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GLOBAL EXISTENCE FOR THE CAUCHY PROBLEM FOR THE VISCOUS SHALLOW WATER EQUATIONS

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ABSTRACT. A global existence and uniqueness theorem of strong solutions for the initial-value problem for the viscous shallow water equations is established for small initial data and no forcing. Polynomial L^2 and L^∞ decay rates are established and the solution is shown to be classical for $t > 0$.

1. Introduction. The numerical solutions of the hyperbolic systems encountered in weather prediction models often develop high-frequency gravity wave solutions which can seriously distort short-term forecasts (on the order of hours to days; typically 12 hours). Various initialization schemes have been studied to control these distortions. The 'slow manifold,' proposed by Leith [**3**], is believed to have an invariance property such that if one begins with initial data on the slow manifold, the solution will remain free of gravity waves for all time. Temam [**11**] suggests a relationship between the mathematical concept of an inertial manifold and slow manifolds.

The shallow water equations are the simplest primitive equation model to exhibit gravity waves. However, before one can study the issues of global attractors and inertial manifolds, the question of global existence and uniqueness must be thoroughly addressed.

1.1. Well-posedness. Bui [**1**] proved local existence and uniqueness of classical solutions to the Dirichlet problem for the unforced viscous shallow water equations using Lagrangian coordinates and Hölder space estimates. He assumed the initial data $h_0 \in C^{1,\alpha}(\Omega)$ and $u_0 \in C^{2,\alpha}(\Omega)$.

Kloeden [**2**] proved global existence and uniqueness of classical solutions to the forced Dirichlet problem using Sobolev space estimates by following the energy method of Matsumura and Nishida [**5, 6**]. In addition to the assumptions 3–6 (Section 2), Kloeden further assumes the

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solutions are spatially periodic and satisfy Dirichlet boundary conditions; the initial data $(h_0, u_0) \in H^4(\Omega)$; and $\Phi \in H^5(\Omega)$. The spatially periodic assumption was made in order to simplify the a priori estimate. This assumption eliminates the need to compute boundary estimates.

Bui and Kloeden do not include the Coriolis force in their equations, but both state that the inclusion of this force will not alter the results of their respective theorems. Indeed the Coriolis force acts perpendicular to the velocity field and hence does no work. Neither paper establishes a decay rate of solutions. Both papers assume a positivity constraint, the fluid height $h(t, x) > 0$ for all $x \in \Omega$ and $t \geq 0$.

Sundbye [**10**] proves global existence of strong solutions for the forced initial value problem with Dirichlet boundary conditions. An exponential C^0 decay rate is established for this Dirichlet problem. The positivity of the fluid height h is also established for all $x \in \Omega$ and $t \geq 0$. The solutions are shown to be classical for $t > 0$.

In this paper we prove global existence of strong solutions for the unforced initial value problem. Polynomial L^2 and L^∞ decay rates are established for the Cauchy problem, and the solutions are shown to be classical for $t > 0$.

1.2. Method of proof. The method of proof of the a priori estimate follows closely the energy method developed by Matsumura [**4**]. This technique essentially consists of considering the differential equation for v^k where

$$v^k(t, x) = (1 + t)^k u(t, x), \quad k \in \mathbf{N}, \tag{1.1}$$

and then deriving energy estimates for v^k. These estimates are weighted a priori estimates for u (Racke [**8**]).

Matsumura [**4**] solved the Cauchy problem for the compressible, viscous, heat-conducting fluids. Matsumura and Nishida [**6,7**] further modified this technique to solve the Dirichlet problem on interior and exterior domains.

Zheng [**12, 13**], Zheng and Chen [**14**] and Zheng and Shen [**15**] used this technique to solve nonlinear parabolic equations and coupled quasilinear hyperbolic-parabolic systems of a specific form.

For a more complete review of energy methods applied to initial value problems, see Racke [**8**].

2. The viscous shallow water equations. We consider a uniformly rotating sheet of fluid with constant, uniform density and unbounded horizontal extent. The height of the fluid surface is given by $h(x, y, t)$. The bottom topography $h_B(x, y)$ is assumed to be zero. The horizontal velocity vector is assumed to be independent of height and the vertical velocity is, in turn, determined by mass continuity and the hydrostatic approximation.

In vector form, the viscous shallow water equations are given by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + g\nabla h + f\mathbf{k} \times \mathbf{u} = \nu \frac{\nabla \cdot (h \nabla u)}{h} \tag{2.1}$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0 \tag{2.2}$$

with initial conditions

$$\mathbf{u}(0, x) = \mathbf{u}_0(x), \qquad h(0, x) = h_0(x), \quad \text{for} \quad x \in \Omega, \tag{2.3}$$

where $t \geq 0$, $x = (x_1, x_2) \in \Omega$; $u(t, x) = (u^1(t, x), u^2(t, x))$ is the horizontal velocity field; $h(t, x)$ is the fluid depth, the quantity gh is the geopotential; $\nu = \mu/\rho > 0$ is the kinematic viscosity; f is the Coriolis parameter; and $\mathbf{k}$ is the unit vector $(0, 0, 1)$.

