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DESCRIPTION OF A CLASS OF LOCALLY PSEUDOCONVEX ALGEBRAS WHICH HAVE AN EQUIVALENT LOCALLY M-PSEUDOCONVEX TOPOLOGY

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ABSTRACT. Let $r \in (0,1]$, p be an r-homogeneous semi-norm on a commutative algebra A and $\phi : [0,\infty) \mapsto [0,\infty)$ be an increasing function such that $p(x^2) \leq \phi(p(x))$ for all $x \in A$. It is shown that such a semi-norm p is (i) A-pseudoconvex, (ii) equivalent with a submultiplicative semi-norm. A description of a class of locally pseudoconvex algebras which have an equivalent locally m-pseudoconvex topology is given.

Introduction. Let A be a topological algebra, that is, a linear topological space over $\mathbf{C}$ in which has been defined a separately continuous (associative) multiplication. If the underlining linear topological space A is locally pseudoconvex, then A is called a locally pseudoconvex algebra. It is known (see [**11**, p. 6]) that the topology of locally pseudoconvex algebras can be given by means of a family $\mathcal{P} = \{p_\lambda \mid \lambda \in \Lambda\}$ of r_λ-homogeneous semi-norms where $r_\lambda \in (0,1]$ is fixed for each $\lambda \in \Lambda$ (r-homogeneity of a semi-norm p on A means that $p(\alpha x) = |\alpha|^r p(x)$ for each $x \in A$ and $\alpha \in \mathbf{C}$). Furthermore, A is called a locally absorbingly pseudoconvex (shortly a locally A-pseudoconvex) algebra if, for each $x \in A$ and $\lambda \in \Lambda$, there exist positive numbers $M_\lambda(x)$ and $N_\lambda(x)$ such that

$$p_\lambda(xy) \leq M_\lambda(x) p_\lambda(y) \tag{1}$$

and

$$p_\lambda(yx) \leq N_\lambda(x) p_\lambda(y) \tag{2}$$

for all $y \in A$. In particular, if $M_\lambda(x) = N_\lambda(x) = p_\lambda(x)$ for each $x \in A$ and $\lambda \in \Lambda$, then A is called a locally multiplicatively pseudoconvex (shortly a locally m-pseudoconvex) algebra. Moreover, if the numbers $M_\lambda(x)$ and $N_\lambda(x)$ do not depend on semi-norms p_λ, $\lambda \in \Lambda$, i.e., (1)

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and (2) hold for some $M_\lambda(x) = M(x)$ and $N_\lambda(x) = N(x)$ for all $x \in A$, then A is called a locally uniformly A-pseudoconvex algebra. In the particular case when $\sup_{\lambda\in\Lambda} p_\lambda(x)$ is finite for each $x \in A$, then a locally pseudoconvex algebra is called a strongly spectrally bounded algebra (see [**1**, p. 72]). It is easy to see that every locally uniformly A-pseudoconvex algebra with unit is a strongly spectrally bounded algebra. Let A be a topological algebra, and let $m(A)$ be the set of all closed regular two-sided ideals of A which are maximal as left or as right ideals. It is known that $m(A)$ is not necessarily a nonempty set even in the commutative case (see [**12**, p. 125]). In this case, when the quotient algebra A/M (with the quotient topology) is topologically isomorphic with $\mathbf{C}$ for each $M \in m(A)$, algebra A is called a Gelfand-Mazur algebra. For the description of Gelfand-Mazur algebras, see [**2**, pp. 123–126].

A semi-norm p on an algebra A is called subquadrative if

$$p(x^2) \leq p(x)^2 \tag{3}$$

for each $x \in A$ (see [**4**] or [**5**]).

1. On A-pseudoconvexity of seminorms. Let A be a commutative algebra over $\mathbf{C}$, $r \in (0,1]$, and let p be an r-homogeneous semi-norm on A.

