

TIME-MAP TECHNIQUES FOR SOME BOUNDARY VALUE PROBLEMS

WALTER DAMBROSIO

ABSTRACT. We illustrate the classical time-map technique for the study of both Dirichlet and periodic boundary value problems for second order ordinary differential equations. We also present a new technique for Picard BVPs, which is then applied to prove existence results for superlinear problems.

1. Introduction. The aim of this paper is to illustrate the concept of time-map and its applications to various boundary value problems associated to ordinary differential equations. To do this, we first survey some classical results; secondly, a new time-map technique is developed and applied together with some recent continuation theorems in order to give existence and multiplicity results for superlinear problems.

The notion of time-map arises from simple considerations of phase-plane analysis for an autonomous equation like

$$(1.1) \quad u'' + g(u) = 0$$

with g continuous. It is well known, see, e.g., [27], that for equation (1.1) we can write the conservation of the energy

$$(1.2) \quad H(u, u') = \frac{1}{2}(u')^2 + G(u) = \text{const.},$$

where $G(u) = \int_0^u g(t) dt$ is the potential energy. Under suitable sign and growth conditions on g , see Section 2.1, in the phase-plane $(x, y) = (u, u')$ the level sets of the function H are closed curves surrounding the origin. According to (1.2), for every $\alpha > 0$ the time $\tau(\alpha; x, y)$ needed to a solution corresponding to the $\alpha^2/2$ energy-level to rotate in the phase-plane from a point of abscissa x to a point of abscissa y is given by

$$\tau(\alpha; x, y) = \int_x^y \frac{1}{\sqrt{\alpha^2 - 2G(s)}} ds.$$

Received by the editors on February 17, 1993, and in revised form July 10, 1997.

Copyright ©1998 Rocky Mountain Mathematics Consortium

The function $\alpha \mapsto \tau(\alpha; x, y)$ is called time-map associated to equation (1.1). According to the boundary conditions that one deals with, a different choice of x and y is needed in order to describe, by means of τ , existence (and/or bifurcation) of solutions of the examined boundary value problem.

The structure of the paper is as follows.

Section 2 is devoted to the definition and main properties of the time-map and to a survey of the various (classical and more recent) applications of this notion. In particular, in Section 2.1 we report an important result contained in the pioneering work of Z. Opial [24], where the asymptotic behavior of the time-map under a sign condition only is described (see Theorem 2.3).

In Section 2.2, following the approach of [3], we explain how the time-map can be used in order to introduce a subset \mathcal{F} of \mathbf{R}^2 , called “generalized Fučík spectrum,” that enables us to characterize the solutions of (1.1) together with two-point homogeneous boundary conditions

$$(1.3) \quad u(0) = u(\pi) = 0$$

and to prove a classical result for (1.1)–(1.3) (see Theorem 2.5) when g is superlinear, i.e.,

$$\lim_{|u| \rightarrow +\infty} \frac{g(u)}{u} = +\infty.$$

In Section 2.3 we produce a result due to Z. Opial [23], based on the time-map introduced before, for the existence of solutions of a periodic problem of the form

$$(1.4) \quad \begin{cases} u'' + g(u) = p(t) \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases}$$

where the function p is T -periodic (for some $T > 0$). Afterwards, we quote the works of several authors, e.g., [6, 7, 10, 13, 22, 26] who, in more recent years, improved the original result of Z. Opial for problem (1.4). These papers are based on refined time-map techniques and on conditions on the nonlinearity g that ensure the “nonresonance” of the autonomous equation

$$u'' + g(u) = 0$$

with the classical Fučík spectrum [14].

In Section 2.4 we conclude this brief overview on the use of the time-map by describing some bifurcation results [28, 30]. In these papers the point of view is totally different and bifurcation points of a problem like

$$\begin{cases} u'' + f(u) = 0 \\ u(0) = u(\lambda) = 0 \end{cases}$$

are obtained by the search of critical points of the time-map.

The rest of the paper (Sections 3 to 5) is devoted to the study of a problem of the form

$$(1.5) \quad \begin{cases} u''(t) + f(u(t)) = p(t, u(t), u'(t)) \\ u(0) = A, \quad u(\pi) = B \end{cases}$$

where $f : \mathbf{R} \rightarrow \mathbf{R}$ is continuous and superlinear, $p : [0, \pi] \times \mathbf{R}^2 \rightarrow \mathbf{R}$ is continuous and satisfies a linear growth condition in the last two arguments, and $(A, B) \in \mathbf{R}^2$.

A problem of this type has been considered in the last few years, among others, by A. Capietto, J. Mawhin and F. Zanolin (see [3, 4, 5] and the expository paper [21]) who used topological methods based on continuation theorems for a coincidence equation of the form

$$(1.6) \quad Lu = N(u, \lambda),$$

with the parameter λ varying in $[0, 1]$, and on the use of a functional φ “evaluated” on the solutions of (1.6).

The idea of using such a functional (which reduces to the winding number of a curve in some concrete problems) has been recently developed also by M. Furi and M. P. Pera, see, e.g., [15] and references therein.

The results in [1, 3, 4, 5] are based on the reduction, through a suitable homotopy, of problem (1.5) to a simplified problem which can be studied with classical time-map techniques. In this way, in [5] the existence of two sequences of solutions of (1.5) with prescribed nodal properties was proved.

In Section 3, we study autonomous problems of the form

$$(1.7) \quad \begin{cases} u''(t) + g(u(t)) = 0 \\ u(0) = A, \quad u(\pi) = B \end{cases}$$

by means of a new technique based on the use of the time-map. In order to study (1.7) we need to introduce *three* time-maps T_1 , T_2 and T_3 that describe the solutions of our problem; more precisely, if $T(\alpha) = (T_1(\alpha), T_2(\alpha), T_3(\alpha))$ for some $\alpha > 0$, we will prove that there exists a set $S \subset \mathbf{R}^3$ (S consists of four families of planes) such that (1.7) has a solution if and only if $T(\alpha) \in S$. This set S is a three-dimensional variant of the two-dimensional Fučík spectrum [14]. The set S leads to the consideration of some regions of the space where we can compute the degree of a suitable map whose zeros correspond to (initial values of) solutions to problem (1.7). In view of the homotopy used in Section 5, it is useful to pay attention to the section of S with the plane $x = z$ in \mathbf{R}^3 , which corresponds to the case $B = -A$. This construction enables us to reduce (1.5) to an autonomous problem by means of a homotopy different from the one used in [5]. In that paper, the authors use a homotopy such that, for $\lambda = 0$, the boundary conditions are homogeneous and the degree associated to the autonomous problem is always different from zero; here we reduce, for $\lambda = 0$, to the boundary condition

$$u(0) = A, \quad u(\pi) = -A$$

which simplifies the structure of the set S , as above described, even if we still have to deal with some regions with zero degree (cf. Figure 2).

The technique based on the three time-maps introduced here has various applications. In this paper we use it for the study of problem (1.5) (in order to give another proof of the result of [5]) and for the discussion of the existence of positive solutions for an autonomous superlinear boundary value problem (see Section 4); we also refer to the paper [2] where an asymmetric nonlinearity is considered.

In Section 4, using the time-maps introduced in Section 3, we deal with a problem like

$$(1.8) \quad \begin{cases} u'' + f(u) = 0 \\ u(0) = tA, \quad u(\pi) = tB \end{cases}$$

where t , A and B are positive real numbers and f is a suitable superlinear map. Then (see Theorem 4.1) we prove that there exists $\bar{t} > 0$ such that problem (1.8) has a solution without zeros and with a prescribed number of maxima if $t < \bar{t}$. The proof of this result involves a detailed study of the time-maps $T_i(\alpha)$, $i = 1, 2, 3$; to do this we have to extend

to our situation the results of Z. Opial on the asymptotic behavior of the time-maps (see also Lemma 3.2).

In Section 5 we finally get to the study of the nonautonomous problem (1.5). Our goal is to obtain the existence of two sequences of solutions, as in [5]. However, in that paper the existence of two solutions was proved with a modification of the original functional φ . In our approach, inspired by [1], we do not modify the functional and we consider a slightly different continuation theorem due to M. Henrard [16] which enables us to find the two solutions through the successive application of this theorem to two disjoint open sets, both containing a solution.

Finally, we point out that in [3, 4, 5] the degree that leads to the existence of solutions for (1.5) is computed only taking into account the regions where the degree corresponding to the autonomous problems is not zero; in our paper, we show that it is possible to reach the result exploiting also the regions of degree zero (see Lemma 5.9). This method of computation of the degree seems to be of some use; it is also applied in the forthcoming paper [2].

2. A survey on some time-map techniques for boundary value problems. In this section we introduce the notion of time-map (as in the work of Z. Opial [24]) and we illustrate some applications to the study of boundary value problems. The aim of the section is to explain how, in the last years, a very simple tool, the time-map, has been used in different applications. For others applications and a more general point of view, we refer to the book [28].

2.1. Definition of time-map and asymptotic properties. Let us start with an equation of the form

$$(2.1) \quad u'' + g(u) = 0$$

where $g : \mathbf{R} \rightarrow \mathbf{R}$ is continuous. Let us set $G(u) = \int_0^u g(s) ds$ and suppose

$$(2.2) \quad g(u)u > 0, \quad \forall u \neq 0$$

and

$$(G) \quad \lim_{|u| \rightarrow +\infty} G(u) = +\infty.$$

We observe that condition (2.2) implies that the “potential energy” G is bounded below, so that every Cauchy problem for equation (2.1) has exactly one global solution.

As in [24], we give the following:

Definition 2.1. We call *time-map* associated to equation (2.1) the function

$$(2.3) \quad \tau_g(\alpha) = \frac{\sqrt{2}}{2} \left| \int_0^\alpha \frac{1}{\sqrt{G(\alpha) - G(s)}} ds \right|,$$

defined for every $\alpha \in \mathbf{R}$, $\alpha \neq 0$.

We remark that if we make the weaker hypothesis

$$\lim_{|x| \rightarrow +\infty} g(x) \operatorname{sgn}(x) = +\infty,$$

then the function τ_g is defined only for sufficiently large α .

If we write equation (2.1) as an autonomous system, then, under our assumptions, the solutions are closed curves in the phase-plane $(x, y) = (u, u')$ and they lie on the level sets of the energy

$$H(u, u') = \frac{1}{2}(u')^2 + G(u)$$

of the system.

