ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 4, Winter 1998

SOME CHARACTERIZATIONS FOR BOX SPLINE WAVELETS AND LINEAR DIOPHANTINE EQUATIONS

DING-XUAN ZHOU

ABSTRACT. Box splines are investigated from the point of view of wavelets. Some characterizations concerning linear independence of integer translates of Box splines are presented in terms of the defining matrices. It is shown that a direct extension of a criterion for linear independence of refinable functions in the univariate case to the multivariate case holds for the Box spline M_Ξ in $\mathbf{R}^s$ when $\operatorname{rank}\Xi = s$ while not any more when $\operatorname{rank}\Xi < s$.

1. Introduction and main results. Stability and linear independence of integer translates of a refinable function or distribution play basic roles in wavelet decompositions and multivariate splines. These properties can be characterized by the Fourier-Laplace transform of this distribution. It was shown by Ron [**17**] that, for a compactly supported distribution ϕ in $\mathbf{R}^s$, $\{\phi(\cdot - \alpha) : \alpha \in \mathbf{Z}^s\}$ are linearly independent if and only if, for any $\omega \in \mathbf{C}^s$, there exists some $\alpha \in \mathbf{Z}^s$ such that $\phi^\wedge(\omega + 2\pi\alpha) \neq 0$, where $\phi^\wedge$ is the Fourier-Laplace transform of ϕ.

Suppose that ϕ is k-refinable, $2 \le k \in \mathbf{N}$, say

$$\phi = \sum_{\alpha\in\mathbf{Z}^s} b_\alpha \phi(k\cdot - \alpha), \tag{1.1}$$

$$\phi^\wedge(0) = 1, \tag{1.2}$$

where $\{b_\alpha\}_{\alpha\in\mathbf{Z}^s}$ is a finitely supported sequence called the mask sequence of the refinement equation (1.1). Then ϕ can be determined by

Received by the editors on March 22, 1995, and in revised form on December 10, 1996.
AMS *Mathematics Subject Classification.* 42C15, 41A63, 41A15.
Key words and phrases. Linear independence, box spline wavelets, refinement equation, linear diophantine equations.
The author is supported by the Alexander von Humboldt Foundation, Research Grants Council and City University of Hong Kong. Part of the paper was written while the author was in the Department of Mathematics, University of Duisburg, D-47048 Duisburg, Germany.

the mask sequence as follows:

$$\phi^\wedge(\omega) = \prod_{j=1}^{\infty} \frac{1}{k^s} \tilde{b}(e^{-i(\omega/k^j)}), \quad \omega \in \mathbf{C}^s, \tag{1.3}$$

where

$$\tilde{b}(z) = \sum_{\alpha \in \mathbf{Z}^s} b_\alpha z^\alpha, \quad z \in (\mathbf{C}\backslash\{0\})^s \tag{1.4}$$

is the symbol of the mask sequence. Thus it is natural to investigate the linear independence of integer translates of ϕ in terms of the mask sequence b. In the univariate case $s = 1$ with $k = 2$, such criteria were given by Jia and Wang [**13**], Cohen [**3**], Cohen, Daubechies and Feauveau [**4**], and also Daubechies [**8, 9**]. The author extended their results to general $k \in \mathbf{N}$ in [**19, 21**] as follows.

Theorem A. *Let $s = 1$, $2 \leq k \in \mathbf{N}$, ϕ a compactly supported distribution satisfying* (1.1) *and* (1.2) *with a finitely supported mask sequence b. Then the integer translates of ϕ are linearly independent if and only if the following two conditions hold*:

(i) *For any $z \in \mathbf{C}\backslash\{0\}$,*

$$\sum_{l=0}^{k-1} |\tilde{b}(e^{-i2\pi(l/k)} z)| > 0; \tag{1.5}$$

(ii) *for any $m \in \mathbf{N}$ and $z \in T := \{z \in \mathbf{C} : |z| = 1\}$ satisfying $z^{k^m} = z \neq 1$, there exists some integer $d \geq 0$ such that*

$$\sum_{l=1}^{k-1} |\tilde{b}(e^{-i2\pi(l/k)} z^{k^d})| > 0. \tag{1.6}$$

The purpose of this paper is to consider the corresponding multivariate problem. We state first that the similar necessity still holds.

Theorem 1. *Let $s \in \mathbf{N}$, $2 \leq k \in \mathbf{N}$, $\mathcal{E}_{k,s} := \{\alpha = (\alpha_1, \dots, \alpha_s)^T \in \mathbf{Z}^s : 0 \leq \alpha_j \leq k-1$ for any $1 \leq j \leq s\}$. Suppose that ϕ is a compact supported distribution in $\mathbf{R}^s$ satisfying* (1.1) *and* (1.2) *with a finitely supported mask sequence b. If the integer translates of ϕ are linearly independent, then the following two conditions hold*:

(i) *For any $z \in (\mathbf{C}\backslash\{0\})^s$,*

$$\sum_{l \in \mathcal{E}_{k,s}} |\tilde{b}(e^{-i2\pi(l/k)} z)| > 0; \tag{1.7}$$

(ii) *for any $m \in \mathbf{N}$ and $z \in T^s$ satisfying $z^{k^m} = z \neq (1, \dots, 1)^T$, there exists some integer $d \geq 0$ such that*

$$\sum_{l \in \mathcal{E}_{k,s} \backslash \{(0, \dots, 0)^T\}} |\tilde{b}(e^{-i2\pi(l/k)} z^{k^d})| > 0. \tag{1.8}$$

From this result we may hope that the converse is also true as in the univariate case. However, we present examples of Box splines to show that this is not always the case. To this end, we shall give some characterizations for Box splines, especially concerning the second condition (1.8). Let us mention here that, as an important class of multivariate wavelets, Box splines, especially the constructions of Box spline wavelets and pre-wavelets, have been investigated by a series of papers [**1, 2, 12, 15, 16, 18, 20**].

