ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 3, Fall 1998

MULTIPLE POSITIVE SOLUTIONS FOR HIGHER ORDER BOUNDARY VALUE PROBLEMS

ERIC R. KAUFMANN

ABSTRACT. Multiple positive solutions are shown to exist for the boundary value problem $u^{(n)} + f(t,u) = 0$, $\alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0$, $\gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0$, $u^{(i)}(0) = 0$, $0 \le i \le n-3$, when f is sublinear at one end point (zero or infinity) and superlinear at the other. The methods involve applications of a fixed point theorem for operators on a cone in a Banach space.

1. Introduction. In this paper we consider the two-point boundary value problem,

$$u^{(n)} + f(t,u) = 0, \quad 0 \le t \le 1, \tag{1}$$

$$\begin{gathered} \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \\ \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0, \\ u^{(i)}(0) = 0, \quad 0 \le i \le n-3, \end{gathered} \tag{2}$$

where $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, $\alpha, \beta, \gamma, \delta \ge 0$, and $\rho = \beta\gamma + \alpha\gamma + \alpha\delta > 0$. Notice if $u(t)$ is a nonnegative solution of (1), (2), then $u^{(n-2)}(t)$ is concave on $[0,1]$.

When $n = 2$ the boundary value problem (1), (2), arises in nonlinear elliptical equations on an annulus, see [**2, 3, 11, 13, 14**]. In many physical and biological problems only positive solutions are of interest. Cones provide an elegant means to define positive elements in a Banach space. In [**4**] and [**5**] fixed point theorems with respect to a cone were used to find positive solutions for higher order boundary value problems. For a thorough treatment of cones in a Banach space, see Deimling [**6**] or Krasnosel'skii [**12**].

Received by the editors on October 25, 1995, and in revised form on August 26, 1996.
1991 AMS *Mathematics Subject Classification.* 34B15, 34B27.
Key words and phrases. Boundary value problems, cones in a Banach space, Green's functions, positive solutions.

Our goal is to extend the work of Erbe, Hu, and Wang [**9**] to obtain two positive solutions of (1) and (2), when f is superlinear at one endpoint (zero or infinity) and sublinear at the other; that is, when either

(i) $f_{0,m} = +\infty$ and $f_{\infty,m} = +\infty$, or

(ii) $f_{0,M} = 0$ and $f_{\infty,M} = 0$,

where,

$$f_{0,m} = \lim_{x \to 0^+} \min_{0 \le t \le 1} \frac{f(t,x)}{x},$$

$$f_{\infty,m} = \lim_{x \to +\infty} \min_{0 \le t \le 1} \frac{f(t,x)}{x},$$

$$f_{0,M} = \lim_{x \to 0^+} \max_{0 \le t \le 1} \frac{f(t,x)}{x},$$

and,

$$f_{\infty,M} = \lim_{x \to +\infty} \max_{0 \le t \le 1} \frac{f(t,x)}{x}.$$

We will make our assumptions on f more precise in Section 3. The results herein are also related to those by Atici [**1**] and Eloe and Henderson [**8**].

In Section 2 we present some preliminary results involving the Green's function of (1) and (2). We also state a fixed point theorem due to Krasnosel'skii [**12**] which will be used to yield multiple positive solutions of (1) and (2). In Section 3 we provide an appropriate Banach space and cone in order to apply the fixed point theorem to obtain solutions of (1) and (2).

2. Preliminaries. In this section we state a fixed point theorem due to Krasnosel'skii which utilizes cones in a Banach space. Our cone will be constructed based on properties of the Green's function, $G(t,s)$, for the boundary value problem,

$$-u^{(n)} = 0, \quad 0 \le t \le 1, \tag{3}$$

$$\begin{aligned} \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) &= 0, \\ \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) &= 0, \\ u^{(i)}(0) = 0, \quad 0 \le i &\le n-3. \end{aligned} \tag{4}$$

It can be shown, using arguments similar to those in Eloe [**7**], that

$$\frac{\partial^i}{\partial t^i} G(t,s) > 0, \quad (0,1)\times(0,1), \quad 0 \le i \le n-3,$$

and

$$\frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) = K(t,s),$$

where $K(t,s)$ is the Green's function for

$$-u'' = 0, \quad 0 \le t \le 1, \tag{5}$$

$$\begin{aligned} \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0. \end{aligned} \tag{6}$$

In [**10**] it was shown that $K(t,s)$ satisfies

$$0 \le K(t,s) \le K(s,s), \qquad 0 \le t,s \le 1, \tag{7}$$

as well as

$$\frac{K(t,s)}{K(s,s)} \ge \sigma, \quad \frac{1}{4} \le t \le \frac{3}{4}, \quad 0 \le s \le 1, \tag{8}$$

where $\sigma = \min\{(\gamma+4\delta)/(4(\gamma+\delta)), (\alpha+4\beta)/(4(\alpha+\beta))\}$.