For every $\alpha \in \mathbf{R}$, $\alpha \neq 0$, the function $\alpha \mapsto \tau_g(\alpha)$ represents the time needed to a solution u of energy $\alpha^2/2$, i.e., to a solution such that

$$H(u, u') = \frac{1}{2}\alpha^2,$$

to rotate, in the upper half-plane if $\alpha > 0$, in the lower half-plane if $\alpha < 0$, from the point $(0, \alpha)$ to the point $(C(\alpha), 0)$, where

$$G(C(\alpha)) = \frac{1}{2}\alpha^2, \quad \operatorname{sgn} C(\alpha) = \operatorname{sgn} \alpha.$$

With this notation, the time map (2.3) can also be written as

$$\tau_g(\alpha) = \left| \int_0^{C(\alpha)} \frac{1}{\sqrt{\alpha^2 - 2G(s)}} ds \right|.$$

We set, for every $\alpha > 0$,

$$\tau_g^+(\alpha) = \int_0^{C_2(\alpha)} \frac{1}{\sqrt{\alpha^2 - 2G(s)}} ds$$

and

$$\tau_g^-(\alpha) = \int_{-C_1(\alpha)}^0 \frac{1}{\sqrt{\alpha^2 - 2G(s)}} ds$$

where $C_1(\alpha) > 0$ and $C_2(\alpha) > 0$ are such that

$$G(C_2(\alpha)) = \frac{1}{2}\alpha^2 = G(-C_1(\alpha)).$$

In [24], the author proved the following theorems that illustrate the asymptotic behavior of the time-maps:

Theorem 2.2. *If $g(u)/u$ is increasing (decreasing) in $(0, +\infty)$, then $\tau_g^+(\alpha)$ is decreasing (increasing) for $\alpha > 0$.*

If $g(u)/u$ is decreasing (increasing) in $(-\infty, 0)$, then $\tau_g^-(\alpha)$ is decreasing (increasing) for $\alpha > 0$.

Theorem 2.3. *The following relations hold:*

(i)

$$\begin{aligned} \lim_{u \rightarrow \pm\infty} \frac{g(u)}{u} &= k^\pm, \quad 0 \leq k^\pm \leq +\infty, \\ \implies \lim_{\alpha \rightarrow +\infty} \tau_g^\pm(\alpha) &= \frac{\pi}{2\sqrt{k^\pm}}; \end{aligned}$$

(ii)

$$\begin{aligned} \lim_{u \rightarrow 0^\pm} \frac{g(u)}{u} &= h^\pm, \quad 0 \leq h^\pm \leq +\infty, \\ \implies \lim_{\alpha \rightarrow 0} \tau_g^\pm(\alpha) &= \frac{\pi}{2\sqrt{h^\pm}}. \end{aligned}$$

Remark 2.4. We emphasize that Theorem 2.2 and Theorem 2.3 hold only under the conditions (2.2) and (G) and that these results are independent of (possible) boundary conditions associated with equation (2.1).

2.2. *An application to an autonomous Dirichlet problem.* Now, moving away from the papers of Z. Opial, where a periodic problem was studied, let us consider equation (2.1) together with two-point homogeneous boundary conditions

$$(2.4) \quad u(0) = u(\pi) = 0.$$

Because of the meaning of the time-map in the phase-plane, it is easy to check that a solution $u(\cdot; \alpha)$ of the initial value problem

$$\begin{cases} u'' + g(u) = 0 \\ u(0) = 0, \quad u'(0) = \alpha \end{cases}$$

is a solution of the boundary value problem (2.1)–(2.4) if and only if there exist two integers $m, n \in \mathbf{N}$ with $|m - n| \leq 1$ such that

$$2m\tau_g^+(\alpha) + 2n\tau_g^-(\alpha) = \pi.$$

This leads to consideration of the following subset of \mathbf{R}^2 [4]:

$$\mathcal{F} = \{(x, y) \in \mathbf{R}^2 : x \geq 0, y \geq 0, 2mx + 2ny = \pi \\ \text{for some integers } m, n \in \mathbf{N} \text{ with } |m - n| \leq 1\},$$

called the “generalized Fučík spectrum” for problem (2.1)–(2.4). Then problem (2.1)–(2.4) has a solution of energy $\alpha^2/2$ if and only if

$$(\tau_g^+(\alpha), \tau_g^-(\alpha)) \in \mathcal{F}.$$

The name “generalized Fučík spectrum” is due to the fact that the set \mathcal{F} reduces to the well-known Fučík spectrum $\{(\mu, \nu) \in \mathbf{R}^2 : \mu > 0, \nu > 0, m/\sqrt{\mu} + n/\sqrt{\nu} = 1 \text{ for some integers } m, n \in \mathbf{N} \text{ with } |m - n| \leq 1\}$ in the particular case of $g(u) = \mu u^+ - \nu u^-$, where $u^+ = \max(u, 0)$, $u^- = \max(-u, 0)$ and $\mu > 0, \nu > 0$, see [14].

By means of these considerations we can prove an existence theorem for problem (2.1)–(2.4) with

$$\lim_{|u| \rightarrow +\infty} \frac{g(u)}{u} = +\infty,$$

i.e., a “superlinear” problem. We observe that this superlinear condition implies that

$$\lim_{|u| \rightarrow +\infty} g(u) \operatorname{sgn}(u) > 0,$$

so that the time-maps $\tau_g^\pm(\alpha)$ are defined for sufficiently large α . Since a function $u(\cdot; \alpha)$ is a solution of (2.1)–(2.4) if and only if $(\tau_g^+(\alpha), \tau_g^-(\alpha)) \in \mathcal{F}$, the solutions of (2.1)–(2.4) correspond to the intersections between the lines belonging to the set \mathcal{F} and the support of the curve in \mathbf{R}^2 defined by

$$\tau : \alpha \longmapsto (\tau_g^+(\alpha), \tau_g^-(\alpha)).$$

We know that the set \mathcal{F} consists of infinitely many straight lines; moreover, let us denote by $P_m(\pi/(2m), 0)$ and $Q_n(0, \pi/(2n))$ the intersections of each line with the coordinate axes. Then, the distances between P_m and $O(0, 0)$ and between Q_n and $O(0, 0)$ tend to 0 when m and n go to infinity. This means that every neighborhood of $(0, 0)$ in \mathbf{R}^2 contains infinitely many points of \mathcal{F} .

Using Theorem 2.3, in the superlinear case we deduce that

$$\lim_{\alpha \rightarrow +\infty} \tau_g^+(\alpha) = \lim_{\alpha \rightarrow +\infty} \tau_g^-(\alpha) = 0.$$

Thus, the support of the curve τ crosses infinitely many times the set \mathcal{F} before getting to $(0, 0)$. Every intersection gives rise to a solution of (2.1)–(2.4). Thus, we have proved the following classical “phase-plane analysis” result:

Theorem 2.5. *If*

$$\lim_{|u| \rightarrow +\infty} \frac{g(u)}{u} = +\infty$$

then problem (2.1)–(2.4) has infinitely many solutions. Moreover

$$\lim_{|(u, u')| \rightarrow +\infty} H(u, u') = +\infty.$$

The importance of the time-map does not come down to the possibility of proving Theorem 2.5. In the next sections, we will see how to extend the notion of time-map, in order to prove the existence of infinitely many solutions for some two-point nonhomogeneous nonautonomous boundary value problems as well.

2.3. An application to a nonautonomous periodic problem. In this subsection, we present a classical approach [23] to a nonautonomous periodic problem. The main tool used is the time-map associated to the autonomous problem and introduced in Section 2.1. We will see then how the original result of Z. Opial has been improved in more recent years by several authors, see, e.g., [6, 7, 8, 13, 26].

Let us consider the equation

$$(2.5) \quad u'' + g(u) = p(t)$$

with $g : \mathbf{R} \rightarrow \mathbf{R}$ and $p : \mathbf{R} \rightarrow \mathbf{R}$ continuous. Let us assume condition (2.2) and

$$(2.6) \quad \lim_{|u| \rightarrow +\infty} g(u) = +\infty.$$

Let us look for T -periodic (for some $T > 0$) solutions of (2.5), i.e., for solutions of (2.5) satisfying the boundary conditions

$$(2.7) \quad u(0) - u(T) = u'(0) - u'(T) = 0.$$

Using the previously defined maps $\tau_g^+(\alpha)$ and $\tau_g^-(\alpha)$, we set

$$\tau_{g,+} = \liminf_{\alpha \rightarrow +\infty} \tau_g^+(\alpha)$$

and

$$\tau_{g,-} = \liminf_{\alpha \rightarrow -\infty} \tau_g^-(\alpha).$$

Thus, the following holds:

Theorem 2.6 [23]. *Under hypotheses (2.2) and (2.6), if for some $T_0 > 0$ and $T'_0 > 0$ we have*

$$\tau_{g,+} \geq T_0 \quad \text{and} \quad \tau_{g,-} \geq T'_0,$$

then for every $T < T_0 + T'_0$ problem (2.5)–(2.7) has at least one solution for every T -periodic function p .

The proof of this result is based on the concept of index of a vector field with respect to a zero and on the homotopy invariance of this index. It is meaningful to observe that in the proof the author reduced the nonautonomous problem to an autonomous one and used a continuation theorem in order to carry to the starting problem the results for the autonomous problem. This kind of process is still used nowadays in order to obtain existence results for problems like (2.5)–(2.7) via topological degree methods. For the complete proof of Theorem 2.6 we refer to [23]; here we recall only a preparatory lemma where the properties of the time-maps are used.

Lemma 2.7. *Under the hypotheses of Theorem 2.6, if K is an integral curve of equation (2.5), then for every $T < T_0$ we have*

$$T_1 > T/4,$$

being $T_1 = b - a$, where for every $t \in [a, b]$ we have $x(t) > 0$, $y(t) > 0$, $(x(t), y(t)) \in K$.

Proof. We only need to prove the result for a solution $(x(t), y(t)) = (u(t), u'(t))$ of (2.5) such that

$$x(0) = 0, \quad y(T_1) = 0, \quad y(t) > 0 \quad \forall t \in [0, T_1].$$

Under these assumptions, the solution $x(t)$ is increasing in $[0, T_1)$. If we denote by $t = t(x)$ its inverse and write $p(x) = p(t(x))$, then there exists $p > 0$ such that

$$|p(x)| \leq p \quad \forall x \in [0, x_0], \quad x_0 = x(T_1).$$

With respect to the function $y(x) = y(t(x))$, we have

$$y \frac{dy}{dx} = -g(x) + p(x), \quad \forall x \in [0, x_0]$$

and

$$-g(x) - p \leq y(x)y'(x).$$

Integrating, we infer

$$T_1 = \int_0^{x_0} \frac{dx}{y(x)} \geq \frac{1}{\sqrt{2}} \int_0^{x_0} \frac{dx}{\sqrt{G(x_0) - G(x) + p(x - x_0)}};$$

from Theorem 2.3(i) we deduce

$$\begin{aligned} \liminf_{x_0 \rightarrow +\infty} \frac{1}{\sqrt{2}} \int_0^{x_0} \frac{dx}{\sqrt{G(x_0) - G(x) + p(x - x_0)}} \\ = \liminf_{x_0 \rightarrow +\infty} \frac{1}{\sqrt{2}} \int_0^{x_0} \frac{dx}{\sqrt{G(x_0) - G(x)}} \geq \frac{T_0}{4} > \frac{T}{4}, \end{aligned}$$

and the lemma is proved. \square

In order to apply Theorem 2.6, conditions are needed that ensure the validity of the inequalities

$$\tau_{g,+} \geq T_0, \quad \tau_{g,-} \geq T'_0.$$

Using Theorem 2.3 and some considerations on the asymptotic behavior of the time-maps, Z. Opial proved the following:

Theorem 2.8 [23]. *If the function g satisfies conditions (2.2) and (2.6) and if*

$$T < \frac{\pi\sqrt{2}}{\sqrt{k}} + \frac{\pi\sqrt{2}}{\sqrt{h}}$$

where

$$\lim_{x \rightarrow +\infty} \frac{G(x)}{x^2} = k, \quad \lim_{x \rightarrow -\infty} \frac{G(x)}{x^2} = h,$$

then problem (2.5)–(2.7) has a solution for every T -periodic continuous function p .