Let $\Xi := (\xi_1, \dots, \xi_n)$ be an $s \times n$ integer matrix. Denote Ξ also as the set of nonzero integer vectors $\{\xi_1, \dots, \xi_n\}$. The Box spline M_Ξ associated with Ξ is the distribution, see [**1**], given by the rule

$$\langle f, M_\Xi \rangle := \int_{[0,1)^n} f(\Xi u)\, du, \quad f \in \mathcal{D}(\mathbf{R}^s), \tag{1.9}$$

or, equivalently, by the Fourier-Laplace transform,

$$M_\Xi^\wedge(\omega) = \prod_{j=1}^{n} \frac{1 - e^{-i\xi_j^T \omega}}{i\xi_j^T \omega}, \quad \omega \in \mathbf{C}^s. \tag{1.10}$$

It is easily seen that M_Ξ is k-refinable, i.e.,

$$M_\Xi^\wedge(\omega) = \frac{1}{k^s}\tilde{b}_\Xi(e^{-i(\omega/k)})M_\Xi^\wedge\left(\frac{\omega}{k}\right) \tag{1.11}$$

with

$$\tilde{b}_\Xi(e^{-i\omega}) = k^{s-n}\prod_{j=1}^{n}\frac{1-e^{-ik\xi_j^T\omega}}{1-e^{-i\xi_j^T\omega}}, \quad \omega\in\mathbf{C}^s. \tag{1.12}$$

For $l\in\mathcal{E}_{k,s}$, we denote

$$\Xi_l := \{\xi\in\Xi : l\cdot\xi\equiv l^T\xi\notin k\mathbf{Z}\}. \tag{1.13}$$

For an $m\times n$ integer matrix A, we denote by $d_{A,p}$ the greatest common divisor of all $p\times p$ minors of A for $1\le p\le\min\{m,n\}$. For $J\subset\{1,2,\dots,n\}$, we denote by $A(J)$ the matrix made up of the columns of A indicated by J.

Now we can give the characterization of the condition (1.8) for Box splines as follows.

Theorem 2. *Let $s,n\in\mathbf{N}$, $2\le k\in\mathbf{N}$, Ξ be an $s\times n$ integer matrix with $\xi_j\ne 0$, $1\le j\le n$, $\tilde{b}_\Xi$ be given by* (1.12). *Then the following statements are equivalent*:

(i) *For any $m\in\mathbf{N}$ and $z\in T^s$ satisfying $z^{k^m}=z\ne(1,\dots,1)^T$, there exists some integer $d\ge 0$ such that*

$$\sum_{l\in\mathcal{E}_{k,s}\backslash\{(0,\dots,0)^T\}}|\tilde{b}_\Xi(e^{-i2\pi(l/k)}z^{k^d})|>0. \tag{1.14}$$

(ii) *One of the following two conditions holds*:

(a) *There is some $l\in\mathcal{E}_{k,s}\backslash\{(0,\dots,0)^T\}$ such that*

$$\Xi_l=\varnothing,\ \text{the empty set}; \tag{1.15}$$

(b) *for any $s\times(k^s-1)$ matrix X whose columns are $x_l\in\Xi_l$, $l\in\mathcal{E}_{k,s}\backslash\{(0,\dots,0)^T\}$ and any prime $p\in\mathbf{N}$,*

$$p\left|\frac{d_{X,s}}{d_{X,s-1}}\right.\quad\text{implies}\quad p|k. \tag{1.16}$$

(iii) *For any* $m \in \mathbf{N}$ *and* $z \in T^s$ *satisfying* $z^{k^m} = z \neq (1, \dots, 1)^T$, *the following holds*

$$\sum_{l \in \mathcal{E}_{k,s} \setminus \{(0,\dots,0)^T\}} |\tilde{b}_\Xi(e^{-i2\pi(l/k)} z)| > 0. \tag{1.17}$$

Thus, for the Box spline distributions, the second condition (1.8) of Theorem 1 can be reduced to two conditions on the defining matrix Ξ. We can simplify condition (ii)(a) further to the following characterization.

Theorem 3. *Let* k, s, Ξ *be given as in Theorem* 2. *If* rank $(\Xi) = s$, *then* (ii)(a) *of Theorem* 2 *holds if and only if*

$$\left(\frac{d_{\Xi,s}}{d_{\Xi,s-1}}, k \right) > 1. \tag{1.18}$$

If rank $(\Xi) < s$, *then* (ii)(a) *of Theorem 2 always holds.*

Condition (ii)(b) of Theorem 2 can also be simplified by means of the lemmas in Section 2, see Lemma 5.

The condition (1.7) for Box spline distributions is closely related with linear independence of integer translates of discrete Box splines, which has been completely characterized in [**1, 7, 11**]. Let us recall the so-called discrete Box splines. For the scaling matrix $H :=$ diag $\{1/k, \dots, 1/k\}$ and the integer matrix Ξ, the discrete Box spline $b_H(\cdot|\Xi)$ can be defined by its Fourier-Laplace transform as

$$\begin{aligned} b_H(\cdot|\Xi)^\wedge(\omega) &= \prod_{j=1}^{n} \frac{1 - e^{-i\xi_j^T \omega}}{1 - e^{-i\xi_j^T \omega/k}} \\ &= k^{n-s} \tilde{b}_\Xi(e^{-i\omega/k}). \end{aligned} \tag{1.19}$$

From these formulas we can see that the condition (1.7) with $b = b_\Xi$ is equivalent to the linear independence of integer translates of the discrete Box spline $b_H(\cdot|\Xi)$, while the latter problem has been completely solved by Jia [**11**], Dahmen and Micchelli [**7**], de Boor, Höllig and Riemenschneider [**1**]. Jia showed in [**11**] that, when $\operatorname{rank}\Xi = s$, the integer translates of $b_H(\cdot|\Xi)$ are linearly independent if and only if k is relatively prime to $|\det B|$ for any $\mathbf{R}^s$-basis $B \subset \Xi$. De Boor, Höllig and Riemenschneider [**1**] extended this result to the case when $\operatorname{rank}\Xi < s$ and proved that the integer translates of $b_H(\cdot|\Xi)$ are linearly independent if and only if k is relatively prime to $d_{Z,\operatorname{rank}(Z)}$ for any linearly independent subset $Z \subset \Xi$.

Combining these results with Theorems 2 and 3, we state that, for Box spline distributions, the converse of Theorem 1 holds when $\operatorname{rank}\Xi = s$ while not any more when $\operatorname{rank}\Xi < s$.