The existence of the multiple positive solutions of (1) and (2) is based on an application of the following fixed point theorem [**12**].

Theorem 1. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a cone. Assume Ω_1, Ω_2 are bounded open subsets of $\mathcal{B}$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose that*

$$T\colon \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$$

is a completely continuous operator such that, either

(i) $\|Tu\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$, *or*

(ii) $\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$ *and* $\|Tu\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Multiple positive solutions. In this section we use the fixed point theorem from Section 2 to find two positive solutions of (1) and (2). It is well known that $u(t)$ is a solution of (1) and (2) if and only if

$$u(t) = \int_0^1 G(t,s) f(s, u(s))\, ds, \quad 0 \le t \le 1.$$

Define

$$\mathcal{B} = \{x \in C^{(n-2)}[0,1] : x^{(i)}(0) = 0,\ 0 \le i \le n-3\},$$

with norm $\|x\| = |x^{(n-2)}|_\infty$, where $|\cdot|_\infty$ denotes the supremum norm on $[0,1]$. Then, $(\mathcal{B}, \|\cdot\|)$ is a Banach space.

Remark. For each $x \in \mathcal{B}$, $|x^{(i)}|_\infty \le \|x\|$, $0 \le i \le n-2$.

Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \left\{ x \in \mathcal{B} : x^{(n-2)}(t) \ge 0 \text{ and } \min_{1/4 \le t \le 3/4} x^{(n-2)}(t) \ge \sigma \|x\| \right\}.$$

If $x \in \mathcal{P}$ then, $x^{(i)}(t) \ge 0$, $0 \le i \le n-2$, and $x^{(i)}(t) \ge \sigma\|x\|((t - 1/4)^{n-i-2})/(n-i-2)!$, $1/4 \le t \le 3/4$, $0 \le i \le n-2$. Hence, if $x \in \mathcal{P}$ then,

$$x^{(i)}(t) \ge \frac{\sigma}{(n-i-2)!4^{n-i-2}} \|x\|, \quad \frac{1}{2} \le t \le \frac{3}{4}, \quad 0 \le i \le n-2. \tag{9}$$

Consider the operator $T\colon \mathcal{P} \to \mathcal{B}$ given by

$$Tu(t) = \int_0^1 G(t,s) f(s, u(s))\, ds, \quad 0 \le t \le 1.$$

We seek a fixed point of the operator T in $\mathcal{P}$.

Lemma 2. *The operator T is completely continuous and $T\colon \mathcal{P} \to \mathcal{P}$.*

Proof. Let $u \in \mathcal{P}$. From (7) we have, for $0 \le t \le 1$,

$$\begin{aligned}(Tu)^{(n-2)}(t) &= \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) f(s,u(s))\,ds \\ &= \int_0^1 K(t,s) f(s,u(s))\,ds \\ &\le \int_0^1 K(s,s) f(s,u(s))\,ds,\end{aligned}$$

and, hence,

$$\|Tu\| = |(Tu)^{(n-2)}|_\infty \le \int_0^1 K(s,s) f(s,u(s))\,ds. \tag{10}$$

If $u \in \mathcal{P}$ then, by (8) and (10),

$$\begin{aligned}\min_{1/4 \le t \le 3/4} (Tu)^{(n-2)}(t) &= \min_{1/4 \le t \le 3/4} \int_0^1 K(t,s) f(s,u(s))\,ds \\ &\ge \sigma \int_0^1 K(s,s) f(s,u(s))\,ds \ge \sigma\,\|Tu\|.\end{aligned}$$

Finally from (7), we have $(Tu)^{(n-2)}(t) \ge 0$ for $u \in \mathcal{P}$, and so, $Tu \in \mathcal{P}$. Standard arguments can be used to show that T is completely continuous. This completes the proof of the lemma. □

For our first theorem we will require that f satisfies the following conditions:

(A) $f_{0,m} = +\infty$, $f_{\infty,m} = +\infty$ and

(B) there exists a $p > 0$ such that, if $0 \le x \le p$, $0 \le t \le 1$, then

$$f(t,x) \le \eta p,$$

where

$$\eta = \left(\int_0^1 K(s,s)\,ds \right)^{-1} = \frac{6\rho}{6\delta\beta + 3\gamma\beta + \alpha\gamma + 3\alpha\delta}.$$

Theorem 3. *Assume* $f(t,u)$ *satisfies conditions* (A) *and* (B). *Then, the boundary value problem,* (1), (2), *has at least two positive solutions,* $u_1, u_2 \in \mathcal{P}$, *such that*

$$0 \le \|u_1\| \le p \le \|u_2\|.$$

Proof. Choose $M > 0$ so that

$$\frac{\sigma M}{(n-2)!4^{n-2}} \int_{1/2}^{3/4} K\left(\frac{1}{2}, s\right) ds \ge 1. \tag{11}$$

By condition (A), there exists a $0 < r < p$ such that

$$f(t,u) \ge Mu, \tag{12}$$

for $0 \le u \le r$, $0 \le t \le 1$.

Let $u \in \mathcal{P}$ with $\|u\| = r$. From (9) and (12) we have

$$\begin{aligned}(Tu)^{(n-2)}\left(\frac{1}{2}\right) &= \int_0^1 K\left(\frac{1}{2}, s\right) f(s, u(s))\, ds \\ &\ge M \int_{1/2}^{3/4} K\left(\frac{1}{2}, s\right) u(s)\, ds \\ &\ge \frac{\sigma M}{(n-2)!4^{n-2}} \int_{1/2}^{3/4} K\left(\frac{1}{2}, s\right) ds\, \|u\| \\ &\ge \|u\|.\end{aligned}$$

If we define $\Omega_1 = \{u \in \mathcal{B} : \|u\| < r\}$, then the above argument shows that

$$\|Tu\| \ge \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_1. \tag{13}$$

Now consider $u \in \mathcal{P}$ with $\|u\| = p$. By the remark, $|u|_\infty \le p$, and so, from condition (B),

$$\begin{aligned}(Tu)^{(n-2)}(t) = \int_0^1 K(t,s) f(s,u(s))\, ds &\le \int_0^1 K(s,s) f(s,u(s))\, ds \\ \le \int_0^1 K(s,s)\, ds\, \eta\, p &\le p = \|u\|.\end{aligned}$$

If we define $\Omega_2 = \{u \in \mathcal{B} : \|u\| < p\}$, then

$$\|Tu\| \leq \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_2. \tag{14}$$

Theorem 1, together with (13) and (14), implies that there exists a fixed point, u_1, of T in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. This fixed point satisfies $r < \|u_1\| < p$.

Using condition (A) again, we know there exists an $R_1 > 0$ such that

$$f(t,u) \geq \varepsilon u, \tag{15}$$

for all $u \geq R_1$, where $\varepsilon > 0$ was chosen so that

$$\frac{\sigma\varepsilon}{(n-2)!4^{n-2}} \int_{1/2}^{3/4} K\left(\frac{1}{2}, s\right) ds \geq 1.$$

Set $R = \max\{2p, ((n-2)!4^{n-2}/\sigma)R_1\}$ and pick $u \in \mathcal{P}$ with $\|u\| = R$. Notice, by (9), that $u(t) \geq (\sigma/((n-2)!4^{n-2}))\|u\| \geq R_1$ on $[1/2, 3/4]$. And so,

$$\begin{aligned}
(Tu)^{(n-2)}\left(\frac{1}{2}\right) &= \int_0^1 K\left(\frac{1}{2}, s\right) f(s, u(s))\, ds \\
&\geq \int_{1/2}^{3/4} K\left(\frac{1}{2}, s\right) \varepsilon\, u(s)\, ds \\
&\geq \frac{\sigma\varepsilon}{(n-2)!4^{n-2}} \int_{1/2}^{3/4} K\left(\frac{1}{2}, s\right) ds\, \|u\| \\
&\geq \|u\|.
\end{aligned}$$

Set $\Omega_3 = \{u \in \mathcal{B} : \|u\| < R\}$. Then

$$\|Tu\| \geq \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_3. \tag{16}$$