The equations form a semi-linear mixed hyperbolic-parabolic system. For global existence for the Cauchy problem, we assume the following conditions on equations (2.1)–(2.3):

1. $\Omega = \mathbf{R}^2$;
2. $h_0(x) > 0$ for all $x \in \Omega$;
3. $(h_0, u_0) \in H^3(\Omega)$;
4. $\|(h_0 - \bar{h}, u_0)\|_{H^3}$ is small, i.e., the initial data are smooth functions close to a constant state $(\bar{h}, 0)$ where $\bar{h} = \int_\Omega h_0(x)\, dx = \int_\Omega h(x)\, dx$;
5. $|h_0 - \bar{h}| \ll \bar{h}$, i.e., the perturbation is much smaller than the average height.

The shallow water equations in index notation are:

$$u_t^i + u^j u_{x_j}^i - \nu \frac{(h u_{x_j}^i)_{x_j}}{h} + g h_{x_i} + f \varepsilon_{ij\ell} k^j u^\ell = 0 \tag{2.4}$$

$$h_t + (hu^j)_{x_j} = 0 \tag{2.5}$$

$$u(0,x) = u_0(x), \qquad h(0,x) = h_0(x) \quad \text{for} \quad x \in \mathbf{R}^2 \tag{2.6}$$

where $\varepsilon_{ij\ell}$ is the alternating unit tensor defined by

$$\varepsilon_{ij\ell} = \begin{cases} +1 & \text{if } ij\ell = 123,\ 231,\text{ or } 312; \\ -1 & \text{if } ij\ell = 321,\ 132,\text{ or } 213; \\ 0 & \text{if any two indices are repeated.} \end{cases} \tag{2.7}$$

Equations (2.4)–(2.5) are perturbed about the constant steady state by applying the change of variables $(h,u) \to (h+\bar{h}, u)$.

For Equation (2.5), we have

$$h_t + ((h+\bar{h})u^j)_{x_j} = 0 \tag{2.8}$$

and for Equation (2.4), we have

$$u_t^i + u^j u_{x_j}^i - \nu \frac{((h+\bar{h})u_{x_j}^i)_{x_j}}{h+\bar{h}} + g(h+\bar{h})_{x_i} + f\varepsilon_{ij\ell}k^j u^\ell = 0 \tag{2.9}$$

Without loss of generality, we normalize the acceleration of gravity since it is not a fluid property and rewrite the perturbed problem as follows:

$$L^0(h,u) = h_t + \bar{h}u_{x_j}^j = G^0 \tag{2.10}$$

$$L^i(h,u) = u_t^i - \nu u_{x_j x_j}^i + h_{x_i} + f\varepsilon_{ij\ell}k^j u^\ell = G^i, \quad i = 1,2 \tag{2.11}$$

$$u(0,x) = u_0(x), \qquad h(0,x) = h_0(x) - \bar{h} \tag{2.12}$$

where the terms on the right include all nonlinearities and have the form

$$G^0(h,u) = -(hu^j)_{x_j} \tag{2.13}$$

$$G^i(h,u) = -u^j u_{x_j}^i + \nu \frac{h_{x_j} u_{x_j}^i}{h+\bar{h}} \tag{2.14}$$

for $i = 1, 2$.

A solution of (2.10)–(2.12) is sought in the set of functions $X(0, \infty; E)$ for some $E \le E_0$, where for $0 \le t_1 \le t_2 \le \infty$,

(2.15)

$$\begin{aligned} X(t_1,t_2;E) &= \{(h,u): \\ &\quad h \in C^0(t_1,t_2;H^3) \cap C^1(t_1,t_2;H^2) \cap L^2(t_1,t_2;H^3), \\ &\quad (1+t)^{1/2}D^2h \in L^2(t_1,t_2;L^2), (1+t)D^3h \in L^2(t_1,t_2;L^2), \\ &\quad u \in C^0(t_1,t_2;H^3) \cap C^1(t_1,t_2;H^1) \cap L^2(t_1,t_2;H^4), \\ &\quad (1+t)^{1/2}D^2u \in L^2(t_1,t_2;L^2), (1+t)D^3u \in L^2(t_1,t_2;H^1), \\ &\qquad \text{and } N(t_1,t_2) \le E\} \end{aligned}$$

where the energy N is defined by

(2.16)

$$\begin{aligned} N^2(t_1,t_2) = \sup_{t_1 \le t \le t_2} &\{\|(h,u)(t)\|_{H^3}^2 + t\|D(h,u)(t)\|^2 \\ &\qquad + t^2\|D^2(h,u)(t)\|_{H^1}^2\} \\ &+ \int_{t_1}^{t_2} \{\|h(s)\|_{H^3}^2 + \|u(s)\|_{H^4}^2 + s\|D^2(h,u)(s)\|^2 \\ &\qquad + s^2\|D^3h(s)\|^2 + s^2\|D^3u(s)\|_{H^1}^2\}\, ds. \end{aligned}$$