Starting from the work of Z. Opial, in more recent years several authors studied the existence of solutions of periodic boundary value problems under suitable hypotheses on the nonlinearity. Here, we present some of these results that improve the original one explained above.

In [13], A. Fonda and F. Zanolin, making use of a continuation lemma based on topological degree arguments, proved the following:

Theorem 2.9. *Let $\tau_g^\pm = \limsup_{\alpha \rightarrow \pm\infty} \tau_g^\pm(\alpha)$. Assume that*

$$\lim_{|x| \rightarrow +\infty} g(x) \operatorname{sgn}(x) = +\infty$$

and that

$$(2.8) \quad \tau_{g,-} + \tau_g^+ > T$$

or

$$(2.9) \quad \tau_{g,+} + \tau_g^- > T.$$

Then problem (2.5)–(2.7) has at least one solution for every T -periodic integrable function p .

As already observed, the crucial point in the applicability of Theorem 2.9 is to verify condition (2.8) or (2.9). In order for (2.8) or (2.9) to be fulfilled, we have to give some conditions on the nonlinearity g ; in this direction, a lot of papers [22, 29, 31] deal with the so-called “one-sided growth restrictions.” Following this approach, more recently, first in [10] and then in [8] in a more general setting, the existence of solutions of (2.5)–(2.7) has been proved under conditions like

$$(2.10) \quad \liminf_{x \rightarrow +\infty} \frac{g(x)}{x} = 0, \quad \frac{xg'(x)}{g(x)} \leq M \quad \text{for } x \geq d > 0$$

or

$$(2.11) \quad \liminf_{x \rightarrow +\infty} \frac{2G(x)}{x^2} < \left(\frac{\pi}{T}\right)^2.$$

On the same lines, in [13], the authors proved the following existence result:

Theorem 2.10. *Let us assume*

$$(g(x) - \bar{p}) \geq 0 \quad \text{for } |x| \geq d, \quad \bar{p} = \frac{1}{T} \int_0^T p(x) dx$$

and

$$(2.12) \quad \liminf_{x \rightarrow +\infty} \frac{g(x)}{x} = \rho < \left(\frac{\pi}{T}\right)^2.$$

Let us also suppose that for positive large x the map $g(x) - \rho x$ is nondecreasing. Then problem (2.5)–(2.7) has a solution.

Conditions like (2.10), (2.11) or (2.12) guarantee a “nonresonant” asymptotic behavior of the nonlinearity with respect to the spectrum $\text{spec}_T(-x'')$ of the differential operator $x \mapsto -x''$ subject to T -periodic boundary conditions. The necessity of ensuring such a fact led some authors to a different approach to (2.5)–(2.7), see, e.g., [6, 7, 19, 26].

In [19], the existence of solutions to (2.5)–(2.7) was proved under the nonresonant assumption

$$(2.13) \quad \begin{aligned} \omega^2 j^2 < g_* &= \liminf_{|x| \rightarrow +\infty} \frac{g(x)}{x} \leq \limsup_{|x| \rightarrow +\infty} \frac{g(x)}{x} \\ &= g^* < \omega^2 (j+1)^2, \end{aligned}$$

where $\omega = 2\pi/T$ and $j \in \mathbf{N}$. In more recent years, in [7] T. Ding and W. Ding defined the equation

$$(2.14) \quad u'' + g(u) = 0$$

to be *asymptotically resonant* if

$$\lim_{\alpha \rightarrow +\infty} \tau_g^+(\alpha) = \frac{T}{j}$$

for some integer $j \in \mathbf{N}$. Under the assumptions

(g_1) g globally Lipschitz in \mathbf{R}

(g_2) $g(x)/x \geq \delta > 0$ for $|x| \geq d > 0$,

they proved that (2.5)–(2.7) has a solution if (2.14) is not asymptotically resonant. Again, the problem is to find conditions on g that ensure (2.14) not to be asymptotically resonant. Sufficient conditions are given in [7] and [9]. Under assumptions (g_1) and (g_2), if

$$[g_*, g^*] \cap \text{spec}_T(-x'') \neq \emptyset,$$

then (2.14) is not asymptotically resonant. In this situation, assumptions (g_1) and (g_2) cannot be dropped; without these restrictions on g , in the last few years D. Qian [26] proved the nonasymptotical resonance of (2.14), assuming

$$\limsup_{|x| \rightarrow +\infty} \frac{G(x)}{g(x)^2} < +\infty.$$

We observe that the last condition is satisfied for instance if g is asymptotically linear, i.e.,

$$0 < \liminf_{|x| \rightarrow +\infty} \frac{g(x)}{x} \leq \limsup_{|x| \rightarrow +\infty} \frac{g(x)}{x} < +\infty.$$

2.4. Bifurcation results. Another way of using the concept of time-map has been developed by several authors; we quote, e.g., the lecture note by R. Schaaf [28] and its references. This way consists of the search of critical points of the time-map in order to get bifurcation results for some differential equations.

More precisely, let us consider the following reaction-diffusion problem

$$(2.15) \quad \begin{cases} u_t = u_{xx} + \lambda^2 f(u) & \lambda > 0, \\ u(t, 0) = u(t, 1) = 0 & \forall t. \end{cases}$$

In combustion problems, for instance, it is useful to study an equation of the form

$$(2.16) \quad \begin{cases} u''(x) + \lambda^2 f(u(x)) = 0 & \lambda > 0, \\ u(0) = u(1) = 0 \end{cases}$$

describing intermediate steady-states of (2.15) for the temperature distribution u , where λ measures the amount of unburnt substance. In this context turning points (with respect to the l -direction) of a branch of solutions, i.e., of a connected component of nontrivial solutions, correspond to ignition and extinction of the process, and it is important to know whether they exist or not, see [12]. In order to treat this situation, the time-map related to (2.16) can be used, see [28].

First of all, we can make a scaling setting $x = \lambda t$ and obtaining, from (2.16),

$$(2.17) \quad \begin{cases} u''(t) + f(u(t)) = 0 \\ u(0) = u(\lambda) = 0. \end{cases}$$

Let us suppose that $f(0) = 0$ and $f'(0) > 0$; assume also that

$$f(u) = uf^+(u)$$

with

$$f^+ : (a^-, a^+) \longrightarrow \mathbf{R}^+, \quad a^- < 0 < a^+$$

locally Lipschitz. If $F(u) = \int_0^u f(t) dt$, $b^- = -\sqrt{2F(a^-)}$ and $b^+ = \sqrt{2F(a^+)}$, then, using the fact that the energy is conserved, as in Section 2.1, we can introduce the time-map $\tau(\alpha)$, defined for $\alpha \in (b^-, b^+)$, see [28].

Now, (λ, u) is a positive solution of (2.17) if and only if u is a solution of

$$\begin{cases} u''(t) + f(u(t)) = 0 \\ u(0) = 0, \quad u'(\lambda) = \alpha \end{cases}$$

and

$$\lambda = \tau(\alpha)$$

for some $\alpha > 0$. This means that, in order to study bifurcations of problem (2.17), we have to know the behavior of the map τ ; more precisely, bifurcation points of (2.17) correspond to critical values of τ . In this framework, R. Schaaf proved the following:

Theorem 2.11. *Under the previous assumptions, $(\tau(0), 0)$ is a bifurcation point of (2.17) from the trivial solution set $u \equiv 0$, $\lambda \in \mathbf{R}^+$. Moreover,*

$$\tau(0) = \frac{\pi}{\sqrt{f'(0)}}.$$

Remark 2.12. The second part of the statement of Theorem 2.11 agrees with Theorem 2.3(ii), in the case where $f'(0)$ exists.

2. For the proof of Theorem 2.11 the following integral representation of τ is used:

$$(2.18) \quad \tau(\alpha) = \int_0^\pi C'(\alpha \sin \theta) d\theta,$$

where, as usual, the map C is implicitly defined by

$$F(C(\alpha)) = \frac{1}{2}\alpha^2, \quad \text{sgn } C(\alpha) = \text{sgn } (\alpha).$$

Using Theorem 2.11, in [28] the author studied problem (2.17) with $f^+(u) = e^{-(u-\alpha)^2}$, $\alpha \geq 0$. It can be meaningful to view problem (2.17) as the stationary equation for a population of size u diffusing on $[0, 1]$, which has a hostile environment forcing $u(t, 0) = u(t, 1) = 0$. The quantity $1/\lambda^2$ can be thought of as the diffusion coefficient of the population, whereas $f^+(u)$ models the reproduction rate, i.e., the birth rate minus the death rate. The bifurcation diagram for this stationary equation governs the behavior of the population also for problem (2.15).

From Theorem 2.11, we deduce that $(\pi\sqrt{e^{\alpha^2}}, 0)$ is a bifurcation point for (2.17). If we let $\lambda^* = \min_{\alpha \geq 0} \tau(\alpha) > 0$, then no nontrivial steady-state solutions exist for $\lambda < \lambda^*$. For the study of the stability of the branches of solutions, we refer to [11] and [17].

We end this brief excursus on the use of the time-maps in the study of bifurcations by illustrating a result of J. Smoller and A. Wasserman [30] for problem (2.17) with

$$f(u) = -(u-a)(u-b)(u-c)$$

where $a < b < c$. First of all we notice that, in general, $f(0) \neq 0$; nevertheless, we can define the time-map $\tau(\alpha)$ for $0 < \alpha < \sqrt{2F(c)}$. In [30] the authors proved the following:

Theorem 2.13. *If $f(u) = -(u-a)(u-b)(u-c)$ and $0 \leq a < b < c$ or $a < 0 \leq b < c$, then the map τ has exactly one critical point (a minimum). Then problem (2.17) undergoes exactly one bifurcation.*

The proof of Theorem 2.13 is based on some estimates on the derivative of τ , obtained by the use of the integral representation (2.18).

In order to emphasize the fact that the time-map techniques apply to various problems, we quote the following result [30] for Neumann boundary conditions:

Theorem 2.14. *Let us consider $f(u) = -(u - a)(u - b)(u - c)$ and the problem*

$$(2.19) \quad \begin{cases} u''(t) + f(u(t)) = 0 \\ u'(0) = u'(\lambda) = 0. \end{cases}$$

Then the time-map associated to (2.19) is monotone, so that bifurcation never occurs.