Theorem 4. *Let Ξ and $\tilde{b}_\Xi$ be given as in Theorem* 2. *If* $\operatorname{rank}\Xi = s$, *then the integer translates of M_Ξ are linearly independent if and only if the conditions* (1.7) *and* (1.8) *of Theorem* 1 *hold for $b = b_\Xi$.*

Theorem 5. *Let Ξ be an $s \times n$ integer matrix with* $\operatorname{rank}\Xi < s$. *If, for any linearly independent subset $Z \subset \Xi$, $(d_{Z,\operatorname{rank}(Z)}, k) = 1$, while for some linearly independent subset $Y \subset \Xi, d_{Y,\operatorname{rank}(Y)} > 1$, then the conditions* (1.7) *and* (1.8) *of Theorem* 1 *hold for $b = b_\Xi$, while the integer translates of M_Ξ are linearly dependent.*

Finally we mention a relation between stability and linear independence of integer translates of refinable distributions. The integer translates of a compactly supported distribution ϕ in $\mathbf{R}^s$ are said to be r-linearly independent, $0 \le r \le \infty$, if for any $\omega := (\omega_1, \dots, \omega_s)^T$ in $\mathbf{C}^s$ with $-r \le \operatorname{Im}\omega_j \le r$, $1 \le j \le s$, there is some $\alpha \in \mathbf{Z}^s$ such that $\phi^\wedge(\omega + 2\pi\alpha) \ne 0$. We say that ϕ has stable integer translates if the integer translates of ϕ are zero-linearly independent, see also the definition of stability given by Jia and Wang in [**13**].

Theorem 6. *Let ϕ and b satisfy the assumptions of Theorem* 1 *and* $0 \le r \le \infty$. *Then the integer translates of ϕ are r-linearly independent if and only if the following two conditions hold*:

(i) *For any* $z := (z_1, \dots, z_s)^T \in \mathbf{C}^s$ *with* $e^{-r/k} \le |z_j| \le e^{r/k}$,

$$\sum_{l \in \mathcal{E}_{k,s}} |\tilde{b}(e^{-i2\pi(l/k)} z)| > 0;$$

(ii) *the integer translates of* ϕ *are stable.*

2. Lemmas. In the proofs of the main results, we need some lemmas related to the following system of linear diophantine equations

$$Ay = b, \tag{2.1}$$

where A is an $m \times n$ integer matrix and b is an integer m-vector.

The following preliminary result can be found in [**11**, Theorem 3.2].

Lemma 1. *Let A be an integer matrix of full row rank. Then the system* (2.1) *has an integer solution for y if and only if $d_{A,m} = d_{[A,b],m}$.*

Using Lemma 1, we have

Lemma 2. *Let A be an $m \times n$ integer matrix of full row rank, $d \in \mathbf{N}$ a divisor of $d_{A,m}$. Then the following system of linear diophantine equations*

$$Ay = db \tag{2.2}$$

has an integer solution $y \in \mathbf{Z}^n$ for any $b \in \mathbf{Z}^m$ if and only if

$$\frac{d_{A,m}}{d_{A,m-1}} \Big| d. \tag{2.3}$$

Proof of Lemma 2. We use the method of Jia [**11**].

By Lemma 1, the sufficiency is trivial since $d_{A,m} = d_{[A,db],m}$.

To prove the necessity, we note that

$$d_{A,m-1} = \text{g.c.d.}\{d_{X,m-1} : X \text{ is an } m \times (m-1) \text{ submatrix of } A\}. \tag{2.4}$$

By Lemma 1, for any $b \in \mathbf{Z}^m$, the following holds

$$d_{A,m} = d_{[A,db],m},$$

which is equivalent to that for any $m \times (m-1)$ submatrix X of A,

$$d_{A,m} | \det(X, db),$$

i.e.,

$$\frac{d_{A,m}}{d} \Big| \det(X, b).$$

For any fixed X, we choose $b \in \mathbf{Z}^m$ such that

$$\det(X, b) = d_{X,m-1},$$

hence

$$\frac{d_{A,m}}{d} \Big| d_{X,m-1},$$

and, by (2.4),

$$\frac{d_{A,m}}{d} \Big| d_{A,m-1}.$$

Therefore, we have

$$\frac{d_{A,m}}{d_{A,m-1}} \Big| d.$$

The proof of Lemma 2 is complete. $\square$

Lemma 3. *Let $q \in \mathbf{Z}^s$ be such that $d_{q,1} = 1$. Then, for any $r \in \mathbf{C}$, $rq \in \mathbf{Z}^s$ if and only if $r \in \mathbf{Z}$.*

Proof of Lemma 3. The sufficiency is trivial.

Suppose that $r \in \mathbf{C}$ is such that $rq \in \mathbf{Z}^s$. By [**1**, Lemma 6.23], there exists an $s \times (s-1)$ integer matrix X such that $\det[X, q] = 1$. Hence,

$$[X, q]^{-1} = [Y_1, Y_2]^T \in \mathbf{Z}^{s \times s},$$

where $Y_1 \in \mathbf{Z}^{s \times (s-1)}$, $Y_2 \in \mathbf{Z}^{s \times 1}$. Therefore, $r = Y_2^T rq \in \mathbf{Z}$.

The proof of Lemma 3 is complete. $\square$

Lemma 4. *Let* $\Xi \in \mathbf{Z}^{s\times n}$, X *be an* $s \times (k^s - 1)$ *matrix whose columns are* $x_l \in \Xi_l$, $l \in \mathcal{E}_{k,s} \backslash \{(0, \ldots, 0)^T\}$. *Then* $\operatorname{rank}(X) = s$.

Proof of Lemma 4. Suppose to the contrary that $\operatorname{rank}(X) := p < s$. We choose $J \subset \mathcal{E}_{k,s} \backslash \{(0, \ldots, 0)^T\}$ such that $\operatorname{rank}(X) = \operatorname{rank}(X(J)) = p = \#J$, let $\tilde{X}$ be an $s \times (s - p)$ integer matrix such that $\operatorname{rank}[X(J), \tilde{X}] = s$. Denote $[X(J), \tilde{X}]^{-1} = [Y_1, Y_2]^T$ with $Y_1 \in \mathbf{R}^{s\times(s-1)}$ and $Y_2 \in \mathbf{R}^{s\times 1}$. We know that $\det[X(J), \tilde{X}][Y_1, Y_2] \in \mathbf{Z}^{s\times s}$, $Y_2 \neq (0, \ldots, 0)^T$ and $Y_2^T X(J) = 0$. Therefore, we can choose $r \in \mathbf{Z}^{s\times 1}$ such that $d_{r,1} = 1$ and

$$\frac{r^T}{k} X(J) = 0,$$

which implies

$$\frac{r^T}{k} X = 0.$$

Choose $l \in \mathcal{E}_{k,s} \backslash \{(0, \ldots, 0)^T\}$ and $\alpha \in \mathbf{Z}^s$ such that

$$r = l + k\alpha.$$

Then, for any $x \in X$,

$$\frac{l}{k} \cdot x = \left(\frac{r}{k} - \alpha\right) \cdot x \in \mathbf{Z}.$$

In particular,

$$l \cdot x_l \in k\mathbf{Z},$$

which is a contradiction.