We denote by $W^{k,2}(\Omega) = H^k(\Omega)$ the Sobolev space of functions f which, along with its mth order generalized spatial derivatives $D^m f$ for $m = 1, 2, \ldots, k$, belong to $L^2(\Omega)$, with norm given by

$$\|f\|_k = \left(\sum_{m=0}^{k} \int_\Omega |D^m f(x)|^2\, dx \right)^{1/2},$$

by

$$D^m f = \left(\frac{\partial}{\partial x}\right)^\alpha f^i, \qquad |\alpha| = m, \quad i = 1, 2, \ldots, n,$$

which is a vector composed of all mth partial derivatives, by $\|\cdot\| = \|\cdot\|_{L^2}$, by $C^k(a,b;X)$ the space of functions $f : [a,b] \to X$, which are k-times continuously differentiable functions in time and by $L^k(a,b;X)$ the space of functions $f : [a,b] \to X$ for which $\|f(t)\|_k$ is square integrable for $t \in [a,b]$.

3. Local and global existence. We prove the following global existence theorem for the Cauchy problem:

Theorem 3.1 (Global existence and uniqueness for the Cauchy problem). *Consider the initial-value problem* (2.1)–(2.3) *with the assumptions* 1–6 *and initial data* $(h_0 - \bar{h}, u_0) \in H^3(\mathbf{R}^2)$. *Then there exists* $\varepsilon > 0$ *and* $C > 0$ *such that if*

$$\|(h_0 - \bar{h}, u_0)\|_{H^3} \le \varepsilon, \tag{3.1}$$

then the problem (2.1)–(2.3) *has a unique global solution in time satisfying*

$$h - \bar{h} \in C^0(0, \infty; H^3) \cap C^1(0, \infty; H^2), \tag{3.2}$$

$$u \in C^0(0, \infty; H^3) \cap C^1(0, \infty; H^1) \tag{3.3}$$

with the following decay rates

$$\|(h - \bar{h}, u)\| \le C\|(h_0 - \bar{h}, u_0)\|_{H^3} \tag{3.4}$$

$$\|D(h - \bar{h}, u)\| \le C(1 + t)^{-1/2}\|(h_0 - \bar{h}, u_0)\|_{H^3} \tag{3.5}$$

$$\|D^2(h - \bar{h}, u)\| \le C(1 + t)^{-1}\|(h_0 - \bar{h}, u_0)\|_{H^3} \tag{3.6}$$

$$\|(h - \bar{h}, u)(t)\|_{L^\infty} \le C(1 + t)^{-1/2}\|(h_0 - \bar{h}, u_0)\|_{H^3} \tag{3.7}$$

$$\|D(h - \bar{h}, u)(t)\|_{L^\infty} \le C(1 + t)^{-1}\|(h_0 - \bar{h}, u_0)\|_{H^3} \tag{3.8}$$

$$\|D^2 u(t)\|_{L^\infty} \le C(1 + t)^{-3/2}\|(h_0 - \bar{h}, u_0)\|_{H^3}. \tag{3.9}$$

Theorem 3.1 is proved by a combination of a local existence result and an a priori estimate.

Theorem 3.2 (Local existence). *Suppose the problem* (2.10)–(2.12) *has a unique solution* $(h, u) \in X(0, T; E_0)$ *for some* $T \ge 0$, *and consider the problem for* $t \ge T$. *Then there exist positive constants* τ, ε_0 *and* C_0 *with* $\varepsilon_0\sqrt{1 + C_0^2} \le E_0$, *which are independent of* T *such that, if* $N(T, T) \le \varepsilon_0$, *then the problem* (2.10)–(2.12) *has a unique solution*

$$(h, u) \in X(T, T + \tau; C_0 N(T, T)). \tag{3.10}$$

Proof. (See Sundbye [**9**] and Matsumura and Nishida [**5**].) □

Theorem 3.3 (A priori estimate). *Suppose the problem* (2.10)–(2.12) *has a solution* $(h, u) \in X(0, T; E_0)$ *for some* $T > 0$. *Then there exist positive constants* ε_1 *and* C_1 *with* $\varepsilon_1 \le \varepsilon_0$ *and* $\varepsilon_1 C_1 \le E_0$, *which are independent of* T *such that, if* $N(0, T) \le \varepsilon_1$, *then*

$$N(0, T) \le C_1 N(0, 0). \tag{3.11}$$

We will prove Theorem 3.3 in Section 4, but note here the proof of Theorem 3.1.