3. Time-maps and computation of the degree for Picard problems. In this section we study the following Picard problem

$$(3.1) \quad \begin{cases} u''(t) + f(u(t)) = 0 \\ u(0) = A, \quad u(\pi) = B, \end{cases}$$

A, B being two real numbers and $f : \mathbf{R} \rightarrow \mathbf{R}$ being a continuous function such that

$$(H1) \quad f(x)x > 0 \quad \text{for all } x \neq 0.$$

As in Section 2.1, we introduce the “potential energy” $F(x) = \int_0^x f(u) du$ and the “global energy” $H(x, y) = y^2/2 + F(x)$. Let us suppose

$$(F) \quad \lim_{|x| \rightarrow +\infty} F(x) = +\infty.$$

For $\alpha > 0$ we define by F^α the sub-levels of energy $\alpha^2/2$, i.e.,

$$F^\alpha = \left\{ (x, y) \in \mathbf{R}^2 : H(x, y) < \frac{1}{2}\alpha^2 \right\}$$

and we denote by $-C_1(\alpha) < 0$ and $C_2(\alpha) > 0$ the two solutions of the equation

$$F(x) = \frac{1}{2}\alpha^2,$$

see Figure 1. Let Γ^α be the boundary of F^α . For α big enough we have

$$(3.2) \quad \min\{C_1(\alpha), C_2(\alpha)\} > \max\{|A|, |B|\}.$$

From now on, we shall assume that α is large enough to satisfy condition (3.2).

Now let us consider a fixed orbit Γ^α and let us assume, without loss of generality, that $B \leq A$. We define, for each energy level Γ^α , the following three time-maps that will enable us to describe the solutions of energy $\alpha^2/2$. Indeed, we set:

$$\begin{aligned} T_1(\alpha) &= \int_A^{C_2(\alpha)} \frac{1}{\sqrt{\alpha^2 - 2F(s)}} ds, \\ T_2(\alpha) &= \int_B^A \frac{1}{\sqrt{\alpha^2 - 2F(s)}} ds, \\ T_3(\alpha) &= \int_{-C_1(\alpha)}^B \frac{1}{\sqrt{\alpha^2 - 2F(s)}} ds. \end{aligned}$$

If $A < B$, analogous definitions can be given by swapping A and B , $T_1(\alpha)$ and $T_3(\alpha)$. As already noticed, $T_1(\alpha)$ is the time needed by a solution of energy $\alpha^2/2$ to rotate in the upper half-plane from the point $(A, \sqrt{\alpha^2 - 2F(A)})$ to the point $(C_2(\alpha), 0)$. The quantities $T_2(\alpha)$ and $T_3(\alpha)$ have a similar meaning. We also remark that the symmetry of the orbits with respect to the x -axis implies that each $T_i(\alpha)$, $i = 1, 2, 3$, is also the time needed for a rotation between the corresponding points in the half-plane $y < 0$.

Let $P_1(A, \sqrt{\alpha^2 - 2F(A)})$ be one of the intersection points between Γ^α and the straight line of equation $x = A$; let $u(\cdot; x, y)$ be the only solution of the Cauchy problem

$$(3.3) \quad \begin{cases} u'' + f(u) = 0 \\ u(0) = x \\ u'(0) = y. \end{cases}$$

Then we observe that the solution $u(\cdot; A, \sqrt{\alpha^2 - 2F(A)})$ of (3.3) is a solution of (3.1) if and only if there exists an integer $m \geq 1$ such that

$$2mT_1(\alpha) + (2m - 1)T_2(\alpha) + 2mT_3(\alpha) = \pi$$

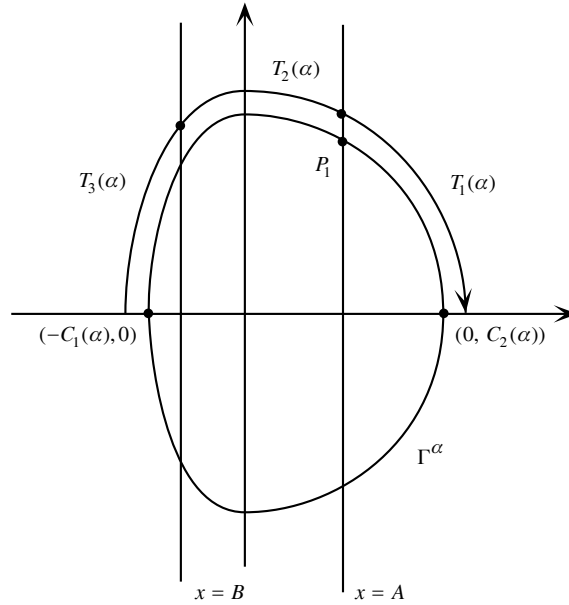


FIGURE 1. Time-maps for Picard problems.

or

$$2mT_1(\alpha) + (2m - 1)T_2(\alpha) + 2(m - 1)T_3(\alpha) = \pi.$$

Likewise, the solution $u(\cdot; A, -\sqrt{\alpha^2 - 2F(A)})$ of (3.3) is a solution of (3.1) if and only if there exists an integer $m \geq 0$ such that

$$2mT_1(\alpha) + (2m + 1)T_2(\alpha) + 2(m + 1)T_3(\alpha) = \pi$$

or

$$2mT_1(\alpha) + (2m + 1)T_2(\alpha) + 2mT_3(\alpha) = \pi.$$

In the case $A < B$ it is easy to see that we end up to the first two equations for solutions with $u'(0) < 0$, to the last two for solutions with $u'(0) > 0$.

Now we consider the set $S = S_1 \cup S_2$, where $S_1 = \{(x, y, z) \in \mathbf{R}^3, x > 0, y \geq 0, z > 0\}$: there exists an $m \in \mathbf{N}$ such that

$2mx + (2m - 1)y + 2mz = \pi$ or $2mx + (2m - 1)y + 2(m - 1)z = \pi$ and $S_2 = \{(x, y, z) \in \mathbf{R}^3, x > 0, y \geq 0, z > 0: \text{there exists } m \in \mathbf{N} \text{ such that } 2mx + (2m + 1)y + 2mz = \pi \text{ or } 2mx + (2m + 1)y + 2(m + 1)z = \pi\}$.

Then, problem (3.1) has a solution of energy $\alpha^2/2$ if and only if for the triple $T(\alpha) = (T_1(\alpha), T_2(\alpha), T_3(\alpha))$ we have $T(\alpha) \in S$.

Remark 3.1. We note that in the case $A = B$, so that $T_2(\alpha) = 0$, the set S reduces to the set \mathcal{F} corresponding to homogeneous Dirichlet boundary conditions, described in Section 2.2.

For future discussion it is useful to distinguish among the regions of the space defined by the planes constituting S , the following: $S_{2m+1} = \{(x, y, z) \in \mathbf{R}^3, x > 0, y \geq 0, z > 0: 2mx + (2m - 1)y + 2mz > \pi, 2mx + (2m - 1)y + 2(m - 1)z < \pi, 2mx + (2m + 1)y + 2mz > \pi, 2(m - 1)x + (2m - 1)y + 2mz < \pi, m \in \mathbf{N}\}$ and $S_{2m} = \{(x, y, z) \in \mathbf{R}^3, x > 0, y \geq 0, z > 0: 2mx + (2m - 1)y + 2mz < \pi, 2(m + 1)x + (2m + 1)y + 2mz > \pi, 2mx + (2m + 1)y + 2mz < \pi, 2mx + (2m + 1)y + 2(m + 1)z > \pi, m \in \mathbf{N}\}$. The careful reader is invited to draw some of the regions S_{2m} and S_{2m+1} . In Figure 2, we represent the projection S^* of the set S and the projections S_n^* of the regions S_n in the plane $x = z$ (cf. also Remark 5.5): they correspond to the case f odd and $B = -A$.

We shall show that in the regions described above some degree associated to (3.1) is always ± 1 (see Theorem 3.7).

Using Theorem 2.3 we can prove the following:

Lemma 3.2. *Let f be a continuous function satisfying (H1) and*

$$(H2) \quad \lim_{|x| \rightarrow +\infty} \frac{f(x)}{x} = +\infty.$$

Then

$$\lim_{\alpha \rightarrow +\infty} T_1(\alpha) = \lim_{\alpha \rightarrow +\infty} T_2(\alpha) = \lim_{\alpha \rightarrow +\infty} T_3(\alpha) = 0.$$

Proof. Let us suppose for simplicity $0 \leq B \leq A$; the other cases are similar. We start with $T_2(\alpha)$, since

$$T_2(\alpha) \leq \frac{A - B}{\sqrt{\alpha^2 - 2F(A)}}, \quad \forall \alpha \gg 0,$$

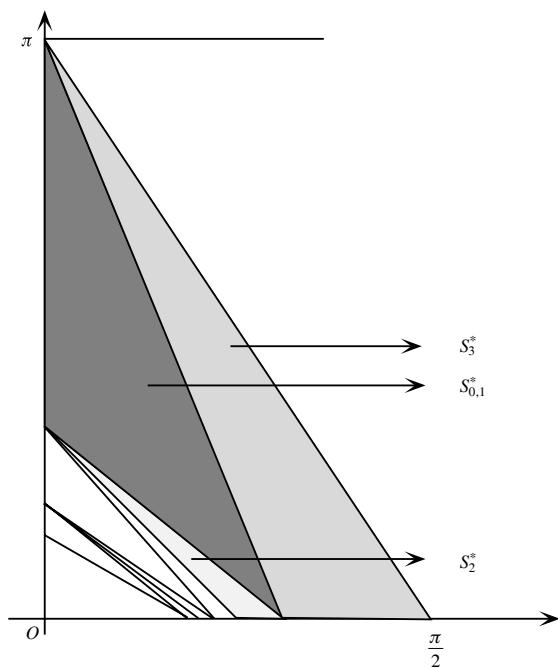


FIGURE 2. The set S^* .

the result immediately follows.

Now, for $T_1(\alpha)$, let us write

$$\begin{aligned} T_1(\alpha) &= \int_0^{C_2(\alpha)} \frac{1}{\sqrt{\alpha^2 - 2F(s)}} ds - \int_0^A \frac{1}{\sqrt{\alpha^2 - 2F(s)}} ds \\ &= \tau_f^+(\alpha) - r_1(\alpha). \end{aligned}$$

From Theorem 2.3 we deduce that

$$\lim_{\alpha \rightarrow +\infty} \tau_f^+(\alpha) = 0;$$

as before, being

$$r_1(\alpha) \leq \frac{A}{\sqrt{\alpha^2 - 2F(A)}},$$

we have

$$\lim_{\alpha \rightarrow +\infty} r_1(\alpha) = 0.$$

Analogously, if we write

$$\begin{aligned} T_3(\alpha) &= \int_{-C_1(\alpha)}^0 \frac{1}{\sqrt{\alpha^2 - 2F(s)}} ds + \int_0^B \frac{1}{\sqrt{\alpha^2 - 2F(s)}} ds \\ &= \tau_f^-(\alpha) + r_3(\alpha), \end{aligned}$$

the result is immediate. \square

Using Lemma 3.2, we can prove the following.