It is known (see [**6**, p. 58], [**7**, p. 91] and [**8**, p. 491]) that if an r-homogeneous semi-norm p with $r \in (0,1]$ satisfies the condition $p(x^2) = p(x)^2$ for all $x \in A$, then p is submultiplicative and, moreover, p is equivalent with a submultiplicative 1-homogeneous semi-norm. It follows from Theorem 3.1 of [**5**] that every subquadrative homogeneous (that is the case where $r = 1$) semi-norm p on A is equivalent with a homogeneous submultiplicative semi-norm q (that is, there exist positive numbers C_1 and C_2 such that $C_1 p(x) \leq q(x) \leq C_2 p(x)$ for all $x \in A$). We shall now show the following more general result.

Theorem 1. *Let A be a commutative algebra over $\mathbf{C}$, $r \in (0,1]$, p be an r-homogeneous semi-norm on A and $\phi : [0,\infty) \to [0,\infty)$ be an increasing function such that*

$$p(x^2) \leq \phi(p(x)) \tag{4}$$

for all $x \in A$. Then p is equivalent with an r-homogeneous submultiplicative semi-norm.

Proof. By the commutativity of algebra A, we have $4xy = (x+y)^2 - (x-y)^2$ for all $x, y \in A$. Thus

$$\begin{aligned} 4^r p(xy) &= p(4xy) = p((x+y)^2 - (x-y)^2) \\ &\le p((x+y)^2) + p((x-y)^2) \\ &\le \phi(p(x+y)) + \phi(p(x-y)) \\ &\le \phi(p(x)+p(y)) + \phi(p(x)+p(y)) \\ &= 2\phi(p(x)+p(y)), \end{aligned}$$

for all x and $y \in A$. So we have $p(xy) \le (2/4^r)\phi(p(x)+p(y))$ for all $x, y \in A$. Now let $x, y \in A$ and $\varepsilon > 0$ be given. If we choose $\alpha = p(x)+\varepsilon$ and $\beta = p(y)+\varepsilon$, then

$$\begin{aligned} \alpha^{-1}\beta^{-1}p(xy) &= p((\alpha^{-1/r}x)(\beta^{-1/r}y)) \\ &\le \frac{2}{4^r}\phi(p(\alpha^{-1/r}x) + p(\beta^{-1/r}y)) \\ &= \frac{2}{4^r}\phi\left(\frac{p(x)}{p(x)+\varepsilon} + \frac{p(y)}{p(y)+\varepsilon}\right) \\ &\le \frac{2}{4^r}\phi(2) \end{aligned}$$

or $p(xy) \le (2/4^r)\phi(2)(p(x)+\varepsilon)(p(y)+\varepsilon)$. This implies that

$$p(xy) \le \frac{2}{4^r}\phi(2)p(x)p(y)$$

for all x and $y \in A$.

If we take $q = (2/4^r)\phi(2)p$, then it is easy to see that q is an r-homogeneous submultiplicative semi-norm on A which is equivalent with p.

As a corollary, we have the following results.

Corollary 1. *Let A and p be as in Theorem 1. Then p is an A-pseudoconvex semi-norm.*

Corollary 2. *Let A be a commutative algebra over $\mathbf{C}$, $r \in (0,1]$, and let p be an r-homogeneous subquadrative semi-norm on A. Then p is an A-pseudoconvex semi-norm on A which is equivalent with a submultiplicative pseudoconvex semi-norm on A.*

2. A characterizing m-pseudoconvex algebra among the pseudoconvex algebras.

Let A be a locally pseudoconvex algebra, for which the topology has been given by a family $\mathcal{P} = \{p_\lambda | \lambda \in \Lambda\}$ of r_λ-homogeneous semi-norms, where $r_\lambda \in (0,1]$ is fixed for each $\lambda \in \Lambda$. Denote by $T(\mathcal{P})$ the Hausdorff topology on A defined by the system $\mathcal{P}$. By Theorem 1 we have