Choose the initial data $(h, u)(0)$ sufficiently small in order that

$$N(0, 0) \le \min\left\{\varepsilon_0, \frac{\varepsilon_1}{C_0}, \frac{\varepsilon_1}{C_1\sqrt{1 + C_0^2}}\right\}. \tag{3.12}$$

Theorem 3.2 with $T = 0$ gives a local solution

$$(h, u) \in X(0, \tau; C_0\ N(0, 0)). \tag{3.13}$$

Since $C_0\ N(0, 0) \le \varepsilon_1 \le \varepsilon_0$, Theorem 3.3 with $T = \tau$ implies

$$N(0, \tau) \le C_1 N(0, 0). \tag{3.14}$$

Then Theorem 3.2 with $T = \tau$ implies there exists an extension to the solution

$$\begin{aligned} (h, u) &\in X(\tau, 2\tau; C_0\ N(\tau, \tau)) \\ &\in X(0, 2\tau; \sqrt{1 + C_0^2}\ N(0, \tau)), \end{aligned} \tag{3.15}$$

since

$$\begin{aligned} N^2(0, 2\tau) &\le N^2(0, \tau) + N^2(\tau, 2\tau) \\ &\le N^2(0, \tau) + C_0^2 N^2(\tau, \tau) \\ &\le (1 + C_0^2)\ N^2(0, \tau). \end{aligned} \tag{3.16}$$

Therefore, since

$$\sqrt{1+C_0^2}\ N(0,\tau) \le C_1\sqrt{1+C_0^2}\ N(0,0) \le \varepsilon_1, \tag{3.17}$$

Theorem 3.3 with $T = 2\tau$ yields

$$N(0,2\tau) \le C_1\ N(0,0), \tag{3.18}$$

and Theorem 3.2 with $T = 2\tau$ gives the extension to the solution

$$\begin{aligned}(h,u) &\in X(2\tau,3\tau;C_0\ N(2\tau,2\tau))\\ &\in X(0,3\tau;\sqrt{1+C_0^2}\ N(0,2\tau)).\end{aligned} \tag{3.19}$$

Repetition of this process gives

Proposition 3.4 (Global existence). *There exist positive constants ε and C with $\varepsilon C \le E_0$ such that if $N(0,0) \le \varepsilon$, then the initial-value problem* 2.10–2.12 *has a unique solution $(h,u) \in X(0,\infty;CN(0,0))$.*

To complete the proof of Theorem 3.1, it remains to show the decay rate of the solution and to show the solutions are in fact classical solutions for $t > 0$.

Since

$$\begin{aligned}h(t) &\in C^0(0,T;H^3) \cap C^1(0,T;H^2)\\ &\subset C^0(0,T;C^{1,\alpha}) \cap C^1(0,T;C^{0,\alpha})\end{aligned} \tag{3.20}$$

for any $\alpha \in (0,1)$, $h(t)$ is a classical solution for all $t \ge 0$. The reader is referred to Matsumura and Nishida [**5**, pp. 101–103] for the proof that $u(t)$ is a classical solution for $t > 0$.

The decay rates (3.4)–(3.6) follow directly from Theorem 3.3. The estimate (3.7) follows from Nirenberg's inequality,

$$\begin{aligned}\sup_x |(h-\bar{h},u)(t)| &\le \|(h-\bar{h},u)(t)\|_{L^\infty}\\ &\le C\|(h-\bar{h},u)(t)\|^{1/2}\|D^2(h-\bar{h},u)(t)\|^{1/2}\\ &\le C(1+t)^{-1/2}\|(h_0-\bar{h},u_0)\|.\end{aligned} \tag{3.21}$$

Nirenberg's inequality can be modified to

$$\sup_x |D(h,u)| \le C\|D(h,u)\|^{1/2}\|D^3(h,u)\|^{1/2}; \tag{3.22}$$

this implies

$$\begin{aligned}\sup_x |D(h-\bar h,u)(t)| &\le \|D(h-\bar h,u)(t)\|_{L^\infty}\\ &\le C\|D(h-\bar h,u)(t)\|^{1/2}\|D^3(h-\bar h,u)(t)\|^{1/2}\\ &\le C(1+t)^{-1}\|(h_0-\bar h,u_0)\|.\end{aligned} \tag{3.23}$$

See Zheng [**12**] for proof that

$$\|D^2u(t)\|_{L^\infty} \le C(1+t)^{-3/2}\|(h_0-\bar h,u_0)\|. \tag{3.24}$$

4. The a priori estimate. The proof of Theorem 3.3 follows from Proposition 4.3 and Theorem 4.4. Lemmas 4.1 and 4.2 establish bounds on the derivatives in terms of the nonlinearities. In Proposition 4.3, these nonlinearities are then related to the energy norm given by Equation (2.16). Theorem 4.4 is the heart of the proof of the a priori estimate and shows the nonlinearities and, hence, the norm remains bounded independently of time. The remainder of Section 4 is the proof of Theorem 4.4 which follows from Propositions 4.5, 4.6, 4.7, 4.8 and 4.9.

