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SOME QUESTIONS AND CONJECTURES IN THE THEORY OF UNIVALENT FUNCTIONS

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ABSTRACT. The main object of this paper is first to answer a question of Campbell and Singh in the affirmative, and then to show that Komatu's conjecture and Thomas's conjecture are false at least in some cases.

1. Introduction. Let $\mathcal{S}$ denote the class of *normalized* analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are univalent in the *open* unit disk $\mathcal{U}$. Also let $\mathcal{S}_R$ denote its subclass consisting of functions with real coefficients. The set of all odd functions in $\mathcal{S}$ is denoted by $\mathcal{S}^{(2)}$.

A function $f \in \mathcal{S}$ is said to be starlike of order α, denoted by $f \in \mathcal{S}^*(\alpha)$, if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad 0 \le \alpha < 1; \quad z \in \mathcal{U}.$$

Let

$$\mathcal{S}^*(0) = \mathcal{S}^* \quad \text{and} \quad \mathcal{K} = \{f : zf'(z) \in \mathcal{S}^*\}.$$

In this paper we first answer a question of Campbell and Singh [**1**] in the affirmative. We then show that Komatu's conjecture [**5**] and Thomas's conjecture [**7**, p. 166] are false at least in some cases.

2. A question of Campbell and Singh. Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$$

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and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}.$$

Define their integral convolution $f \otimes g$ by

$$F(z) = (f \otimes g)(z) := z + \sum_{n=2}^{\infty} a_n b_n \frac{z^n}{n}.$$

Campbell and Singh [**1**] asked the following question:

Do there exist univalent functions $f(z)$ and $g(z)$ such that the coefficients of

$$F(z) = (f \otimes g)(z) = z + c_2 z^2 + c_3 z^3 + \cdots$$

satisfy

$$|c_3 - c_2^2| > 1?$$

Here we give an explicit example which leads to an affirmative answer to this question.

Let α be a real number with $0 < |\alpha| < (1/2)\pi$. It is known that the function

$$f(z) = z(1-z)^{-2e^{i\alpha}\cos\alpha} \tag{1}$$

is α-spirallike but not close-to-convex [**2**, p. 72].

Consider the rotation of the function (1), and set

$$f(z) = z(1 - e^{-i\theta} z)^{-2e^{i\alpha}\cos\alpha}. \tag{2}$$

Now choose

$$\alpha = \frac{1}{2}\arg(15 + 8i)$$

and

$$\theta = \arg(15 + 8i), \tag{3}$$

and expand the corresponding function (2) in powers of z; we thus obtain

$$f(z) = z + \left(\frac{32}{17} - \frac{8}{17}i\right)z^2 + \left(\frac{688}{289} - \frac{444}{289}i\right)z^3 + \cdots . \tag{4}$$

We consider the integral convolution of the function (4) with itself, that is,

$$F(z) = (f \otimes f)(z) = z + c_2 z^2 + c_3 z^3 + \cdots ;$$

then

$$c_2 = \frac{32}{289}(15 - 8i)$$

and

$$c_3 = \frac{16}{250563}(17263 - 38184i). \tag{5}$$

Therefore,

$$|c_3 - c_2^2| = \frac{1}{250563}|218384 - 126336i| = 1.0069\cdots > 1. \tag{6}$$

We denote the set of all α-spirallike functions by $\mathcal{SP}(\alpha)(-(1/2)\pi < \alpha < (1/2)\pi)$. It is known that the class of close-to-convex functions and $\mathcal{S}_R$ are not closed under the operation $\otimes$ [**2**, p. 247]. In contrast with Ruscheweyh and Sheil-Small's proof, cf. [**8**], of the Pólya-Schoenberg conjecture that $\mathcal{S}^* = \mathcal{SP}(0)$ is closed under the operation $\otimes$, we should like to point out that the above discussion yields the following result.

Theorem 1. *For each* α $(\alpha \neq 0; |\alpha| < \alpha_0 = 0.3138\cdots (= 17.986\cdots \text{degrees}))$, *there exist functions* $f(z)$ *and* $g(z)$ *in the class* $\mathcal{SP}(\alpha)$ *such that the coefficients of*

$$F(z) = (f \otimes g)(z) = z + c_2 z^2 + c_3 z^3 + \cdots$$

satisfy

$$|c_3 - c_2^2| > 1,$$

where

$$(\tan \alpha_0)^2 = 0.1054\cdots$$

is the smallest positive root of the equation:

$$9x^3 + 36x^2 + 53x - 6 = 0. \tag{7}$$

In fact, if we let $f(z) = g(z)$ be the function given by (2) with $\theta = 2\alpha$ and $\gamma = \tan\alpha$, then

$$|c_3 - c_2^2| = \frac{\{4\gamma^2 + (1 + 3^{-1}\gamma^2)^2\}^{1/2}}{(1+\gamma^2)^2}, \tag{8}$$

and the assertion of Theorem 1 is readily established by letting $\gamma^2 = x$.

