γ be a primitive Heronian triangle. Denote by Γ_i, $i = 1, 2, 3$, the primitive Heronian triangles in the similarity classes of the remaining three rational triangles in the orthocentric quadrangle containing Γ. The four triangles Γ and Γ_i, $i = 1, 2, 3$, are either all decomposable or all indecomposable.*

Example. From the orthocentric quadrangle of each of the indecomposable Heronian triangles (15, 34, 35; 252) and (25, 34, 39; 420), we obtain three other indecomposable primitive Heronian triangles.

(a_1, b_1, c_1)	(g_1, g_2, g_3)	(a_1, b_1, c_1)	(g_1, g_2, g_3)
(15, 34, 35; 252)	(5, 17, 5)	(25, 34, 39; 420)	(5, 17, 13)
(55, 17, 60; 462)	(5, 17, 10)	(285, 187, 364; 26334)	(5, 17, 26)
(119, 65, 180; 1638)	(5, 17, 10)	(700, 561, 169; 30030)	(10, 17, 13)
(65, 408, 385; 12012)	(5, 34, 5)	(855, 952, 169;62244)	(5, 34, 13)

9. Construction of Heronian triangles with given simplifying factors.

Theorem 15. *Let* g_1, g_2, g_3 *be* odd *numbers satisfying the following conditions.*

(i) *At least two of* g_1, g_2, g_3 *exceed* 1.

(ii) *The prime divisors of* g_i, $i = 1, 2, 3$, *are all congruent to* 1 (mod 4).

(iii) $\gcd(g_1, g_2, g_3) = 1$.

Suppose g_1, g_2, g_3 *together contain* λ *distinct rational* (*odd*) *prime divisors. Then there are* $2^{\lambda-1}$ *distinct, primitive Heronian triangles with simplifying factors* (g_1, g_2, g_3).

Proof. Suppose (g_1, g_2, g_3) satisfies these conditions. By (ii), there are primitive Gaussian integers θ_i, $i = 1, 2, 3$, such that $g_i = N(\theta_i)$. Since $\gcd(g_1, g_2, g_3) = 1$, if a rational prime $p \equiv 1 \pmod 4$ divides g_i and g_j, then, in the ring $\mathbf{Z}[\sqrt{-1}]$, the prime factorizations of θ_i and θ_j contain powers of the same Gaussian prime π of $\bar{\pi}$.

Therefore, if g_1, g_2, g_3 together contain λ rational prime divisors, then there are 2^λ choices of the triple of primitive Gaussian integers $(\theta_1, \underline{\theta_2, \theta_3})$, corresponding to a choice between the Gaussian primes $\pi(p)$ and $\overline{\pi(p)}$ for each of these rational primes.

Choose units ε_1 and ε_2 such that $z_1 = \varepsilon_1\theta_2\overline{\theta_3}$ and $z_2 = \varepsilon_2\overline{\theta_1}\theta_3$ are positive.

Two positive Gaussian integers z_1 and z_2 define a positive Gaussian integer z_3 via (8) if and only if

$$0 < \phi(z_1) + \phi(z_2) < \frac{\pi}{2}. \tag{10}$$

Since $\phi(z_1^*) + \phi(z_2^*) = \pi - (\phi(z_1) + \phi(z_2))$, it follows that exactly one of the two pairs (z_1, z_2) and (z_1^*, z_2^*) satisfies condition (10). There are, therefore, $2^{\lambda-1}$ Heronian triangles with (g_1, g_2, g_3) as simplifying factors. □

Making use of Theorems 13, 15 and Proposition 12, it is now easy to construct indecomposable primitive Heronian triangles from any triples of odd integers (g_1, g_2, g_3), each greater than 1, and satisfying the conditions of Theorem 15. For example, by choosing g_1, g_2, g_3 from the first few primes of the form $4k+1$, we obtain the following primitive Heronian triangles, all indecomposable.

(g_1, g_2, g_3)	(d_1, n_1)	(d_2, n_2)	(d_3, n_3)	$(a, b, c; \Delta)$
(5, 13, 17)	(14, 5)	(7, 6)	(7, 4)	(25, 39, 34; 420)
	(5, 14)	(9, 2)	(8, 1)	(175, 117, 68; 2520)
	(11, 10)	(7, 6)	(8, 1)	(275, 273, 68; 9240)
	(10, 11)	(9, 2)	(7, 4)	(275, 117, 238; 13860)
(5, 13, 29)	(4, 19)	(12, 1)	(8, 1)	(95, 39, 58; 456)
	(16, 11)	(8, 9)	(8, 1)	(110, 117, 29; 1584)
	(11, 16)	(12, 1)	(7, 4)	(220, 39, 203; 3696)
	(19, 4)	(8, 9)	(7, 4)	(95, 234, 203; 9576)
(5, 17, 29)	(22, 3)	(12, 1)	(2, 9)	(55, 34, 87; 396)
	(18, 13)	(9, 8)	(9, 2)	(65, 68, 29; 936)
	(18, 13)	(12, 1)	(6, 7)	(195, 34, 203; 3276)
	(22, 3)	(9, 8)	(7, 6)	(55, 204, 203; 5544)
(13, 17, 29)	(22, 3)	(16, 11)	(10, 11)	(39, 136, 145; 2640)
	(22, 3)	(19, 4)	(5, 14)	(429, 646, 1015; 87780)
	(18, 13)	(19, 4)	(11, 10)	(1521, 646, 1595; 489060)
	(18, 13)	(16, 11)	(14, 5)	(1521, 1496, 1015; 720720)

Further examples can be obtained by considering the orthocentric quadrangle of each of these triangles.

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Department of Mathematics, Florida Atlantic University, Boca Raton, FL 33431-0991
E-mail address: yiu@fau.edu