Define for $k=0,1,2,3$

$$A^k(t) = \int (D^kG^0\cdot D^kh + \bar h D^kG^i\cdot D^ku^i)\,dx \tag{4.1}$$

$$B^k(t) = \int \left(D^kG^0_{x_i}\cdot D^kh_{x_i} + \frac{\bar h}{\nu}D^kG^i\cdot D^kh_{x_i}\right)dx \tag{4.2}$$

$$C^k(t) = \int \frac{\bar h}{\nu}D^kG^0_{x_i}\cdot D^ku^i\,dx. \tag{4.3}$$

Lemma 4.1. *There exists a positive constant C independent of t such that*

$$\begin{aligned}\|D^m(h,u)(t)\|^2 + \nu\int_0^t \|D^{m+1}u(\tau)\|^2\,d\tau\\ \le C\left\{\|D^m(h_0,u_0)\|^2 + \int_0^t |A^m(\tau)|\,d\tau\right\}\end{aligned} \tag{4.4}$$

for $0 \le m \le 3$ *and*

$$(4.5)\quad t^\ell \|D^m(h,u)(t)\|^2 + \nu \int_0^t \tau^\ell \|D^{m+1}u(\tau)\|^2\,d\tau \le C\int_0^t (\tau^{\ell-1}\|D^m(h,u)(\tau)\|^2 + |\tau^\ell A^m(\tau)|)\,d\tau$$

for $\ell = 1,2,\ m = 1,2,3.$

Proof. Consider the equality

$$(4.6)\quad \begin{aligned} &\int_0^t \tau^\ell \left(\int D^m L^0 \cdot D^m h + \bar h D^m L^i \cdot D^m u^i\,dx \right) d\tau \\ &= \int_0^t \tau^\ell A^m(\tau)\,d\tau \\ &= \int_0^t \tau^\ell \int D^m(h_t + \bar h u^i_{x_i}) D^m h \\ &\quad + \bar h (D^m(u^i_t - \nu u^i_{x_j x_j} + h_{x_i} + f\varepsilon_{ij\ell} k^j u^\ell) D^m u^i)\,dx\,d\tau \\ &= \int_0^t \tau^\ell \int \frac{1}{2}\frac{\partial}{\partial t}(D^m h)^2 + \frac{\bar h}{2}\frac{\partial}{\partial t}(D^m u^i)^2 + \bar h D^m u^i_{x_i} D^m h \\ &\qquad - \bar h \nu D^m u^i_{x_j x_j} D^m u^i + \bar h D^m h_{x_i} D^m u^i \\ &\qquad + \bar h D^m u^i \cdot f \varepsilon_{ij\ell} k^j D^m u^\ell\,dx\,d\tau. \end{aligned}$$

Integrating by parts and combining, we have

$$(4.7)\quad \begin{aligned} &\frac{\tau^\ell}{2}(\|D^m h(\tau)\|^2 + \bar h \|D^m u^i(\tau)\|^2)\Big|_0^t + \int_0^t \tau^\ell \bar h \nu \|D^{m+1}u^i\|^2\,d\tau \\ &= \int_0^t \frac{\ell \tau^{\ell-1}}{2}(1-\delta^{\ell,0})(\|D^m h\|^2 + \bar h\|D^m u^i\|^2)\,d\tau \\ &\quad + \int_0^t \tau^\ell A^m(\tau)\,d\tau \end{aligned}$$

where $\delta^{i,j}$ is the Kronecker delta.

Equations (4.4) and (4.5) readily follow. □

Lemma 4.2. *There exists a positive constant C independent of t such that*

$$(4.8)\quad \|D^m h(t)\|^2 + \frac{\bar{h}}{\nu}\int_0^t \|D^m h(\tau)\|^2\,d\tau \le C\Big\{\|D^m h_0\|^2 + \|D^{m-1}u_0\|^2 + \|D^{m-1}u(t)\|^2 + \int_0^t (\|D^m u(\tau)\|^2 + |B^{m-1}(\tau)| + |C^{m-1}(\tau)|)\,d\tau\Big\}$$

for $1 \le m \le 3$ *and*

$$(4.9)\quad t^\ell\|D^m h(t)\|^2 + \frac{\bar{h}}{\nu}\int_0^t \tau^\ell\|D^m h(\tau)\|^2\,d\tau \le C\Big\{t^\ell\|D^{m-1}u(t)\|^2 + \int_0^t (\tau^{\ell-1}\|D^m h(\tau)\|^2 + \tau^{\ell-1}\|D^{m-1}u(\tau)\|^2 + \tau^\ell\|D^m u(\tau)\|^2 + |\tau^\ell B^{m-1}(\tau)| + |\tau^\ell C^{m-1}(\tau)|)\,d\tau\Big\}$$