Theorem 1, together with (14) and (16), implies that there exists a fixed point, u_2, of T such that $p < \|u_2\| < R$ and the proof is complete. □

For our second theorem we will require that f satisfies the following conditions:

(C) $f_{0,M} = 0$, $f_{\infty,M} = 0$, and

(D) there exists a $q > 0$ such that, if $(\sigma/((n-2)!4^{n-2}))q \le x \le q$, $0 \le t \le 1$, then

$$f(t,x) \ge \lambda q,$$

where

$$\lambda = \bigg(\int_{1/2}^{3/4} K\bigg(\frac{1}{2}, s\bigg)\, ds \bigg)^{-1}.$$

Theorem 4. *Assume $f(t,u)$ satisfies conditions* (C) *and* (D). *Then the boundary value problem* (1), (2), *has at least two positive solutions, $u_1, u_2 \in \mathcal{P}$, such that*

$$0 \le \|u_1\| \le q \le \|u_2\|.$$

Proof. From condition (C), there exists an $0 < r < p$ such that $f(t,u) < \eta u$ for all $0 \le u \le r$, $0 \le t \le 1$, where $\eta = (\int_0^1 K(s,s)\, ds)^{-1}$. Define

$$\Omega_1 = \{u \in \mathcal{B} : \|u\| < r\}.$$

For $u \in \mathcal{P} \cap \partial\Omega_1$ we have, by the remark, $|u|_\infty \le \|u\| = r$. Hence,

$$\begin{aligned}(Tu)^{(n-2)}(t) &= \int_0^1 K(t,s) f(s,u(s))\, ds \le \int_0^1 K(s,s)\, \eta\, u(s)\, ds \\ &\le \eta \int_0^1 K(s,s)\, ds \|u\| \le \|u\|.\end{aligned}$$

That is,

$$\|Tu\| \le \|u\| \quad \text{for} \quad u \in \mathcal{P} \cap \partial\Omega_1. \tag{17}$$

Now let

$$\Omega_2 = \{u \in \mathcal{B} : \|u\| < q\}.$$

Notice that, for $u \in \mathcal{P} \cap \partial\Omega_2$,

$$\min_{1/2 \le t \le 3/4} u(t) \ge \frac{\sigma}{(n-2)!4^{n-2}} \|u\| = \frac{\sigma q}{(n-2)!4^{n-2}}.$$

Thus, by (D),

$$(Tu)^{(n-2)}\left(\frac{1}{2}\right) = \int_0^1 K\left(\frac{1}{2}, s\right) f(s, u(s))\, ds$$
$$\geq \int_{1/2}^{3/4} K\left(\frac{1}{2}, s\right) ds\, \lambda\, q \geq q = \|u\|.$$

Hence,

$$\|Tu\| \geq \|u\| \quad \text{on} \quad \mathcal{P} \cap \partial\Omega_2. \tag{18}$$

Returning to condition (C), we know that for any $\varepsilon > 0$, there exists an $M > 0$ such that,

$$f(t, u) \leq M + \varepsilon u \quad \text{for} \quad u \geq 0, \quad 0 \leq t \leq 1.$$

And so

$$(Tu)^{(n-2)}(t) \leq \int_0^1 K(t, s)[M + \varepsilon u(s)]\, ds$$
$$\leq M \int_0^1 K(s, s)\, ds + \varepsilon \int_0^1 K(s, s) u(s)\, ds$$
$$\leq \frac{M}{\eta} + \varepsilon \int_0^1 K(s, s) u(s)\, ds$$

By choosing $\varepsilon > 0$ sufficiently small and $R > M/\eta$ sufficiently large, we have for $u \in \mathcal{P} \cap \partial\Omega_3$

$$\|Tu\| \leq R = \|u\|, \tag{19}$$

where

$$\Omega_3 = \{x \in \mathcal{B} : \|u\| < R\}.$$

Applying Theorem 1 to (17), (18) and (19) yields the desired results. This completes the proof of the theorem. □

As an example, consider the boundary value problem

$$u''' + \frac{1}{3}(u^{1/6} + u^7) = 0, \tag{20}$$

$$\begin{aligned} \frac{6079700}{123517}u'(0) - 49u''(0) &= 0, \\ 7u'(1) + 220u''(1) &= 0, \\ u(0) &= 0. \end{aligned} \tag{21}$$

Note that $f_{0,m} = \lim_{u \to 0^+}(u^{1/6} + u^6)/(3u) = +\infty$ and $f_{\infty,m} = +\infty$. Also

$$\eta \approx 0.701$$

and

$$\sigma \approx 0.624.$$

It can be shown that $f(t,u)$ satisfies condition (B) for $p = 1.003$.