The proof of Lemma 4 is complete. $\square$

Lemma 5. *Let* B *be an* $s \times s$ *submatrix of* $\Xi \in \mathbf{Z}^{s\times n}$ *with* $|\det B| > 1$ *and* $(|\det B|, k) = 1$. *Then, for any* $l \in \mathcal{E}_{k,s} \backslash \{(0, \ldots, 0)^T\}$, *there exists some* $\xi \in B \subset \Xi$ *such that* $l \cdot \xi \notin k\mathbf{Z}$, *i.e.,* $\xi \in \Xi_l$.

Proof of Lemma 5. Suppose to the contrary that, for some $l \in \mathcal{E}_{k,s} \backslash \{(0, \ldots, 0)^T\}$,

$$B^T l \in k\mathbf{Z}^s.$$

Then
$$\frac{l}{k} \in (B^T)^{-1}\mathbf{Z}^s \subset \frac{1}{|\det B|}\mathbf{Z}^s.$$
Hence,
$$\frac{|\det B| d_{l,1}}{k}\frac{l}{d_{l,1}} \in \mathbf{Z}^s.$$

By Lemma 3, the following holds
$$\frac{|\det B| d_{l,1}}{k} \in \mathbf{Z},$$
which implies $d_{l,1}/k \in \mathbf{Z}$ since $(|\det B|, k) = 1$. Therefore, we have
$$\frac{l}{k} = \frac{d_{l,1}}{k}\frac{l}{d_{l,1}} \in \mathbf{Z}^s,$$
which is a contradiction. The proof of Lemma 5 is complete. □

3. Proofs of the main results. The proof of Theorem 1 is similar to that given in [**19**].

Proof of Theorem 1. Suppose that the integer translates of ϕ are linearly independent. Then the first condition (1.7) must be satisfied since otherwise there is some $z_0 = e^{-i\omega_0}$ with $\omega_0 \in \mathbf{C}^s$ such that $\sum_{l\in\mathcal{E}_{k,s}} |\tilde{b}(e^{-i2\pi(l/k)}e^{-i\omega_0})| = 0$, which implies, by (1.1), for any $\alpha \in \mathbf{Z}^s$, i.e., any $\beta \in \mathbf{Z}^s$ and $l \in \mathcal{E}_{k,s}$,
$$\begin{aligned}\phi^\wedge(k\omega_0 + 2\pi\alpha) &= \phi^\wedge(k\omega_0 + 2\pi l + 2\pi k\beta)\\ &= \frac{1}{k^s}\tilde{b}(e^{-i2\pi(l/k)}e^{-i\omega_0})\phi^\wedge\left(2\pi\frac{l}{k} + \omega_0 + 2\pi\beta\right)\\ &= 0.\end{aligned}$$

Now we prove the second condition (1.8). Suppose to the contrary that, for some $m \in \mathbf{N}$ and $z \in T^s$ satisfying $z^{k^m} = z \neq (1,\dots,1)^T$, and any integer $d \geq 0$, the following holds
$$\sum_{l\in\mathcal{E}_{k,s}\setminus\{(0,\dots,0)^T\}} |\tilde{b}(e^{-i2\pi(l/k)}z^{k^d})| = 0.$$

Then, for some $n \in \mathbf{Z}^s$ with $n/(k^m-1) \notin \mathbf{Z}^s$, $z = e^{-i2\pi(n/(k^m-1))}$. We show that, for $\alpha \in \mathbf{Z}^s$,

$$\phi^\wedge\left(2\pi\frac{n}{k^m-1} + 2\pi\alpha\right) = 0. \tag{3.1}$$

To this end, set $n+(k^m-1)\alpha = k^p q$ where $0 \le p \in \mathbf{Z}$ and $q \in \mathbf{Z}^s \backslash k\mathbf{Z}^s$. Hence

$$\begin{aligned}\phi^\wedge\left(2\pi\frac{n}{k^m-1} + 2\pi\alpha\right) &= \phi^\wedge\left(2\pi\frac{k^p q}{k^m-1}\right) \\ &= \prod_{j=1}^{p+1}\left\{\frac{1}{k^s}\tilde{b}(e^{-i2\pi(k^{p-j}q/(k^m-1))})\right\} \\ &\quad \cdot \phi^\wedge\left(2\pi\frac{q}{k(k^m-1)}\right).\end{aligned}$$

We state that

$$\tilde{b}(e^{-i2\pi(q/(k(k^m-1)))}) = 0. \tag{3.2}$$

To prove (3.2), choose $r = (k^{m(p+1)}-1)/(k^m-1) \in \mathbf{N}$. It is easily seen that $(r,k)=1$. Let

$$-rq = ku + v, \tag{3.3}$$

where $u \in \mathbf{Z}^s$, $v \in \mathcal{E}_{k,s}$. We must have $v \neq (0,\dots,0)^T$, since otherwise $-rq \in k\mathbf{Z}^s$ which implies $q \in k\mathbf{Z}^s$, a contradiction. Therefore,

$$\begin{aligned}\tilde{b}(e^{-i2\pi(q/(k(k^m-1)))}) &= \tilde{b}(e^{-i2\pi q(k^m(p+1)-r(k^m-1))/(k(k^m-1))}) \\ &= \tilde{b}(e^{-i2\pi(-rq/k)}(e^{-i2\pi(k^p q/(k^m-1))})^{k^{(m-1)(p+1)}}) \\ &= \tilde{b}(e^{-i2\pi(v/k)}(e^{-i2\pi(n/(k^m-1))})^{k(m-1)(p+1)}) \\ &= \tilde{b}(e^{-i2\pi(v/k)}z^{k^{(m-1)(p+1)}}) \\ &= 0.\end{aligned}$$

Thus (3.2) is valid, hence (3.1) holds, which is a contradiction. The proof of Theorem 1 is complete. $\square$

The proof of Theorem 2 is much more involved. Here we must use the lemmas given in Section 2.