for $\ell = 1, 2,\ m = 2, 3.$

Proof. Consider the equality

$$\begin{aligned}(4.10)\quad &\int_0^t \tau^\ell\Big(\int D^{m-1}L^0_{x_i}\cdot D^{m-1}h_{x_i} + \frac{\bar{h}}{\nu}D^{m-1}L^i\cdot D^{m-1}h_{x_i}\,dx\Big)\,d\tau \\ &= \int_0^t \tau^\ell B^{m-1}(\tau)\,d\tau \\ &= \int_0^t \tau^\ell \int D^{m-1}(h_{tx_i} + \bar{h}u^i_{x_jx_j})D^{m-1}h_{x_i} + \frac{\bar{h}}{\nu}D^{m-1}(u^i_t - \nu u^i_{x_jx_j} + h_{x_i} + f\varepsilon_{ij\ell}k^j u^\ell)D^{m-1}h_{x_i}\,dx\,d\tau \\ &= \int_0^t \tau^\ell\Big(\int\Big(\frac{1}{2}\frac{\partial}{\partial t}(D^m h)^2 + \frac{\bar{h}}{\nu}D^{m-1}u^i_t D^{m-1}h_{x_i} + \frac{f\bar{h}}{\nu}\varepsilon_{ij\ell}k^j D^{m-1}u^\ell D^m h\Big)\,dx + \frac{\bar{h}}{\nu}\|D^m h\|^2\Big)\,d\tau.\end{aligned}$$

Integrating by parts and combining, we have

$$\begin{aligned}(4.11)\quad &\left\{\frac{\tau^\ell}{2}\|D^m h(\tau)\|^2+\tau^\ell\frac{\bar h}{\nu}\int D^{m-1}u^i(\tau)D^m h(\tau)\,dx\right\}\Big|_0^t\\ &\quad + C\int_0^t \tau^\ell\int D^{m-1}u^i D^m h\,dx\,d\tau\\ &\quad + \frac{\bar h}{\nu}\int_0^t \tau^\ell\|D^m h\|^2\,d\tau\\ &= \int_0^t\left(\frac{\bar h^2}{\nu}\tau^\ell\|D^m u\|^2+\frac{\ell\tau^{\ell-1}}{2}(1-\delta^{\ell,0})\|D^m h\|^2\right)d\tau\\ &\quad + \frac{\bar h}{\nu}\int_0^t \ell\tau^{\ell-1}(1-\delta^{\ell,0})\int D^{m-1}u^i D^m h\,dx\,d\tau\\ &\quad + \int_0^t(\tau^\ell B^{m-1}(\tau)+\tau^\ell C^{m-1}(\tau))\,d\tau.\end{aligned}$$

Equations 4.8 and 4.9 readily follow. □

Proposition 4.3. *There exists a positive constant C independent of t such that*

$$\begin{aligned}N^2(0,t)\le C\Bigg\{&\|(h_0,u_0)\|_{H^3}^2+\int_0^t\Bigg(\sum_{m=0}^3|A^m(\tau)|\\ (4.12)\qquad &+\tau^2(|A^2(\tau)|+|A^3(\tau)|)+\tau|A^1(\tau)|\\ &+\sum_{m=0}^2(1+\tau^m)(|B^m(\tau)|+|C^m(\tau)|)\,d\tau\Bigg)\Bigg\}.\end{aligned}$$

Proof. For any positive constant $\varepsilon>0$ consider the form

$$\begin{aligned}(4.13)\quad &\sum_{k=0}^3(4.4)_k+\varepsilon\sum_{m=1}^3(4.8)_m+\varepsilon^2(4.5)_{\ell=1,k=1}\\ &+\varepsilon^3(4.9)_{\ell=1,m=2}+\varepsilon^4(4.5)_{\ell=2,k=2}+\varepsilon^5(4.9)_{\ell=2,m=3}\\ &+\varepsilon^6(4.5)_{\ell=2,k=3}.\end{aligned}$$

Equation 4.12 follows by taking ε sufficiently small. □

Theorem 4.4. *Suppose* $(h,u) \in X(0,T;E)$ *for some* $E \le E_0$. *Then there exists a positive constant* C *which is independent of* t *such that*

$$\begin{aligned}(4.14)\quad \int_0^t \sum_{m=0}^{3} |A^m(\tau)| + \tau|A^1(\tau)| + \tau^2(|A^2(\tau)| + |A^3(\tau)|) \\ + \sum_{m=0}^{2} (1+\tau^m)(|B^m(\tau)| + |C^m(\tau)|)\, d\tau \\ \le CN^2(0,t) \sum_{i=1}^{3} (N(0,t))^i.\end{aligned}$$

Proof. The proof follows from Propositions 4.5, 4.6, 4.7, 4.8 and 4.9. □

Proposition 4.5.

$$(4.15)\qquad \int_0^t \sum_{m=0}^{3} |A^m(\tau)| \le CN^2(0,t) \sum_{i=1}^{3} (N(0,t))^i.$$