Theorem 3.3. *Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a continuous function satisfying (H1) and (H2). Then, for every $(A, B) \in \mathbf{R}^2$, problem (3.1) has infinitely many solutions such that*

$$\lim_{|(u, u')| \rightarrow +\infty} H(u, u') = +\infty.$$

Proof. By Lemma 3.2

$$\lim_{\alpha \rightarrow +\infty} T(\alpha) = (0, 0, 0).$$

As in the previous section, if we denote with $x_m^i, y_m^i, z_m^i, i = 1, \dots, 4$, the intersections of the four planes belonging to S , for each fixed integer m , with the coordinate axes, we have:

$$\lim_{m \rightarrow +\infty} x_m^i = \lim_{m \rightarrow +\infty} y_m^i = \lim_{m \rightarrow +\infty} z_m^i = 0, \quad \forall i.$$

This means that the distance between these planes and the origin tends to zero as m goes to infinity.

Then the curve $T(\alpha)$ in the space \mathbf{R}^3 intersects infinitely many times the planes constituting S as α goes to infinity; each of these intersections corresponds to a solution of (3.1). \square

Our next step consists of writing problem (3.1) as an abstract equation of the form

$$Lx = Nx.$$

We use the following notations:

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let us consider $x(t) = (u(t), u'(t))$, $g : \mathbf{R}^2 \times \mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined by $g(z_1, z_2) = (A, B)$ and $f_1(x, y) = (f(x), -y)$. Now problem (3.1) can be written as

$$(3.4) \quad \begin{cases} x'(t) + f_1(x(t)) = 0 \\ M_1x(0) + M_2x(\pi) = g(x(0), x(\pi)) \end{cases}$$

that reduces to the equation

$$(3.5) \quad Lx = Nx$$

where

$$\begin{aligned} L : X = C^1([0, \pi], \mathbf{R}^2) &\longrightarrow Z = C^0([0, \pi], \mathbf{R}^2) \times \mathbf{R}^2 \\ x &\longmapsto (x', M_1x(0) + M_2x(\pi)) \end{aligned}$$

and $Nx = (\mathcal{F}x, g(x(0), x(\pi)))$, \mathcal{F} being the Nemytskii operator associated to $-f_1$.

It is well-known, see [20], that L is a Fredholm operator of index zero and that N is L -completely continuous.

Now we consider the open bounded set of $C^1([0, \pi])$ defined by

$$\Omega^\alpha = \{u \in C^1([0, \pi]) : (u(t), u'(t)) \in F^\alpha, \forall t \in [0, \pi]\}.$$

If we consider $\alpha > 0$ such that $T(\alpha) \notin S$, then equation (3.5) has no solution in $\partial\Omega^\alpha$, so the degree $D_{\mathcal{L}}(L - N, \Omega^\alpha)$ is well defined.

In order to compute this degree, let us introduce the following map:

$$\begin{aligned} \mathcal{U} : \mathbf{R}^2 &\longrightarrow \mathbf{R}^2 \\ \mathcal{U}(z_1, z_2) &= (2z_1 - A, z_2 + u(\pi; z_1, z_2) - B) \end{aligned}$$

whose fixed points coincide with the initial values of solutions of the equation $u'' + f(u) = 0$. By a standard procedure, let us define $\mathcal{M} = P + J^{-1}QN + K_{PQ}N$, where P and Q are continuous projectors on X and Z , J is an isomorphism of $\text{Im } Q$ on $\ker L$ and K_{PQ} is the generalized inverse of L , see [20].

We are now in a position to recall the following:

Definition 3.4. Let $\Omega \subset C^1$ and $G \subset \mathbf{R}^2$ be bounded open sets; we say that Ω and G have a *common core* with respect to (3.1) if there are neither fixed points of \mathcal{M} on $\partial\Omega$ nor of \mathcal{U} on ∂G , and each solution u of (3.1) belongs to Ω if and only if $(u(0), u'(0)) \in G$.

Then, using a classical lemma due to Krasnosel'skii, see [18], we have

Lemma 3.5 [3]. *If $T(\alpha) \notin S$, the following property holds:*

$$D_{\mathcal{L}}(L - N, \Omega^\alpha) = \text{deg}_B(I - \mathcal{U}, F^\alpha, 0).$$

Following the ideas of [3] for Sturm-Liouville homogeneous boundary conditions, we now have to compute

$$\begin{aligned} \text{deg}_B(I - \mathcal{U}, F^\alpha) &= (-1)^2 \text{deg}_B(\mathcal{U} - I, F^\alpha) \\ &= \text{deg}_B(\mathcal{U} - I, F^\alpha). \end{aligned}$$

Let us denote by D the straight line $x = A$, and consider the sets

$$B_D = \{z_2 \in \mathbf{R} : (A, z_2) \in F^\alpha\}$$

and

$$\mathcal{R} = \{(z_1, z_2) \in \mathbf{R}^2 : |z_1 - A| < R, z_2 \in B_D\}$$

for some $R > 0$. We set

$$\mathcal{V}(z_1, z_2) = (\mathcal{U} - I)(z_1, z_2) = (z_1 - A, u(\pi; z_1, z_2) - B);$$

from the condition

$$\mathcal{V}(z_1, z_2) = 0 \implies z_1 = A \implies (z_1, z_2) \in D$$

by excision we obtain that

$$\deg_B(\mathcal{V}, F^\alpha, 0) = \deg_B(\mathcal{V}, \mathcal{R}, 0).$$

We consider the following homotopy

$$h(z, \lambda) = (z_1 - A, \tilde{U}_1(\lambda z_1, z_2)),$$

where $\tilde{U}_1(\lambda z_1, z_2) = u(\pi; (1-\lambda)A + \lambda z_1, z_2) - B$, so that $h(z, 1) = \mathcal{V}(z)$. With our choice of \mathcal{R} and using the hypothesis $T(\alpha) \notin S$, it is easy to check that the homotopy is admissible, i.e.

$$h(z, l) \neq 0, \quad \forall \lambda \in [0, 1], \quad \forall z \in \partial\mathcal{R}.$$

Now from the homotopy invariance and the multiplicative property of the degree we have

$$\begin{aligned} \deg_B(\mathcal{V}, \mathcal{R}, 0) &= \deg_B(h(\cdot, 0), \mathcal{R}, 0) \\ &= \deg_B((\text{id} - A) \times \phi, (A - R, A + R) \times B_D, 0) \\ &= \deg_B(\text{id} - A, (A - R, A + R), 0) \deg_B(\phi, B_D, 0) \\ &= \deg_B(\phi, B_D, 0), \end{aligned}$$

where

$$\begin{aligned} \phi : \mathbf{R} &\longrightarrow \mathbf{R} \\ \xi &\longmapsto u(\pi; A, \xi) - B \end{aligned}$$

is the “shooting map.”

From the definition of F^α we have that $B_D = (-l, l)$, where $l > 0$ is such that

$$F(A) + \frac{1}{2}l^2 = \frac{1}{2}\alpha^2.$$

In this way we have proved the following:

Theorem 3.6. *If $\alpha > 0$ is such that $T(\alpha) \notin S$, then*

$$D_{\mathcal{L}}(L - N, \Omega^\alpha) = \deg_B(\phi, (-l, l), 0)$$

where

$$\begin{aligned} \phi : \mathbf{R} &\longrightarrow \mathbf{R} \\ \xi &\longmapsto u(\pi; A, \xi) - B \end{aligned}$$

and $l > 0$ is given by

$$l = \sqrt{\alpha^2 - 2F(A)}.$$

Let us define $\sigma = \operatorname{sgn}(A - B)$ if $A \neq B$, $\sigma = 1$ otherwise. Then we have (cf. the result in [16] corresponding to Sturm-Liouville homogeneous boundary conditions):

Theorem 3.7. *If $T(\alpha) \in S_k$ for some $k > 0$, then*

$$D_{\mathcal{L}}(L - N, \Omega^\alpha) = \sigma(-1)^k$$

whereas if $T(\alpha) \notin S \cup (\cup_k S_k)$

$$D_{\mathcal{L}}(L - N, \Omega^\alpha) = 0.$$

Proof. It is sufficient to show that $\deg_B(\phi, (-l, l), 0) = \sigma(-1)^k$. Observe that the degree of ϕ is given by

$$\deg_B(\phi, (-l, l), 0) = \frac{\operatorname{sgn}\phi(l) - \operatorname{sgn}\phi(-l)}{2}.$$

Let k be even and $A \geq B$. When $T(\alpha) \in S_k$ the function $u(\cdot; A, \xi)$ meets at least $k - 1$ times and at most $k + 1$ times the line $x = B$; consequently, the points $(u(\pi; A, \pm l), u'(\pi; A, \pm l))$ will be at opposite sides of $x = B$ and $u(\pi; A, -l) < 0 < u(\pi; A, l)$. In other words, $\phi(-l) < 0$ and $\phi(l) > 0$. Hence, $\deg(\phi) = +1$. A completely analogous argument can be repeated for k odd or $A < B$.

When $T(\alpha) \notin \cup_k S_k$, the points $(u(\pi; A, \pm l), u'(\pi; A, \pm l))$ will be on the same side of $x = B$, so $\deg_B(\phi, (-l, l), 0) = 0$. \square

4. An existence result for an autonomous superlinear problem. Let us first consider $f(u) = u^3$ and $A = B = 0$: for the problem

$$(4.1) \quad \begin{cases} u'' + u^3 = 0 \\ u(0) = u(\pi) = 0 \end{cases}$$

we have

$$T_1(\alpha) = T_3(\alpha) = \sqrt{2} \int_0^{\sqrt[4]{2\alpha^2}} \frac{1}{\sqrt{2\alpha^2 - s^4}} ds = \tau_f^\pm(\alpha)$$

and

$$T_2(\alpha) \equiv 0.$$

We observe that the numbers $\tau_f^\pm(\alpha)$ are defined for each $\alpha > 0$. By Lemma 3.2 and Theorem 2.3 (ii),

$$\lim_{\alpha \rightarrow +\infty} T_1(\alpha) = \lim_{\alpha \rightarrow +\infty} T_3(\alpha) = 0$$

and

$$\lim_{\alpha \rightarrow 0^+} T_1(\alpha) = \lim_{\alpha \rightarrow 0^+} T_3(\alpha) = +\infty :$$

hence there exists $\bar{\alpha} > 0$ such that

$$T_1(\bar{\alpha}) = T_3(\bar{\alpha}) = \frac{\pi}{2}.$$

Then, for such $\bar{\alpha}$, we have a solution \bar{u} of (4.1), with

$$\frac{1}{2}(\bar{u}')^2 + F(\bar{u}) = \frac{1}{2}\bar{\alpha}^2,$$

and with no zeros in $(0, \pi)$, i.e., a positive solution of (4.1).

Starting from this observation, the aim of this section is to prove that if we consider a more general problem as

$$(4.2) \quad \begin{cases} u'' + f(u) = 0 \\ u(0) = A, \quad u(\pi) = B \end{cases}$$

with a suitable choice of f , then, depending on the values of A and B , a solution with no zeros in $(0, \pi)$ can either exist or not exist.