Proof of Theorem 2. (i) $\Rightarrow$ (ii). If (ii)(a) holds, there is nothing to prove. If (ii)(a) does not hold, i.e., Ξ_l is not empty for any $l \in \mathcal{E}_{k,s}\backslash\{(0,\dots,0)^T\}$, we prove (ii)(b). Suppose to the contrary that, for some $s\times(k^s-1)$ matrix X whose columns are $x_l \in \Xi_l$, $l \in \mathcal{E}_{k,s}\backslash\{(0,\dots,0)^T\}$, and some prime $p \in \mathbf{N}$, the following holds

$$p\Big|\frac{d_{X,s}}{d_{X,s-1}}, \tag{3.4}$$

while

$$(p,k)=1. \tag{3.5}$$

By Lemma 4, let

$$d_{X,s}=p^{r_1}d_1 \tag{3.6}$$

and

$$d_{X,s-1}=p^{r_2}d_2, \tag{3.7}$$

where $r_1, d_1, d_2 \in \mathbf{N}$, $(d_1,d_2,p)=1$, $r_2 \in \mathbf{N}\cup\{0\}$, $r_1 \geq r_2+1$.

By (2.4) and Lemma 4, we can find an $s\times(s-1)$ submatrix $\tilde{B}$ of X such that

$$d_{\tilde{B},s-1}=p^{r_2}d_3 \tag{3.8}$$

while $d_3 \in \mathbf{N}$, $(d_3,p)=1$.

By Lemma 4, choose an $s\times s$ submatrix B of X containing $\tilde{B}$ as its submatrix and $\det B \neq 0$. Then

$$|\det B| = d_{B,s} = p^{r_3}d_4, \tag{3.9}$$

where r_3, $d_4 \in \mathbf{N}$, $r_3 \geq r_1$, $(d_4,p)=1$, $d_1|d_4$.

Let us mention that, by (3.7), (3.8) and (3.9),

$$d_{B^T,s-1}=d_{B,s-1}=p^{r_2}d_5 \tag{3.10}$$

with $d_5 \in \mathbf{N}$, $(d_5, p) = 1$, $d_5 | d_4$.

Using Lemma 2, we choose some $b \in \mathbf{Z}^s$ such that

$$B^T y = p^{r_3 - r_2 - 1} d_4 b \tag{3.11}$$

has no integer solutions $y \in \mathbf{Z}^s$. Define $y_0 \in \mathbf{Z}^s$ to be the unique solution to the following system of linear diophantine equations

$$B^T y = p^{r_3 - r_2} d_4 b, \tag{3.12}$$

i.e.,

$$y_0 = p^{r_3 - r_2} d_4 (B^T)^{-1} b. \tag{3.13}$$

We know that $(1/p) y_0 \notin \mathbf{Z}^s$.

We state that

$$\frac{1}{p} X^T y_0 \in \mathbf{Z}^{k^s - 1}, \tag{3.14}$$

i.e., for any $x \in X$,

$$x^T \frac{y_0}{p} \in \mathbf{Z}. \tag{3.15}$$

To prove this statement, it is sufficient to show that, for $x \in X$,

$$P_x := p^{r_3 - r_2 - 1} d_4 B^{-1} x \in \mathbf{Z}^s. \tag{3.16}$$

To this end, we note that P_x is the unique solution to the system of equations

$$By = p^{r_3 - r_2 - 1} d_4 x. \tag{3.17}$$

Thus, by Lemma 1 we only need to prove

$$d_{B,s} = d_{[B, p^{r_3 - r_2 - 1} d_4 x], s},$$

i.e., for any $s \times (s-1)$ submatrix B_1 of B,

$$d_{B,s} | p^{r_3 - r_2 - 1} d_4 \det [B_1, x]. \tag{3.18}$$

From the definition of $d_{X,s}$ and (3.6), we have

$$p^{r_3-r_2-1}d_4\det[B_1,x] = p^{r_3-r_2-1+r_1}d_4d_1d_6$$

with $d_6 \in \mathbf{Z}$. Hence

$$p^{r_3-r_2-1}d_4\det[B_1,x] = p^{r_1-r_2-1}d_1d_6d_{B,s}.$$

Therefore, (3.18) holds, which implies (3.15) and (3.14).

By (3.5) and Euler's theorem, there exists an $m \in \mathbf{N}$ such that $p|(k^m-1)$. Hence,

$$\frac{y_0}{p} = \frac{q}{k^m-1},$$

where $q \in \mathbf{Z}^s$, $q/(k^m-1) \notin \mathbf{Z}^s$.

Thus, for $z = e^{-i2\pi(q/(k^m-1))} \in T^s$ satisfying $z^{k^m} = z \neq (1,\dots,1)^T$, any integer $d \geq 0$, and any $l \in \mathcal{E}_{k,s}\backslash\{(0,\dots,0)^T\}$, we have

$$\begin{aligned}\tilde{b}_\Xi(e^{-i2\pi(l/k)}z^{k^d}) &= \tilde{b}_\Xi(e^{-i2\pi(l/k+k^dq/(k^m-1))}) \\ &= 0,\end{aligned} \tag{3.19}$$

since $\tilde{b}_\Xi(e^{-i\omega}) = 0$ if and only if, for some $\xi \in \Xi$, $k\xi^T\omega \in 2\pi\mathbf{Z}\backslash 2\pi k\mathbf{Z}$ while, by (3.14), for $x_l \in X$,

$$\begin{aligned}kx_l^T 2\pi\left(\frac{l}{k} + \frac{k^dq}{k^m-1}\right) &= 2\pi x_l^T l + 2\pi k^{d+1}\frac{x_l^T q}{k^m-1} \\ &= 2\pi x_l^T l + 2\pi k^{d+1}x_l^T\frac{y_0}{p} \in 2\pi\mathbf{Z}\backslash 2\pi k\mathbf{Z}.\end{aligned}$$

The conclusion (3.19) is a contradiction to (1.14).

The proof of the first implication is complete.

(ii) $\Rightarrow$ (iii). If (ii)(a) is satisfied, say for some $l \in \mathcal{E}_{k,s}\backslash\{(0,\dots,0)^T\}$, Ξ_l is empty, i.e.,

$$\Xi^T l \in k\mathbf{Z}^n.$$

Then, for any $z = e^{-i2\pi(q/(k^m-1))} \in T^s$ satisfying $z^{k^m} = z \neq (1,\dots,1)^T$, and $\xi \in \Xi$, we have

$$2\pi k\xi^T\left(\frac{l}{k} + \frac{q}{k^m-1}\right) = 2\pi\xi^T l + 2\pi k\frac{\xi^T q}{k^m-1} \notin 2\pi\mathbf{Z}\backslash 2\pi k\mathbf{Z},$$

which implies

$$\tilde{b}_\Xi(e^{-i2\pi(l/k)}z) \neq 0.$$

Hence, (1.17) holds.