Proof. Consider the Coriolis contribution for constant f

$$(4.16)\qquad D^m u^i (f \varepsilon_{ij\ell} k^j D^m u^\ell).$$

For $m = 0, 1, 2, 3$, we have

$$(4.17)\qquad -fD^m u D^m v + fD^m v D^m u = 0.$$

We estimate $|\int_0^t A^m(\tau)\,d\tau|$ as follows:

$$\begin{aligned}\left|\int_0^t A^m(\tau)\,d\tau\right| &= \left|\int_0^t\int (D^m G^0\cdot D^m h + \bar{h} D^m G^i \cdot D^m u^i)\,dx\,d\tau\right| \\ &= \left|\int_0^t\int D^m h\cdot D^m[-D(hu)] \right.\\ &\qquad \left. + \bar{h} D^m u \cdot D^m\left[-uDu + \nu\frac{DhDu}{h+\bar{h}}\right]dx\,d\tau\right| \\ &\leq \left|\int_0^t\int D^m h D^{m+1}(hu)\right| \\ &\quad + \left|\bar{h}\int_0^t\int D^m u D^m(uDu)\right| \\ &\quad + \left|\bar{h}\nu\int_0^t\int D^m u D^m\left[\frac{DhDu}{h+\bar{h}}\right]\right|. \end{aligned} \tag{4.18}$$

See Sundbye [**9**, Lemmas 4.2.7–4.2.9] for estimates of these three integrals. ◻

Proposition 4.6.

$$\int_0^t \tau|A^1(\tau)| \leq CN^3(0,t). \tag{4.19}$$

Proof. We estimate $|\int_0^t \tau A^1(\tau)\,d\tau|$ as follows:

$$\begin{aligned}\left|\int_0^t \tau A^1(\tau)\,d\tau\right| &= \left|\int_0^t\int (\tau DG^0\cdot Dh + \bar{h}\tau DG^i\cdot Du^i)\,dx\,d\tau\right| \\ &= \left|\int_0^t\int \tau Dh\cdot D[-D(hu)] + \bar{h}\tau Du \right.\\ &\qquad\left. \cdot D\left[-uDu + \nu\frac{DhDu}{h+\bar{h}}\right]dx\,d\tau\right| \\ &\leq \left|\int_0^t\int \tau DhD^2[hu]\right| + \left|\bar{h}\int_0^t\int \tau DuD(uDu)\right| \\ &\quad + \left|\bar{h}\nu\int_0^t\int \tau DuD\left[\frac{DhDu}{h+\bar{h}}\right]\right|. \end{aligned} \tag{4.20}$$

Using Nirenberg's inequality and Young's inequality with $p = 4/3$ and $q = 4$, the first integral is bounded by

$$\begin{aligned}\left|\int_0^t\int \tau DhD^2[hu]\right| &= \left|\int_0^t\int -\tau D^2hD[hu]\right| \\ &= \left|\int_0^t\int -\tau D^2h(uDh + hDu)\right| \\ &\le C\int_0^t \tau(\|D^2h\|\|u\|_{H^2}\|Dh\| \\ &\quad + \|D^2h\|\|h\|_{H^2}\|Du\|) \\ &\le C\int_0^t \tau(\|D^2h\|^{3/2}\|u\|_{H^2}\|h\|^{1/2} \\ &\quad + \|D^2h\|\|h\|_{H^2}\|D^2u\|^{1/2}\|u\|^{1/2}) \\ &\le C\int_0^t 2\tau(\|D^2h\|^{3/2}\|h\|^{1/2}\|u\|^{1/2}\|D^2u\|^{1/2}) \\ &\le (\sup_t \|h\|^2)^{1/4}(\sup_t \|u\|^2)^{1/4} \\ &\quad \cdot \int_0^t \tau(\|D^2h\|^2 + \|D^2u\|^2) \\ &\le CN^3(0,t).\end{aligned} \tag{4.21}$$

Similar estimates hold for the second and third integral. □

Proposition 4.7.

$$\int_0^t \tau^2|A^2(\tau)| \le CN^2(0,t)\sum_{i=1}^{2}(N(0,t))^i. \tag{4.22}$$

Proof. We estimate $|\int_0^t \tau^2 A^2(\tau)\,d\tau|$ as follows:

$$\begin{aligned}\left|\int_0^t \tau^2A^2(\tau)\,d\tau\right| &= \left|\int_0^t\int(\tau^2D^2G^0\cdot D^2h + \bar{h}\tau^2D^2G^i\cdot D^2u^i)\,dx\,d\tau\right| \\ &= \left|\int_0^t\int \tau^2D^2h\cdot D^2[-D(hu)]\right.\end{aligned}$$

$$
\tag{4.23}
\begin{aligned}
&+ \bar{h}\tau^2 D^2 u \cdot D^2 \left[-uDu + \nu \frac{DhDu}{h+\bar{h}} \right] dx\, d\tau \Bigg| \\
&\leq \left| \int_0^t \int \tau^2 D^2 h D^3 [hu] \right| \\
&\quad + \left| \bar{h} \int_0^t \int \tau^2 D^2 u D^2 (uDu) \right| \\
&\quad + \left| \bar{h}\nu \int_0^t \int \tau^2 D^2 u D^2 \left[\frac{DhDu}{h+\bar{h}} \right] \right|.
\end{aligned}
$$