We prove our result in the case where $A \geq B \geq 0$, the case $B \geq A \geq 0$ being similar. Let us fix a pair (A, B) with $A \geq B \geq 0$; let us take $t > 0$ as a parameter and consider the problem

$$\begin{cases} u'' + f(u) = 0 \\ u(0) = tA, \quad u(\pi) = tB. \end{cases}$$

Then, we have the following:

Theorem 4.1. *Let us consider $f : \mathbf{R} \rightarrow \mathbf{R}$ odd, continuous, satisfying hypothesis (H1) and such that*

$$\lim_{|x| \rightarrow +\infty} \frac{f(x)}{x} = +\infty, \quad \lim_{x \rightarrow 0} \frac{f(x)}{x} = 0$$

and

$$\frac{d}{dx} \left(\frac{f(x)}{x} \right) > 0, \quad \forall x > 0.$$

Then there exist $\tilde{t} > \bar{t} > 0$ such that if $t < \bar{t}$ then one of the solutions of problem (4.3) is decreasing in $(0, \pi)$ and one has no zeros in $(0, \pi)$ and has exactly one global maximum. If $t \geq \tilde{t}$, then all the solutions of (4.3) have at least one zero in $(0, \pi)$.

Remark 4.2. We recall that for every t problem (4.3) always has infinitely many solutions, see Theorem 3.3.

Proof. We give the proof in the case of $f(u) = u^3$; we leave to the reader the details of the general case.

For problem (4.3), we have the three time-maps

$$T_1(\alpha) = \sqrt{2} \int_{tA}^{\sqrt[4]{2\alpha^2}} \frac{1}{\sqrt{2\alpha^2 - s^4}} ds,$$

$$T_2(\alpha) = \sqrt{2} \int_{tB}^{tA} \frac{1}{\sqrt{2\alpha^2 - s^4}} ds$$

and

$$T_3(\alpha) = \sqrt{2} \int_{-\sqrt[4]{2\alpha^2}}^{tB} \frac{1}{\sqrt{2\alpha^2 - s^4}} ds,$$

that are defined if $\sqrt[4]{2\alpha^2} > tA$, i.e., $\alpha > \alpha_0 = t^2 A^2 / \sqrt{2}$, see condition (3.2). Problem (4.3) has a solution without zeros in $(0, \pi)$ if and only if

$$(4.4) \quad 2T_1(\alpha) + T_2(\alpha) = \pi$$

or

$$(4.5) \quad T_2(\alpha) = \pi$$

for some $\alpha > 0$. Let us start with the study of the function T_2 . By Lemma 3.2 we have

$$\lim_{\alpha \rightarrow +\infty} T_2(\alpha) = 0;$$

moreover, T_2 is a decreasing function and

$$\begin{aligned} \lim_{\alpha \rightarrow \alpha_0} T_2(\alpha) &= \sqrt{2} \int_{tB}^{tA} \frac{1}{\sqrt{2\alpha_0^2 - s^4}} ds \\ &= \sqrt{2} \int_{tB}^{tA} \frac{1}{\sqrt{t^4 A^4 - s^4}} ds \\ &= \tau^+ \left(\frac{t^2 A^2}{\sqrt{2}} \right) - \sqrt{2} \int_0^{tB} \frac{1}{\sqrt{t^4 A^4 - s^4}} ds \\ &= \sigma(t). \end{aligned}$$

Then, in order to solve equation (4.5), we have to know the value $\sigma(t)$.

If $\sigma(t) \leq \pi$, then (4.5) has no solutions; if $\sigma(t) > \pi$ it has exactly one solution.

We have

$$\sigma(t) \leq \tau^+ \left(\frac{t^2 A^2}{\sqrt{2}} \right) \longrightarrow 0 \quad \text{if } t \rightarrow +\infty;$$

moreover,

$$\begin{aligned} \sigma(t) &= \sqrt{2} \int_{tB}^{tA} \frac{1}{\sqrt{t^4 A^4 - s^4}} ds \\ &\geq \sqrt{2} \frac{t(A-B)}{t^2 \sqrt{A^4 - B^4}} \longrightarrow +\infty \quad \text{if } t \rightarrow 0^+. \end{aligned}$$

Then there exists a finite number of points $t_i > 0$ such that $\sigma(t_i) = \pi$; if $\bar{t} = \min t_i$ and $\tilde{t} = \max t_i$, the result is proved.

Now, let us denote by

$$\tilde{T}(\alpha) = 2T_1(\alpha) + T_2(\alpha).$$

We have

$$\lim_{\alpha \rightarrow +\infty} \tilde{T}(\alpha) = 0$$

and

$$\begin{aligned} \lim_{\alpha \rightarrow \alpha_0} \tilde{T}(\alpha) &= \lim_{\alpha \rightarrow \alpha_0} 2\sqrt{2} \int_{tA}^{\sqrt[4]{2\alpha^2}} \frac{1}{\sqrt{2\alpha^2 - s^4}} ds + \sigma(t) \\ &= 2\sqrt{2} \int_{tA}^{\sqrt[4]{2\alpha_0^2}} \frac{1}{\sqrt{2\alpha_0^2 - s^4}} ds + \sigma(t) \\ &= \sigma(t), \end{aligned}$$

because $\sqrt[4]{2\alpha_0^2} = tA$. Again, we can conclude that (4.4) has a solution if $0 < t < \bar{t}$ and has no solution if $t > \bar{t}$. \square

5. An application to nonautonomous superlinear problems.

Now we will use the computation of the degree developed at the end of Section 3 to prove the existence of solutions to the problem

$$(5.1) \quad \begin{cases} u''(t) + f(u(t)) = p(t, u(t), u'(t)) \\ u(0) = A, \quad u(\pi) = B \end{cases}$$

when $(A, B) \in \mathbf{R}^2$, $f : \mathbf{R} \rightarrow \mathbf{R}$ is continuous and satisfies (H1) and (H2), and $p : [0, \pi] \times \mathbf{R}^2 \rightarrow \mathbf{R}$ is continuous and satisfies a linear growth condition in the last two arguments, i.e., there exists $K > 0$ such that

$$(5.2) \quad |p(t, x, y)| \leq K(1 + |x| + |y|) \quad \forall (t, x, y) \in [0, \pi] \times \mathbf{R}^2.$$

We will deal with problem (5.1) using the homotopy

$$(5.3) \quad \begin{cases} u''(t) + h(u(t), \lambda) = \lambda p(t, u(t), u'(t)) \\ u(0) = A, \quad u(\pi) = \lambda B - (1 - \lambda)A, \quad \lambda \in [0, 1], \end{cases}$$

where we set

$$h(u, \lambda) = \lambda f(u) + (1 - \lambda)g(u), \quad \lambda \in [0, 1]$$

with $g : \mathbf{R} \rightarrow \mathbf{R}$ odd, continuous and satisfying (H1), (H2) and

$$(H3) \quad \frac{d}{dx} \left(\frac{g(x)}{x} \right) > 0, \quad \forall x > 0.$$

Then for $\lambda = 1$, (5.3) gives the original problem, and for $\lambda = 0$ we are led to study the autonomous problem

$$(5.4) \quad \begin{cases} u''(t) + g(u(t)) = 0 \\ u(0) = A, \quad u(\pi) = -A. \end{cases}$$

As in Section 3, problem (5.3) can easily be written as an abstract equation of the form

$$Lu = N(u, \lambda)$$

with respect to the spaces $X = C^1([0, \pi], \mathbf{R}^2)$ and $Z = C^0([0, \pi], \mathbf{R}^2) \times \mathbf{R}^2$, setting

$$\begin{aligned} Lu &= (u', M_1 u(0) + M_2 u(\pi)), \\ N(u, \lambda) &= (l(\cdot, u, u', \lambda), A, \lambda B - (1 - \lambda)A) \end{aligned}$$

where

$$l(t, x, y, \lambda) = h(x, \lambda) - \lambda p(t, x, y).$$

In order to find solutions of problem (5.3), we need to apply a continuation theorem that we are now going to introduce.

Let X and Z be real Banach spaces, $L : D(L) \subset X \rightarrow Z$ a linear Fredholm mapping of index zero, $I = [0, 1]$ and $N : X \times I \rightarrow Z$ an L -completely continuous operator. We consider the equation

$$(5.5) \quad Lu = N(u, \lambda), \quad u \in D(L), \quad \lambda \in I.$$

Let

$$\Sigma^* = \{(u, \lambda) \in D(L) \times I : Lu = N(u, \lambda)\}.$$

For any set $B \in X \times I$ and any $\lambda \in I$, we denote by B_λ the section $\{u \in X : (u, \lambda) \in B\}$. Let us consider a continuous functional $\varphi : X \times I \rightarrow \mathbf{R}^+$; let Ω be an open set in $X \times I$ and $(c_k)_{k \in \mathbf{N}}$ be an unbounded increasing sequence that satisfies the following conditions:

(i₄) There exists $R > 0$ such that $\varphi(u, \lambda) \neq c_k$ for all $k \in \mathbf{N}$ and $(u, \lambda) \in \Sigma^*$ with $\|u\| \geq R$.

(i₅) $\varphi^{-1}([0, c_n]) \cap \Sigma^*$ is bounded for each $n \in \mathbf{N}$.

Let k_0 be an integer such that

$$(5.6) \quad c_{k_0} > \sup\{\varphi(u, \lambda) : (u, \lambda) \in \Sigma^*, \|u\| \leq R\}.$$

For $k \geq k_0$, let $O^k = \varphi^{-1}((c_k, c_{k+1})) \cap \Omega$ and $\Sigma^k = \bar{O}^k \cap \Sigma^*$.

Let us assume that

$$(i_6) \quad D_{\mathcal{L}}(L - N(\cdot, 0), O_0^k) \neq 0.$$

Thus, we have [16] the following result:

Theorem 5.1. *Assume that conditions (i₄) and (i₅) hold and that (i₆) is satisfied for each integer $k > k_0$. Then, for each of those integers, equation (5.5) has at least one solution u_k such that $\varphi(u_k, 1) \in (c_k, c_{k+1})$. Moreover, $\lim_{j \rightarrow +\infty} \|u_j\| = +\infty$.*

Consistently with the notation introduced, we denote by $\Sigma^* \subset X \times I$ the set of the solutions of the boundary value problem (5.3). In view of applying Theorem 5.1, we shall introduce a functional φ as follows.

Let

$$\delta : \mathbf{R}^2 \longrightarrow \mathbf{R}, \quad (x, y) \longmapsto \min \left\{ 1, \frac{1}{x^2 + y^2} \right\}.$$

Then we define the continuous functional φ on $X \times I$ by

$$\begin{aligned} \varphi(u, \lambda) &= \frac{1}{\pi} \left| \int_0^\pi [u'(t)^2 + u(t)l(t, u(t), u'(t), \lambda)] \delta(u(t), u'(t)) dt \right|. \end{aligned}$$

Using the computations in [5], we can prove the following:

Lemma 5.2. *There exists $R_0 > 1$ such that for every $(u, \lambda) \in \Sigma^*$ there exists $n \in \mathbf{N}$ satisfying*

$$u(t)^2 + u'(t)^2 \geq R_0^2 \quad \implies \quad |\varphi(u, \lambda) - (n + 1)| < \frac{1}{4}.$$

Remark 5.3. As far as solutions of (5.3) with sufficiently large norm are concerned, we first observe that all the zeros of such solutions are simple.