Theorem 2. *Let A be a commutative locally pseudoconvex algebra, for which the topology has been given by a family $\mathcal{P} = \{p_\lambda \mid \lambda \in \Lambda\}$ of r_λ-homogeneous semi-norms where $r_\lambda \in (0,1]$ is fixed for each $\lambda \in \Lambda$. Suppose that, for each $\lambda \in \Lambda$, there is an increasing function $\phi_\lambda : [0,\infty) \mapsto [0,\infty)$ satisfying the condition*

$$p_\lambda(x^2) \leq \phi_\lambda(p_\lambda(x)) \tag{5}$$

for all $x \in A$ and $\lambda \in \Lambda$. Then

(a) *$(A, T(\mathcal{P}))$ is a locally A-pseudoconvex algebra (in the particular case when A is strongly spectrally bounded and $\phi_\lambda = \phi$ for each $\lambda \in \Lambda$, then A is a locally uniformly A-pseudoconvex algebra),*

(b) *there exists a locally m-pseudoconvex topology T on A such that $(A, T(\mathcal{P}))$ and (A, T) are topologically isomorphic algebras.*

Proof. (a) In the same way as in the proof of Theorem 1, we have

$$p_\lambda(xy) \leq 2^{1-2r_\lambda}\phi_\lambda(2)p_\lambda(x)p_\lambda(y)$$

for all $x, y \in A$ and $\lambda \in \Lambda$. For each fixed $x \in A$ and $\lambda \in \Lambda$, let $M_\lambda(x) = 2^{1-2r_\lambda}\phi_\lambda(2)p_\lambda(x) + 1$. Then $M_\lambda(x) > 0$ and

$$p_\lambda(xy) \leq M_\lambda(x)p_\lambda(y)$$

for all $y \in A$. Hence $(A, T(\mathcal{P}))$ is a commutative locally A-pseudoconvex algebra. In particular, if A is strongly spectrally bounded and $\phi_\lambda = \phi$ for each $\lambda \in \Lambda$, then

$$p_\lambda(xy) \leq M(x)p_\lambda(y)$$

for all $y \in A$ where $M(x) = 2^{1-2r}\phi(2)\sup_\lambda p_\lambda(x)$ for each $x \in A$ and $r = \inf_\lambda r_\lambda$. Thus, A is a commutative uniformly A-pseudoconvex algebra.

(b) For each fixed $\lambda \in \Lambda$, let $q_\lambda = 2^{1-2r_\lambda}\phi_\lambda(2)p_\lambda$. Then q_λ is a submultiplicative r_λ-homogeneous semi-norm on A which is equivalent with p_λ. Now putting $\mathcal{Q} = \{q_\lambda \mid \lambda \in \Lambda\}$, we see that $(A, T(\mathcal{P}))$ and $(A, T(\mathcal{Q}))$ are topologically isomorphic algebras.

By Theorem 2, Corollary 2 of [**2**] and Proposition 2.1 of [**3**] we have the following result.

Corollary 3. *Let A be a commutative locally pseudoconvex algebra with unit T, the topology of which has been defined by a family $\mathcal{P} = \{p_\lambda \mid \lambda \in \Lambda\}$ of r_λ-homogeneous semi-norms, where $r_\lambda \in (0, 1]$ is fixed for all $\lambda \in \Lambda$. If, for each $\lambda \in \Lambda$, there is an increasing function $\phi_\lambda : [0, \infty) \mapsto [0, \infty)$ satisfying the condition* (5), *then $(A, T(\mathcal{P}))$ is a commutative Gelfand-Mazur algebra with continuous inversion.*

Corollary 4. *Let A be a commutative locally pseudoconvex algebra, the topology of which has been given by a family $\mathcal{P} = \{p_\lambda \mid \lambda \in \Lambda\}$ of r_λ-homogeneous subquadrative semi-norms. Then*

(a) *$(A, T(\mathcal{P}))$ is a commutative locally A-pseudoconvex algebra* (*in the particular case when A is strongly spectrally bounded, then A is a commutative locally uniformly A-pseudoconvex algebra*),

(b) *there exists a locally m-pseudoconvex topology T on A such that $(A, T(\mathcal{P}))$ and (A, T) are topologically isomorphic algebras.*

If $p_\lambda(x^2) = p_\lambda(x)^2$ for all $x \in A$ and $\lambda \in \Lambda$ in Corollary 3, then $T(\mathcal{P})$ is equivalent with a locally m-convex topology on A. This condition actually forces A to be commutative (see [**4**]).

Note that, for a certain type of semi-norms, the subquadrativity implies the submultiplicativity as the following example shows.

Example. Let X be the set of real numbers equipped with its usual topology and take $A = C(X)$. Let $C_b(X)^+ = \{g \in C_b(X) | g(t) \geq 0, t \in$

$X\}$. For $n \in \mathbf{N}$ and $g \in C_b(X)^+$ denote by $p_{(n,g)}$ the semi-norm on A defined by

$$p_{(n,g)}(x) = \sup_{t\in[-n,n]} g(t)|x(t)|, \quad x \in A.$$

Now $\mathcal{P} = \{p_{(n,g)} | n \in \mathbf{N}, g \in C_b(X)^+\}$ defines a topology on A which is clearly locally A-convex. Now, if we take $p_{(n,g)}$ from $\mathcal{P}$ and assume that $p_{(n,g)}$ is subquadrative, then we must have

$$\sup_{t\in[-n,n]} g(t)|x(t)|^2 \leq \sup_{t\in[-n,n]} g(t)^2|x(t)|^2 \quad \text{for all } x \in A.$$

But this is possible only if $g(t) \geq 1$ for all $t \in [-n,n]$. If $g(t) \geq 1$ for all $t \in [-n,n]$, then it is easy to see that $p_{(n,g)}$ is submultiplicative. Note that if we assume that $p_{(n,g)}$ is subquadrative and $p_{(n,g)}(e) = 1$, then $g(t) = 1$ for all $t \in [-n,n]$. By using this example it is easy to construct a semi-norm which is locally A-convex but not submultiplicative. (Just take for any $n \in \mathbf{N}$ a function $g \in C_b(X)^+$ for which there is a point $t_0 \in [-n,n]$ such that $g(t_0) < 1$. It is easy to see that if $g(t) > 0$ for all $t \in [-n,n]$, then the semi-norm $p_{(n,g)} \in \mathcal{P}$ is equivalent with a submultiplicative semi-norm q_n where $q_n(x) = \sup_{t\in[-n,n]} |x(t)|$, $x \in A$. This type of semi-norm was studied in [**9**], and it can also be used in functional representation of some locally A-convex algebras, see [**10**].

Open problem. Suppose that A is a commutative algebra with unit and let p be a semi-norm on A. It would be interesting to know whether from the conditions p is subquadrative and $p(e) = 1$ it follows that p is submultiplicative.

REFERENCES

1. M. Abel, *Strongly spectrally bounded algebras*, Act. Comm. Univ. Tartuensis **899** (1990), 71–90.

2. ———, *Gelfand-Mazur algebras*, in *Topological vector spaces, algebras and related areas*, Pitman Research Notes in Math. **316**, London, Longman Group Limited, 1994, 116–129.

3. M. Abel and A. Kokk, *Locally pseudoconvex Gelfand-Mazur algebras*, Proc. Acad. Sci. Estonian Phys.-Math. **37** (1988), 377–386, in Russian.

4. R. Arens and M. Goldberg, *Multiplicativity factors for semi-norms*, J. Math. Anal. Appl. **146** (1990), 469–481.

5. R. Arens, M. Goldberg and W.A.J. Luxemburg, *Multiplicativity factors for seminorms* II, J. Math. Anal. Appl. **170** (1992), 401–413.

6. J. Arhippainen, *On locally convex square algebras*, Funct. App. **22** (1993), 57–63.

7. ———, *On locally pseudoconvex square algebras*, Publ. Matem. **39** (1995), 89–93.

8. S.J. Bhatt and D.J. Karia, *Uniqueness of the uniform norm with an application to topological algebras*, Proc. Amer. Math. Soc. **116** (1992), 499–503.

9. W. Dietrich, *Function algebras on completely regular spaces*, Diss. Northwestern Univ. Evanston, IL, 1971.

10. L. Oubbi, *Representation of locally convex algebras*, Rev. Mat. Univ. Compl. Madrid **7** (1994), 233–243.

11. L. Waelbroeck, *Topological vector spaces and algebras*, Springer-Verlag, Berlin, 1971.

12. W. Żelazko, *Selected topics in topological algebras*, Lecture Notes Series, Aarhus Univ. **31** (1971),

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