If (ii)(a) is not satisfied while (ii)(b) holds, we prove that (iii) is valid. Suppose to the contrary that, for some $m \in \mathbf{N}$ and some $z \in T^s$ satisfying $z^{k^m} = z \neq (1, \dots, 1)^T$, the following holds

$$\sum_{l \in \mathcal{E}_{k,s} \setminus \{(0,\dots,0)^T\}} |\tilde{b}_\Xi(e^{-i2\pi(l/k)}z)| = 0.$$

Let $z = e^{-i2\pi(\eta/(k^m-1))}$ with $\eta \in \mathbf{Z}^s$ and $\eta/(k^m - 1) \notin \mathbf{Z}^s$. Then, for any $l \in \mathcal{E}_{k,s} \setminus \{(0, \dots, 0)^T\}$, we have some $x_l \in \Xi$ such that

$$kx_l^T \left(\frac{l}{k} + \frac{\eta}{k^m - 1} \right) \in \mathbf{Z} \backslash k\mathbf{Z}. \tag{3.20}$$

Since $(k^m - 1, k) = 1$, (3.20) implies

$$\frac{x_l^T \eta}{k^m - 1} \in \mathbf{Z} \tag{3.21}$$

and

$$x_l^T l \in \mathbf{Z} \backslash k\mathbf{Z},$$

i.e.,

$$x_l \in \Xi_l.$$

Let X be the $s \times (k^s - 1)$ integer matrix whose columns are these x_l, $l \in \mathcal{E}_{k,s} \setminus \{(0, \dots, 0)^T\}$. Define $p \in \mathbf{N}$ to be prime and $q \in \mathbf{Z}^s$ such that $(p, d_{q,1}) = 1$ and $q/p = r(\eta/(k^m - 1))$ for some $r \in \mathbf{Z}$, $(p, k) = 1$. Then, from (3.21),

$$X^T \frac{q}{p} = b \in \mathbf{Z}^{k^s - 1}. \tag{3.22}$$

We state that

$$p \Big| \frac{d_{X,s}}{d_{X,s-1}}. \tag{3.23}$$

To prove (3.23) by Lemma 4, let

$$d_{X,s} = p^{r_1} d_1, \tag{3.24}$$

$$d_{X,s-1} = p^{r_2} d_2, \tag{3.25}$$

where $r_1, r_2 \geq 0$, $d_1, d_2 \in \mathbf{N}$, $(d_1 d_2, p) = 1$, $d_2 | d_1$.

Let B be an arbitrary $s \times s$ submatrix of X such that $\det B \neq 0$. Since $d_{B,s-1} | d_{B,s}$, let

$$d_{B^T,s} = d_{B,s} = p^{r_3} d_3,$$
$$d_{B^T,s-1} = d_{B,s-1} = p^{r_4} d_4,$$

where $r_3, r_4 \in \mathbf{Z}$, $r_3 \geq r_4 \geq 0$, $d_3, d_4 \in \mathbf{N}$, $(d_3 d_4, p) = 1$, $d_4 | d_3$.

If $r_3 = r_4$, by (3.22), we know that

$$B^T \frac{d_3}{d_4} \frac{q}{p} = \frac{d_3}{d_4} b.$$

By Lemma 2, the unique solution $y = (d_3/d_4)(q/p)$ to the system of linear diophantine equations

$$B^T y = \frac{d_3}{d_4} b$$

must be in $\mathbf{Z}^s$, hence by Lemma 3, $d_{q,1} d_3/p \in \mathbf{Z}$, which is a contradiction.

Thus, $r_3 > r_4$. By (3.25), we then have $r_3 \geq r_4 + 1 \geq r_2 + 1$. Since B is arbitrary, from (3.24) we get

$$r_1 \geq r_2 + 1,$$

which implies (3.23).

By condition (ii)(b), (3.23) implies $p | k$, which is a contradiction since by our assumption $(p, k) = 1$. Therefore, (1.17) must be true.

The proof of the second implication is complete.

(iii) $\Rightarrow$ (i). This implication is trivial by choosing $d = 0$.

The proof of Theorem 2 is complete. $\square$

The proof of Theorem 3 depends mainly upon linear diophantine equations, especially Lemma 2.

Proof of Theorem 3. Assume first that $\operatorname{rank}\Xi = s$. Suppose that (ii)(a) of Theorem 2 holds, say for some $l \in \mathcal{E}_{k,s}\backslash\{(0,\dots,0)^T\}$, Ξ_l is empty. Then

$$\Xi^T \frac{l}{k} \in \mathbf{Z}^n.$$

Let $p \in \mathbf{N}$ be prime and $p|k/(k, d_{l,1})$. We show that

$$p\Big|\frac{d_{\Xi,s}}{d_{\Xi,s-1}}. \tag{3.26}$$

For any $s \times s$ submatrix B of Ξ with $\det B \neq 0$, we have

$$d_{\Xi,s-1}|\det B.$$

Let $b = B^T(l/k) \in \mathbf{Z}^s$. Then we know that $(|\det B|/(kd_{\Xi,s-1}))l$ is the unique solution to the following system of equations

$$B^T y = \frac{|\det B|}{d_{\Xi,s-1}} b.$$

By the definition of $d_{\Xi,s-1}, d_{\Xi,s-1}|d_{B^T,s-1}$. Therefore, Lemma 2 implies

$$\frac{|\det B|}{kd_{\Xi,s-1}} l \in \mathbf{Z}^s,$$

and, by Lemma 3,

$$\frac{|\det B|}{d_{\Xi,s-1}}\frac{d_{l,1}}{k} = \frac{(|\det B|/d_{\Xi,s-1})(d_{l,1}/(k,d_{l,1}))}{p(k/((k,d_{l,1})p))} \in \mathbf{Z}.$$

Notice that $p|(k/(k,d_{l,1}))$ implies $(p, d_{l,1}/(k,d_{l,1})) = 1$ and then $(p(k/((k,d_{l,1})p)), d_{l,1}/(k,d_{l,1})) = 1$. Hence,

$$\frac{|\det B|}{d_{\Xi,s-1}} \in p\frac{k}{(k,d_{l,1})p}\mathbf{Z} \subset p\mathbf{Z}.$$

Therefore, (3.26) holds, which implies $p|(k, d_{\Xi,s}/d_{\Xi,s-1})$, i.e., (1.18) is valid.