Secondly, we note that the argument in [5] leading to Lemma 5.2 shows that if we consider the zeros of u starting from the first zero a_1

such that $u'(0)u'(a_1) < 0$ up to the last zero b_1 such that $u'(b_1)u'(\pi) < 0$, then their cardinality is exactly n .

Now, in order to prove the main result of this section, let us consider the following:

Lemma 5.4. *Let u be a solution of (5.4) with $u(t)^2 + u'(t)^2 \geq R_0^2$ for each $t \in [0, \pi]$; then*

$$|\varphi(u, 0) - k| < \frac{1}{4}$$

if and only if $u = u_k$ or $u = \tilde{u}_k$, where u_k and \tilde{u}_k are the solutions of the Cauchy problems

$$(5.7) \quad \begin{cases} u'' + g(u) = 0 \\ u(0) = A \\ u'(0) = \sqrt{(\alpha_{1,k})^2 - 2F(A)} \end{cases}$$

and

$$\begin{cases} u'' + g(u) = 0 \\ u(0) = A \\ u'(0) = -\sqrt{(\alpha_{2,k})^2 - 2F(A)} \end{cases}$$

(respectively) with, for k even ($k = 2m$):

$$\begin{aligned} 4mT_1(\alpha_{1,k}) + (2m - 1)T_2(\alpha_{1,k}) &= \pi, \\ 2mT_1(\alpha_{2,k}) + (2m - 1)T_2(\alpha_{2,k}) &= \pi \end{aligned}$$

and for k odd ($k = 2m - 1$):

$$\begin{aligned} \alpha_{1,k} &= \alpha_{2,k}, \\ 2(2m - 1)T_1(\alpha_{1,k}) + (2m - 1)T_2(\alpha_{2,k}) &= \pi. \end{aligned}$$

Proof. We prove the result only for $A \geq 0$ and $u'(0) > 0$, problem (5.7). The other cases are similar.

Let u_k be the solution of (5.7). This means that u_k is a solution of $u'' + g(u) = 0$ with energy $(\alpha_{1,k})^2/2$. By the definition of $\alpha_{1,k}$, the

solution u_k has exactly k zeros in $(0, \pi)$ if k is odd, $k - 1$ zeros in $(0, \pi)$ if k is even. Lemma 5.2 and Remark 5.3 show that in both cases

$$|\varphi(u_k, 0) - k| < \frac{1}{4}.$$

Now, conversely, let us consider a solution of (5.4) with

$$|\varphi(u, 0) - k| < \frac{1}{4} \quad \text{and} \quad u'(0) > 0.$$

An easy argument proves that u must be the solution u_k of (5.7); in fact, let us distinguish two cases.

Case 1. k odd. u must have at least $k - 1$ zeros in $(0, \pi)$. The conditions $u'(0)u'(a_1) < 0$ and $u'(b_1)u'(\pi) < 0$ imply that $u'(a_1) < 0$, $u'(b_1) > 0$ and $u'(\pi) < 0$. So there must be another zero of u in (b_1, π) .

Then u has k zeros in $(0, \pi)$ and it is $u'(\pi) < 0$. But the only solution of (5.4) with these properties is u_k .

Case 2. k even. u has exactly $k - 1$ zeros in $(0, \pi)$ and $u'(\pi) > 0$. Again, we can conclude that $u = u_k$. \square

Remark 5.5. We resume the nodal properties of u_k and \tilde{u}_k :

	number of zeros of u_k in $(0, \pi)$	number of zeros of \tilde{u}_k in $(0, \pi)$
k even	$k - 1$	$k + 1$
k odd	k	k

We also observe that for k even we have

$$\begin{aligned} (T_1(\alpha_{1,k}), T_2(\alpha_{1,k})) &\in b_k, \\ (T_1(\alpha_{2,k}), T_2(\alpha_{2,k})) &\in c_k \end{aligned}$$

and for k odd

$$(T_1(\alpha_{i,k}), T_2(\alpha_{i,k})) \in a_k, \quad i = 1, 2,$$

where a_k, b_k, c_k are the straight lines in the plane (x, y) of equation:

$$\begin{aligned} a_k &: 2kx + ky = \pi; \\ b_k &: 2kx + (k-1)y = \pi; \\ c_k &: kx + (k-1)y = \pi \end{aligned}$$

(these lines constitute the projection S^* of the set S in the plane $x = z$). For each integer m , let us consider the projections S_{2m+1}^* and S_{2m}^* of the regions S_{2m+1} and S_{2m} in the plane $x = z$, cf. Section 3:

$$\begin{aligned} S_{2m+1}^* &= \{(x, y) \in \mathbf{R}^2 : x \geq 0, y \geq 0, \\ &\quad 2(2m-1)x + (2m-1)y < \pi < 4mx + (2m-1)y\}, \\ S_{2m}^* &= \{(x, y) \in \mathbf{R}^2 : x \geq 0, y \geq 0, \\ &\quad 4mx + (2m+1)y < \pi < 2(2m+1)x + (2m+1)y\}; \end{aligned}$$

finally, let us set

$$\begin{aligned} S_{0,m}^* &= \{(x, y) \in \mathbf{R}^2 : x \geq 0, y \geq 0, \\ &\quad 4mx + (2m-1)y < \pi < 4mx + (2m+1)y\}; \end{aligned}$$

these regions of the plane (see Figure 2) will be useful for the computation of the degree, cf. Lemma 5.9.

The functional φ defined above satisfies some classical properties already proved in [4] for homogeneous boundary conditions. For brevity, we omit the proof of the following two lemmas, which are the straightforward variants of such results.

Lemma 5.6 (The “elastic property”). *For each $R_1 > 0$ there is and $R_2 \geq R_1$ such that, for each $(u, \lambda) \in \Sigma^*$, we have*

$$\begin{aligned} \|u\|_{1,\infty} \geq R_2 &\implies |u(t)|^2 + |u'(t)|^2 \geq R_1^2, \\ &\forall t \in [0, \pi]. \end{aligned}$$

Lemma 5.7 (Fast oscillations of large solutions). *For each $N > 0$ there is an $R_1(N) > 0$ such that for all $(u, \lambda) \in \Sigma^*$*

$$\min_{t \in [0, \pi]} |u(t)|^2 + |u'(t)|^2 \geq R_1^2(N) \implies |\varphi(u, l)| \geq N.$$

From the previous lemmas we obtain the following:

Proposition 5.8. *The functional φ satisfies (i_4) and (i_5) of Theorem 5.1 with respect to the sequence $(c_k)_{k \in \mathbf{N}}$, with $c_k = k - 1/2$.*

Let k_0 be as in the proof of Proposition 5.8, see [5]. As an application of Lemma 5.7 with $N = k_0$, we know that there is a constant $d > 0$ such that, for all $(u, \lambda) \in \Sigma^*$

$$u'(0) > d \implies \varphi(u, \lambda) > k_0.$$

Now we consider

$$\begin{aligned} \Omega^+ &= \{(u, \lambda) \in X \times I \mid u'(0) > d\}, \\ \Omega^- &= \{(u, \lambda) \in X \times I \mid u'(0) < -d\}, \end{aligned}$$

and we define

$$\begin{aligned} O^k &= \varphi^{-1}((c_k, c_{k+1})), \\ O_+^k &= \varphi^{-1}((c_k, c_{k+1})) \cap \Omega^+, \\ O_-^k &= \varphi^{-1}((c_k, c_{k+1})) \cap \Omega^-. \end{aligned}$$

With these assumptions we can prove the following:

Lemma 5.9. *For any $k \in \mathbf{N}$ with $k > k_0$,*

$$|D_{\mathcal{L}}(L - N(\cdot, 0), (O_+^k)_0)| \neq 0$$

and

$$|D_{\mathcal{L}}(L - N(\cdot, 0), (O_-^k)_0)| \neq 0.$$

Proof. First of all we observe that the constant k_0 satisfies condition (5.6) of Theorem 5.1, cf. [5].

Now let $k > k_0$ and consider the set $\Sigma^k = \bar{O}^k \cap \Sigma^* = O^k \cap \Sigma^*$, by (i_4) . Then $(\Sigma^k)_0$ is the set of the solutions of (5.4) such that

$$(5.8) \quad k - 1/2 < \varphi(u, 0) < k + 1/2.$$

By Lemma 5.4 we see that

$$(\Sigma^k)_0 = \{u_k, \tilde{u}_k\}, \quad \text{for each } k \in \mathbf{N}, \quad k > k_0.$$

Now we distinguish two cases:

Case 1. k even ($k = 2m$). We consider the open sets

$$\Omega_{\beta_k^+}^{\gamma_k^+} = \{u \in X \mid \beta_k^{+2} < u'(t)^2 + 2F(u(t)) < \gamma_k^{+2}, \forall t \in [0, \pi]\}$$

and

$$\Omega_{\beta_k^-}^{\gamma_k^-} = \{u \in X \mid \beta_k^{-2} < u'(t)^2 + 2F(u(t)) < \gamma_k^{-2}, \forall t \in [0, \pi]\}$$

with

$$\begin{aligned} \gamma_k^+ &= \alpha^{1,k} + \varepsilon_k, & \beta_k^+ &= \alpha^{1,k} - \varepsilon_k, \\ \gamma_k^- &= \alpha^{2,k} + \varepsilon_k, & \beta_k^- &= \alpha^{2,k} - \varepsilon_k \end{aligned}$$

where ε_k is small enough to satisfy the following conditions:

$(\Sigma^k)_0 \subset (\Omega_{\beta_k^+}^{\gamma_k^+} \cup \Omega_{\beta_k^-}^{\gamma_k^-}) \subset \overline{\Omega_{\beta_k^+}^{\gamma_k^+} \cup \Omega_{\beta_k^-}^{\gamma_k^-}} \subset (O^k)_0$ (we observe that it is ensured by the continuity of the functional φ);

$T(\gamma_k^+) \in S_{0,m}^*$, $T(\beta_k^+) \in S_{2m+1}^*$, $T(\gamma_k^-) \in S_{2m}^*$ and $T(\beta_k^-) \in S_{0,m-1}^*$, see Remark 5.5, (we observe that these conditions are valid by the continuity of $T(\cdot)$).

By Theorem 3.7 and by the excision property of the degree we can conclude that

$$D_{\mathcal{L}}(L - N(\cdot, 0), (O^k)_0) = D_{\mathcal{L}}(L - N(\cdot, 0), \Omega_{\beta_k^+}^{\gamma_k^+} \cup \Omega_{\beta_k^-}^{\gamma_k^-}) = 2\sigma,$$

where σ is defined in Section 3.

Now, by the additivity/excision property of the degree again, we have

$$D_{\mathcal{L}}(L - N(\cdot, 0), (O_+^k)_0) = D_{\mathcal{L}}(L - N(\cdot, 0), (O_-^k)_0) = \sigma.$$

Case 2. k odd ($k = 2m + 1$). We consider the open set

$$\Omega_{\beta_k}^{\gamma_k} = \{u \in X \mid \beta_k^2 < u'(t)^2 + 2F(u(t)) < \gamma_k^2, \forall t \in [0, \pi]\}$$

with $\gamma_k = \alpha^{1,k} + \varepsilon_k$ and $\beta_k = \alpha^{1,k} - \varepsilon_k$ where ε_k is small enough to satisfy the following conditions:

$$(\Sigma^k)_0 \subset \Omega_{\beta_k}^{\gamma_k} \subset \bar{\Omega}_{\beta_k}^{\gamma_k} \subset (O^k)_0;$$

$$T(\gamma_k) \in S_{2m+3}^* \text{ and } T(\beta_k) \in S_{2m}^*, \text{ see Remark 5.5.}$$

Then we can conclude that

$$(5.9) \quad D_{\mathcal{L}}(L - N(\cdot, 0), (O^k)_0) = D_{\mathcal{L}}(L - N(\cdot, 0), \Omega_{\beta_k}^{\gamma_k}) = -2\sigma.$$

Now, using the fact that the degree in (5.9) reduces to the Brouwer degree of a one-dimensional map, see Section 3, and the fact that the solutions u_k and \tilde{u}_k belong to O_+^k and O_-^k , respectively, we see that

$$D_{\mathcal{L}}(L - N(\cdot, 0), (O_+^k)_0) = -\sigma$$

and

$$D_{\mathcal{L}}(L - N(\cdot, 0), (O_-^k)_0) = -\sigma. \quad \square$$

Remark 5.10. In the article [4], A. Capietto, J. Mawhin and F. Zanolin obtained that $D_{\mathcal{L}}(L - N_0, O^k) = 2(-1)^k$. Then, in the paper [5] they obtained the final computation of the degree by means of a slight modification of the functional φ . Here we follow the idea already exploited by A. Capietto in [1] of considering the original functional and of breaking the set O^k using the sign of the derivative of u in 0.

Now we are in a position to apply Theorem 5.1 to O_+^k and O_-^k separately. We will find two sequences of solutions of (5.1) that belong to Ω^+ and to Ω^- . This is exactly the result proved in [5]; however, it is obtained with a different homotopy and by using a refinement of the continuation theorem in [3].

Theorem 5.11. *Let f and p satisfy (H2) and (5.2), respectively. Then there is a $k_0 \in \mathbf{N}$ such that, for each $n > k_0$, the boundary value problem (5.1) has at least two solutions v_n and w_n with $v'_n(0) > 0$ and $w'_n(0) < 0$ such that*

$$(5.10) \quad \lim_{n \rightarrow +\infty} \left(\min_{t \in [0, \pi]} |v_n(t)| + |v'_n(t)| \right) = \lim_{n \rightarrow +\infty} \left(\min_{t \in [0, \pi]} |w_n(t)| + |w'_n(t)| \right) = +\infty.$$

These solutions have the following nodal properties:

For n odd, $v'_n(\pi) < 0$ and, moreover, v_n has exactly $n + 1$ zeros in $[0, \pi]$ if $A \leq 0$ and $B \leq 0$; v_n has exactly n zeros in $[0, \pi]$ if $A \leq 0$ and $B > 0$ or if $A > 0$ and $B \leq 0$; v_n has exactly $n - 1$ zeros in $[0, \pi]$ if $A > 0$ and $B > 0$.

For n even, $v'_n(\pi) > 0$ and, moreover, v_n has exactly $n + 1$ zeros in $[0, \pi]$ if $A \leq 0$ and $B \geq 0$; v_n has exactly n zeros in $[0, \pi]$ if $A \leq 0$ and $B < 0$ or if $A > 0$ and $B \geq 0$; v_n has exactly $n - 1$ zeros in $[0, \pi]$ if $A > 0$ and $B < 0$.

For n odd, $w'_n(\pi) < 0$ and, moreover, w_n has exactly $n + 1$ zeros in $[0, \pi]$ if $A \geq 0$ and $B \geq 0$; w_n has exactly n zeros in $[0, \pi]$ if $A \geq 0$ and $B < 0$ or if $A < 0$ and $B \geq 0$; w_n has exactly $n - 1$ zeros in $[0, \pi]$ if $A < 0$ and $B < 0$.

For n even, $w'_n(\pi) > 0$ and, moreover, w_n has exactly $n + 1$ zeros in $[0, \pi]$ if $A \geq 0$ and $B \leq 0$; w_n has exactly n zeros in $[0, \pi]$ if $A \geq 0$ and $B > 0$ or if $A < 0$ and $B \leq 0$; w_n has exactly $n - 1$ zeros in $[0, \pi]$ if $A < 0$ and $B > 0$.

All the zeros of v_n and w_n are simple and all the local maxima or minima of v_n and w_n are strict. Between any two consecutive zeros of a solution, as well as between 0 and the first zero or between the last zero and π , there is only one critical point of the solution.

Proof. The existence of the sequences of solutions to (5.1) with

$$(5.11) \quad \lim_{n \rightarrow +\infty} \|v_n\| = \lim_{n \rightarrow +\infty} \|w_n\| = +\infty$$

follows directly from Theorem 5.1.

The elastic property yields (5.10) from (5.11).

For the discussion of the nodal properties, see [5]. \square

Acknowledgment. The author thanks Prof. A. Capietto for the useful suggestions and for the supervision of the whole work.

REFERENCES

1. A. Capietto, *An existence result for a two-point superlinear boundary value problem*, Proc. Second Int. Conf. on Differential Equations, Marrakesh, 1995.

2. A. Capietto and W. Dambrosio, *Multiplicity results for some two-point asymmetric boundary value problems*, Nonlinear Analysis, TMA, to appear.
3. A. Capietto, M. Henrard, J. Mawhin and F. Zanolin, *A continuation approach to some forced superlinear Sturm-Liouville boundary value problems*, Topological Methods in Nonlinear Analysis **3** (1994), 81–100.
4. A. Capietto, J. Mawhin and F. Zanolin, *Boundary value problems for forced superlinear second order ordinary differential equations*, in *Nonlinear partial differential equations* (H. Brezis and J.L. Lions, eds.), Collège de France Seminar, Longman, New York, 1994.
5. ———, *On the existence of two solutions with a prescribed number of zeros for a superlinear two-point boundary value problem*, Topological Methods Nonlin. Anal. **6** (1995), 175–188.
6. T. Ding, *An infinite class of periodic solutions of periodically perturbed Duffing equations at resonance*, Proc. Amer. Math. Soc. **86** (1982), 47–54.
7. T. Ding and W. Ding, *Resonance problems for a class of Duffing's equations*, Chinese Ann. Math., Ser. B **6** (1985), 427–432.
8. T. Ding, R. Iannacci and F. Zanolin, *On periodic solutions of sublinear Duffing equations*, J. Math. Anal. Appl. **158** (1991), 316–322.
9. ———, *Existence and multiplicity results for periodic solutions of semilinear Duffing equations*, J. Differential Equations **105** (1993), 364–409.
10. M.L. Fernandez and F. Zanolin, *Periodic solutions of a second order ordinary differential equation with one-sided growth restriction on the restoring term*, Arch. Math. **51** (1988), 151–163.
11. P.C. Fife, *Mathematical aspects of reacting and diffusing systems*, Springer Lecture Notes in Biomath. **28** (1979).
12. D.A. Frank-Kamenskii, *Diffusion and heat transfer in chemical kinetics*, Plenum Press, New York, 1969.
13. A. Fonda and F. Zanolin, *On the use of time-maps for the solvability of nonlinear boundary value problems*, Arch. Math. **59** (1992), 245–259.
14. S. Fučík and V. Lovicar, *Periodic solutions of the equation $x'' + g(x) = p$* , Casopis Pest. Mat. **100** (1975), 160–175.
15. M. Furi and M.P. Pera, *On the notion of winding number for closed curves and applications to forced oscillations on even dimensional spheres*, Boll. Un. Mat. Ital. A **7** (1993), 397–407.
16. M. Henrard, *Topological degree in boundary value problems: existence and multiplicity results for second order differential equations*, Ph.D. Thesis, University of Louvain, 1996.
17. D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math. **840** (1981).
18. M.A. Krasnosel'skii and P.P. Zabreiko, *Geometric methods of nonlinear analysis*, Springer-Verlag, Berlin, 1984.
19. J. Mawhin, *Recent trends in nonlinear BVP*, in *Band Abhandlungen der AdW*, Akademie Verlag, Berlin, 1972.
20. ———, *Topological degree methods in nonlinear boundary value problems*, CBMS Regional Conf. Ser. in Math. **40** (1979).

- 21.** ———, *Leray-Schauder continuation theorems in the absence of a priori bounds*, in *Proceedings of the Schauder Conference*, 1996.
- 22.** J. Mawhin and J. R. Ward, *Nonlinear variational two-point BVP*, in *Proceedings of the Conference Variational Methods*, Paris 1988, (H. Beresticki and I. Ekeland, eds.), Basel-Boston, 1990, 209–219.
- 23.** Z. Opial, *Sur les solutions périodiques de l'équation différentielle $x'' + g(x) = p(t)$* , Bull. Acad. Polish Sci. **8** (1960), 151–156.
- 24.** ———, *Sur les périodes des solutions de l'équation différentielle $x'' + g(x) = 0$* , Ann. Polon. Math. **10** (1961), 49–72.
- 25.** ———, *Sur l'existence des solutions périodiques de l'équation différentielle $x'' + f(x, x')x' + g(x) = p(t)$* , Ann. Polon. Math. **11** (1961), 149–159.
- 26.** D. Qian, *Time-maps and Duffing equations across resonance*, Sci. China, Ser. A **23** (1993), 471–479 (in Chinese).
- 27.** R. Reissig, G. Sansone and R. Conti, *Qualitative Theorie nichtlinearer Differentialgleichungen*, Roma, Cremonese, 1963.
- 28.** R. Schaaf, *Global solution branches of two point boundary value problems*, Lecture Notes in Math. **1458** (1990).
- 29.** K. Schmitt, *Periodic solutions of a forced nonlinear oscillator involving a one-sided restoring force*, Arch. Math. **31** (1978), 70–73.
- 30.** J. Smoller and A. Wasserman, *Global bifurcation of steady-state solutions*, J. Differential Equations **39** (1981), 269–290.
- 31.** J.R. Ward *Periodic solutions for systems of second order ordinary differential equations*, J. Math. Anal. Appl. **81** (1981), 92–98.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TORINO, VIA CARLO ALBERTO
10 - 10123 TORINO, ITALY
E-mail address: dambrosio@unito.it