Let $M = 17.90$. Then (11) holds. Using this M we can show that (12) is valid for all $0 \le u \le r$ where

$$r = 8.33 \times 10^{-3}.$$

Hence there is a positive solution, u_1, of (20), (21) which satisfies

$$8.33 \times 10^{-3} \le \|u_1\| \le 1.003.$$

Inequality (15) holds for $u \ge R_1 = 1.94$. From the proof of Theorem 3 we have $R = \max\{2p, 4R_1/\sigma\} \le 12.44$ as an upper bound on our second solution, u_2. Thus,

$$1.003 \le \|u_2\| \le 12.44.$$

The bounds on the norms of the solutions can be improved. Recall that, if $x \in \mathcal{P}$, then

$$x^{(i)}(t) \ge \sigma\|x\|\frac{(t-1/4)^{n-i-2}}{(n-i-2)!}, \quad \frac{1}{4} \le t \le \frac{3}{4}, \quad 0 \le i \le n-2.$$

Fix $0 < \zeta < 1/2$. Then for all $x \in \mathcal{P}$,

$$x^{(i)}(t) \ge \frac{\sigma\zeta^{n-i-2}}{(n-i-2)!}\|x\|, \quad \zeta + \frac{1}{4} \le t \le \frac{3}{4}, \quad 0 \le i \le n-2. \tag{22}$$

Note that $\zeta = 1/4$ in (9). The values of r, R_1 and R change slightly if (22) is used in place of (9) in the proofs of Theorems 3 and 4. For example, if $\zeta = 27/100$, then $r = 8.39 \times 10^{-3}$ and $R = \max\{2p, 1.94/(\sigma\zeta)\} \leq 11.51$ for the boundary value problem (20), (21). Thus the solutions u_1 and u_2 satisfy

$$8.39 \times 10^{-3} \leq \|u_1\| \leq 1.003,$$

and

$$1.003 \leq \|u_2\| \leq 11.51.$$

REFERENCES

1. F. Atici, *Two positive solutions of a boundary value problem for difference equations*, J. Difference Eqns. Appl., to appear.

2. C. Bandle, C.V. Coffman and M. Marcus, *Nonlinear elliptic problems in annular domains*, J. Differential Equations **69** (1987), 322–345.

3. C. Bandle and M.K. Kwong, *Semilinear elliptic problems in annular domains*, J. Appl. Math. Phys. **40** (1989), 245–257.

4. C.J. Chyan, *Positive solution of a system of higher order boundary value problems*, PanAm. Math. J., in press.

5. C.J. Chyan and J. Henderson, *Positive solutions for singular higher order nonlinear equations*, Differential Equations. Dynamical Systems **2** (1994), 153–160.

6. K. Deimling, *Nonlinear functional analysis*, Springer, New York, 1985.

7. P.W. Eloe, *Sign properties of Green's functions for two classes of boundary value problems*, Canad. Math. Bull. **30** (1987), 28–35.

8. P.W. Eloe and J. Henderson, *Positive solutions for higher order differential equations*, Electronic J. Differential Equations (1995), 1–8.

9. L.H. Erbe, S. Hu and H. Wang, *Multiple positive solutions of some boundary value problems*, J. Math. Anal. Appl. **184** (1994), 640–648.

10. L.H. Erbe and H. Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc. **120** (1994), 743–748.

11. X. Garaizar, *Existence of positive radial solutions for semilinear elliptic problems in the annulus*, J. Differential Equations **70** (1987), 69–92.

12. M.A. Krasnosel'skii, *Positive solutions of operator equations*, Fizmatgiz, Moscow, 1962; English translation, P. Noordhoff Ltd., Groningen, The Netherlands, 1964.

13. H. Wang, *On the existence of positive solutions for semilinear elliptic equations in the annulus*, J. Differential Equations **109** (1994), 1–7.

14. J.S.W. Wong, *On the generalized Emden-Fowler equation*, SIAM Rev. **17** (1975), 339–360.

Department of Mathematics and Statistics, 2801 S. University, University of Arkansas at Little Rock, Little Rock, Arkansas 72204-1099
E-mail address: erkaufmann@ualr.edu