3. A conjecture of Komatu. Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$$

and define the function $f_\lambda(z)$ by

$$f_\lambda(z) = z + \sum_{n=2}^{\infty} a_n n^{-\lambda} z^n. \tag{9}$$

Komatu [**5**] proved that, if $f \in \mathcal{S}$, then $f_\lambda \in \mathcal{S}^*$ at least for $\lambda \geq \lambda_0$, where $\lambda_0 \in (3, 4)$ is the unique root of the equation $\zeta(\lambda - 2) = 2$ (ζ being the Riemann zeta function), and conjectured that

(i) If $f \in \mathcal{S}$, then $f_\lambda \in \mathcal{S}$ at least for $\lambda \geq 1$;

(ii) If $f \in \mathcal{K}$ (or, more generally, $f \in \mathcal{S}^*$), then $f_\lambda \in \mathcal{K}$ at least for $\lambda \geq 1$.

Lewis [**6**] showed that Conjecture (ii) is true. In the case $\lambda = 1$, Conjecture (i) reduces to the Biernacki conjecture which is false [**2**, p. 257]. We shall show that Conjecture (i) is also false in the case $\lambda = 2$. Further remarks on Komatu's conjectures can be found in the work of Srivastava and Owa [**9**, p. 82].

Theorem 2. *There exists a function* $f \in \mathcal{S}$ *such that* $f_1(z)$ *and* $f_2(z) \notin \mathcal{S}$.

Proof. For the function (1), one can obtain

$$f_1(z) = e^{-2i\alpha}\{(1-z)^{-e^{2i\alpha}} - 1\} \\ = e^{-2i\alpha}\{\exp(-e^{2i\alpha}\log(1-z)) - 1\}. \tag{10}$$

The geometric property of $\exp(\beta\log(1-z))$ shows that $f_1(z)$ is univalent in $\mathcal{U}$ if and only if β is either in the closed disk $|\beta - 1| \le 1$ or in the closed disk $|\beta + 1| \le 1$, cf. [**3**, Vol. 2, p. 150].

Since $\beta = -e^{2i\alpha}$, we deduce that, if

$$\frac{\pi}{6} < |\alpha| < \frac{\pi}{3}, \tag{11}$$

then $f_1(z) \notin \mathcal{S}$.

From the definition (9), we have

$$zf_2'(z) = f_1(z). \tag{12}$$

For the function (1) with $\alpha = \pi/4$, we have

$$zf_2'(z) = f_1(z) = -i\{(1-z)^{-i} - 1\}.$$

Hence

$$z_0 f_2'(z_0) = 0 \quad \text{at} \quad z_0 = 1 - e^{-2\pi} \in \mathcal{U}.$$

This shows that $f_2(z) \notin \mathcal{S}$, which completes the proof of Theorem 2. □

4. A conjecture of Thomas. Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}.$$

The Hankel determinant of the coefficients of $f(z)$ is denoted by $H_q(n)$, where

$$H_1(n) = a_n a_{n+2} - a_{n+1}^2.$$

For $f \in \mathcal{S}$, Thomas [**7**, p. 166] conjectured that

$$|H_1(n)| \le 1, \quad n = 2, 3, 4, \dots .$$

It follows from this conjecture that, if

$$h(z) = z + \sum_{k=1}^{\infty} c_{2k+1} z^{2k+1} \in \mathcal{S}_R \cap \mathcal{S}^{(2)}, \tag{13}$$

then

$$|c_{2k+1}| \leq 1 \quad \text{and} \quad |c_{2k+1} c_{2k+3}| \leq 1, \quad k = 1, 2, 3, \dots .$$

Recently, Jakubowski [**4**] proved that, if $h \in \mathcal{S}_R \cap \mathcal{S}^{(2)}$, then $|c_3 c_5| \leq 1$. However, the elementary counterexample of Shaeffer and Spencer [**2**, p. 107] shows that, for $h \in \mathcal{S}_R \cap \mathcal{S}^{(2)}$, the above conclusion is false in the cases when $k \geq 2$. Therefore, Thomas's conjecture is also false in the cases when $n \geq 4$.

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