See Sundbye [**9**] for estimates of these three integrals. □

Proposition 4.8.

$$
\tag{4.24}
\int_0^t \tau^2 |A^3(\tau)| \leq CN^2(0,t) \sum_{i=1}^{3} (N(0,t))^i.
$$

Proof. We estimate $|\int_0^t \tau^2 A^3(\tau)\, d\tau|$ as follows:

$$
\tag{4.25}
\begin{aligned}
\left| \int_0^t \tau^2 A^3(\tau)\, d\tau \right| &= \left| \int_0^t \int (\tau^2 D^3 G^0 \cdot D^3 h + \bar{h}\tau^2 D^3 G^i \cdot D^3 u^i)\, dx\, d\tau \right| \\
&= \Bigg| \int_0^t \int \tau^2 D^3 h \cdot D^3 [-D(hu)] \\
&\qquad + \bar{h}\tau^2 D^3 u \cdot D^3 \left[-uDu + \nu \frac{DhDu}{h+\bar{h}} \right] dx\, d\tau \Bigg| \\
&\leq \left| \int_0^t \int \tau^2 D^3 h D^4 [hu] \right| \\
&\quad + \left| \bar{h} \int_0^t \int \tau^2 D^3 u D^3 (uDu) \right| \\
&\quad + \left| \bar{h}\nu \int_0^t \int \tau^2 D^3 u D^3 \left[\frac{DhDu}{h+\bar{h}} \right] \right|.
\end{aligned}
$$

See Sundbye [**9**] for estimates of these three integrals. □

Proposition 4.9.

(4.26)
$$\int_0^t \sum_{m=0}^{2} |B^m(\tau)| \leq CN^2(0,t) \sum_{i=1}^{3} (N(0,t))^i,$$

(4.27)
$$\int_0^t \sum_{m=0}^{2} |C^m(\tau)| \leq CN^3(0,t),$$

(4.28)
$$\int_0^t |\tau B^1(\tau)| \leq CN^3(0,t),$$

(4.29)
$$\int_0^t |\tau C^1(\tau)| \leq CN^3(0,t),$$

(4.30)
$$\int_0^t |\tau^2 B^2(\tau)| \leq CN^2(0,t)(N(0,t)+N^2(0,t)),$$

(4.31)
$$\int_0^t |\tau^2 C^2(\tau)| \leq CN^2(0,t)(N(0,t)+N^2(0,t)).$$

Proof. See Sundbye [**9**]. □

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REFERENCES

1. A.T. Bui, *Existence and uniqueness of a classical solution of an initial boundary value problem of the theory of shallow waters*, SIAM J. Math. Anal. **12** (1981), 229–241.

2. P. Kloeden, *Global existence of classical solutions in the dissipative shallow water equations*, SIAM J. Math. Anal. **16** (1985), 301–315.

3. C. Leith, *Nonlinear normal mode initialization and quasi-geostrophic theory*, J. Atmos. Sci. **37** (1980), 958–968.

4. A. Matsumura, *An energy method for the equations of motion of compressible viscous and heat conductive fluids*, MRC Technical Summary Report # 2194, University of Wisconsin-Madison, 1981.

5. A. Matsumura and T. Nishida, *The initial value problem for the equations of motion of viscous and heat-conductive gases*, J. Math. Kyoto Univ. **20** (1980), 67–104.

6. ———, *Initial boundary value problems for the equations of motion of general fluids*, in *Computing methods in applied sciences and engineering*, (R. Glowinski and F. Lions, eds.), North Holland, Amsterdam, 1982., 389–406.

7. ———, *Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids*, Comm. Math. Phys. **89** (1983), 445–464.

8. R. Racke, *Lectures on nonlinear evolution equations*: *Initial value problems*, in *Aspects of mathematics*, Vieweg & Sohn, Wiesbaden, 1992.

9. L. Sundbye, *Global existence of solutions for the viscous shallow water equations*, Ph.D. Thesis, University of Colorado, Boulder, Colorado, 1994.

10. ———, *Global existence for the Dirichlet problem for the viscous shallow water equations*, J. Math. Anal. Appl. **202** (1996), 236–258.

11. R. Temam, *Inertial manifolds*, Math. Intel. **12** (1990), 68–74.

12. S. Zheng, *Remarks on global existence for nonlinear parabolic systems*, Nonlinear Anal. **10** (1986), 107–114.

13. ———, *Global existence of solutions to initial boundary value problem for nonlinear parabolic equation*, Acta Math. Sinica **4** (1987), 361–374.

14. S. Zheng and Y. Chen, *Global existence for nonlinear parabolic equations*, Chin. Ann. Math. Ser. B **7** (1986), 57–73.

15. S. Zheng and W. Shen, *Global solutions to the Cauchy problem of quasi-linear hyperbolic parabolic coupled systems*, Sci. China Ser. A **30** (1987), 1133–1149.

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