Suppose conversely that (1.18) holds, say $1 < p$ is prime and

$$p\left|\left(k, \frac{d_{\Xi,s}}{d_{\Xi,s-1}}\right).\right. \tag{3.27}$$

We show that (ii)(a) of Theorem 2 is valid. The method of proof is almost the same as that of the proof of the first implication of Theorem 2.

Let

$$d_{\Xi,s-1} = p^{r_1} d_1$$

with $r_1 \geq 0$, $d_1 \in \mathbf{N}$, $(d_1, p) = 1$. Choose an $s \times (s-1)$ submatrix $\tilde{B}$ of Ξ such that

$$d_{\tilde{B}^T,s-1} = p^{r_1} d_2$$

with $d_2 \in \mathbf{N}$, $(d_2, p) = 1$. Take an $s \times s$ submatrix B of Ξ containing $\tilde{B}$, and $\det B \neq 0$, then

$$p^{r_1+1} \nmid d_{B^T,s-1}$$

and

$$d_{B^T,s} = |\det B| = p^{r_2} d_3,$$

where r_2, $d_3 \in \mathbf{N}$, $r_2 \geq r_1 + 1$, $(d_3, p) = 1$.

By Lemma 2, there exists some $b \in \mathbf{Z}^s$ such that

$$B^T y = p^{r_2 - r_1 - 1} d_3 b$$

has no integer solutions, i.e.,

$$y_0 := p^{r_2 - r_1 - 1} d_3 (B^T)^{-1} b \notin \mathbf{Z}^s.$$

Now we prove that

$$\Xi^T y_0 \in \mathbf{Z}^n. \tag{3.28}$$

To this end, it is sufficient to show that, for any $\xi \in \Xi$,

$$P_\xi := p^{r_2 - r_1 - 1} d_3 B^{-1} \xi \in \mathbf{Z}^s.$$

Since P_ξ is the unique solution to the system of equations

$$By = p^{r_2-r_1-1}d_3\xi,$$

we only need to verify that

$$d_{B,s} = d_{[B,p^{r_2-r_1-1}d_3\xi],s}. \tag{3.29}$$

For any $s \times (s-1)$ submatrix B_1 of B, we have by (3.27) and the definition of $d_{\Xi,s}$,

$$\begin{aligned}\det[B_1, p^{r_2-r_1-1}d_3\xi] &= p^{r_2-r_1-1}d_3 \det[B_1,\xi],\\ &\in p^{r_2-r_1-1}d_3 p\, d_{\Xi,s-1}\mathbf{Z},\end{aligned}$$

i.e.,

$$d_{B,s} | \det[B_1, p^{r_2-r_1-1}d_3\xi].$$

Thus, (3.29), and hence (3.28), holds.

We observe that $(B^T)^{-1} \in (d_{\Xi,s-1}/|\det B|)\mathbf{Z}^{s\times s}$. Hence, by (3.27),

$$ky_0 \in \mathbf{Z}^s.$$

Therefore, we have some $l \in \mathcal{E}_{k,s}\backslash\{(0,\ldots,0)^T\}$ such that

$$y_0 \in \frac{l}{k} + \mathbf{Z}^s.$$

Then, by (3.28),

$$\Xi^T \frac{l}{k} \in \mathbf{Z}^n,$$

i.e., Ξ_l is empty.

We have completed the proof for the case $\operatorname{rank}\Xi = s$.

If $\operatorname{rank}\Xi < s$, the proof of Lemma 4 implies that, for some $l \in \mathcal{E}_{k,s}\backslash\{(0,\ldots,0)^T\}$, Ξ_l is empty. Hence (ii)(a) of Theorem 2 holds.

The proof of Theorem 3 is complete. $\square$

Finally let us turn to prove the main results on Box spline wavelets.

Proof of Theorem 4. By Theorem 1 we only need to prove the sufficiency. Assume that the two conditions (1.7) and (1.8) of Theorem 1 hold for $b = b_\Xi$. By the well-known characterization on linear independence of integer translates of Box splines given by Jia [**10**], Dahmen and Micchelli [**6**], it is sufficient to show that, for any $\mathbf{R}^s$-basis $B \subset \Xi$, $|\det B| = 1$.

Suppose to the contrary that, for some $\mathbf{R}^s$-basis $B \subset \Xi$,

$$|\det B| > 1.$$

Then, by [**11**, Corollary 4.3], (1.7) implies $(|\det B|, k) = 1$. By Lemma 5 and Theorems 2, 3, we conclude that (ii)(b) of Theorem 2 must hold. Then, using Lemma 5 again, we know that, for any prime $p \in \mathbf{N}$, $p|(d_{B,s}/d_{B,s-1})$ implies $p|k$. Since $(|\det B|, k) = 1$, we must have

$$d_{B,s} = d_{B,s-1},$$

from which it follows

$$B^{-1} \in \frac{d_{B,s-1}}{|\det B|}\mathbf{Z}^{s\times s} \subset \mathbf{Z}^{s\times s},$$

and, for any $b \in \mathbf{Z}^s$ the following system of linear diophantine equations

$$By = b$$

always has an integer solution. By [**11**, Corollary 3.3], we have $d_{B,s} = |\det B| = 1$, which is a contradiction.

The proof of Theorem 4 is complete. $\square$

Proof of Theorem 5. By Theorem 3, the second condition (1.8) of Theorem 1 always holds for $b = b_\Xi$ in case $\operatorname{rank}\Xi < s$. By our assumptions on Ξ and [**1**, Theorem 6.30], the first condition (1.7) is also satisfied. Hence the two conditions of Theorem 1 hold for $b = b_\Xi$.

On the other hand, by the result of Dahmen, Jia and Micchelli [**5**, Corollary 3.1], the integer translates of M_Ξ are linearly dependent.

The proof of Theorem 5 is complete. $\square$

To end our discussion, we prove Theorem 6.

Proof of Theorem 6. The necessity can be proved in the same way as in Theorem 1 by noticing that $|z_0| = |e^{-i\omega_0}| \in [e^{-r/k}, e^{r/k}]$ if and only if $\mathrm{Im}\,(k\omega_0) \in [-r, r]$.

Sufficiency. Let

$$N(\phi) := \{\omega := (\omega_1, \dots, \omega_s)^T \in \mathbf{C}^s : \\ 0 \le \mathrm{Re}\,\omega_j < 2\pi, \phi^\wedge(\omega + 2\pi\alpha) = 0,\ \forall \alpha \in \mathbf{Z}^s\}.$$

Denote $\mathrm{Im}\,\omega := (\mathrm{Im}\,\omega_1, \dots, \mathrm{Im}\,\omega_s)^T$. Suppose to the contrary that, for some $\omega_0 := (\omega_{0,1}, \dots, \omega_{0,s})^T \in N(\phi)$, it holds that $\mathrm{Im}\,\omega_0 \in [-r, r]^s$. Then, for any $\beta \in \mathbf{Z}^s$ and $l \in \mathcal{E}_{k,s}$,

$$\phi^\wedge(\omega_0 + 2\pi l + 2\pi k\beta) = \frac{1}{k^s}\tilde{b}(e^{-i2\pi(l/k)}e^{-i(\omega_0/k)}) \\ \cdot \phi^\wedge\left(\frac{\omega_0}{k} + 2\pi\frac{l}{k} + 2\pi\beta\right) = 0.$$

Note that $|e^{-i(\omega_{0,j}/k)}| = e^{(1/k)\mathrm{Im}\,\omega_{0,j}} \in [e^{-r/k}, e^{r/k}]$. By condition (i), there exists some $l_0 \in \mathcal{E}_{k,s}$ such that $\tilde{b}(e^{-i2\pi(l_0/k)}e^{-i(\omega_0/k)}) \ne 0$. Then, for any $\beta \in \mathbf{Z}^s$, $\phi^\wedge(\omega_0/k + 2\pi(l_0/k) + 2\pi\beta) = 0$, i.e., $\omega_1 := \omega_0/k + 2\pi(l_0/k) \in N(\phi)$. Observe that $\mathrm{Im}\,\omega_1 \in [-r/k, r/k]^s$. Repeating the same process, we find a sequence $\{\omega_n\}_{n=0}^\infty \subset \mathbf{C}^s$ such that $\omega_{n+1} \in \omega_n/k + 2\pi(\mathcal{E}_{k,s}/k)$, $\omega_n \in N(\phi)$. Hence, $\mathrm{Im}\,\omega_n = k^{-n}\mathrm{Im}\,\omega_0 \in [-r, r]^s$.

Since, for any $n \ge 0$, $\omega_n \in ([0, 2\pi) + i[-r, r])^s$ which is a compact set, there is a subsequence $\{\omega_{n_l}\}_{l=0}^\infty$ of $\{\omega_n\}_{n=0}^\infty$ such that $\omega_{n_l} \to \xi \in \mathbf{C}^s$ as $l \to \infty$. Then $\mathrm{Re}\,\xi \in [0, 2\pi)^s$ and $\mathrm{Im}\,\xi = (0, \dots, 0)^T$. Also, for any $\alpha \in \mathbf{Z}^s$, $\phi^\wedge(\xi + 2\pi\alpha) = \lim_{l\to\infty} \phi^\wedge(\omega_{n_l} + 2\pi\alpha) = 0$. Therefore, $\xi \in N(\phi)$ and $\xi \in \mathbf{R}^s$. This is a contradiction to the condition (ii). The proof of Theorem 6 is complete. $\square$

REFERENCES

1. C. de Boor, K. Höllig and S.D. Riemenschneider, *Box splines*, Springer-Verlag, Berlin, 1993.

2. C.K. Chui, J. Stöckler and J.D. Ward, *Compactly supported Box spline wavelets*, Approx. Theory Appl. **8** (1992), 77–100.

3. A. Cohen, *Ondelettes, analysis multirésolutions et filtres mirroirs en quadrature*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **7** (1990), 439–459.

4. A. Cohen, I. Daubechies and J.C. Feauveau, *Biorthogonal bases of compactly supported wavelets*, Comm. Pure Appl. Math. **45** (1992), 485–560.

5. W. Dahmen, R.Q. Jia and C.A. Micchelli, *On linear dependence relations for integer translates of compactly supported distributions*, Math. Nachr. **151** (1991), 303–310.

6. W. Dahmen and C.A. Micchelli, *Translates of multivariate splines*, Linear Algebra Appl. **52/53** (1983), 217–234.

7. ———, *Algebraic properties of discrete box splines*, Constr. Approx. **3** (1987), 209–221.

8. I. Daubechies, *Ten lectures on wavelets*, SIAM, Philadelphia, 1992.

9. ———, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math. **41** (1988), 909–996.

10. R.Q. Jia, *Linear independence of translates of a box spline*, J. Approx. Theory **40** (1984), 158–160.

11. ———, *Multivariate discrete splines and linear diophantine equations*, Trans. Amer. Math. Soc. **340** (1993), 179–198.

12. R.Q. Jia and Z.W. Shen, *Multiresolution and wavelets*, Proc. Edinburgh Math. Soc. (2) **37** (1993), 271–300.

13. R.Q. Jia and J.Z. Wang, *Stability and linear independence associated with wavelet decompositions*, Proc. Amer. Math. Soc. **117** (1993), 1115–1124.

14. Y. Meyer, *Ondelettes et Opérateurs* I: *Ondelettes*, Hermann, 1990.

15. S.D. Riemenschneider and Z.W. Shen, *Wavelets and pre-wavelets in low dimensions*, J. Approx. Theory **71** (1992), 18–38.

16. ———, *Box splines, cardinal series and wavelets*, in *Approximation theory and functional analysis* (C.K. Chui, ed.), Academic Press, Boston, 1991, 133–149.

17. A. Ron, *A necessary and sufficient condition for the linear independence of the integer translates of a compactly supported distribution*, Constr. Approx. **5** (1989), 297–308.

18. J. Stöckler, *Multivariate wavelets*, in *Wavelets–A tutorial in theory and applications* (C.K. Chui, ed.), Academic Press, New York, 1992, 325–355.

19. Ding-Xuan Zhou, *Stability of refinable functions, multiresolution analysis and Haar bases*, SIAM J. Math. Anal. **27** (1996), 891–904.

20. ———, *Construction of real-valued wavelets by symmetry*, J. Approx. Theory **81** (1995), 323–331.

21. ———, *Linear dependence relations in wavelets and tilings*, Linear Algebra Appl. **249** (1996), 311–323.

Department of Mathematics, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong