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NORMAL LIMITS IN STAR-INVARIANT SUBSPACES IN MULTIPLY CONNECTED DOMAINS

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ABSTRACT. In this paper we give a necessary and sufficient condition for the normal limit to exist at a point on the smooth boundary of a multiply connected domain in the plane, for a function in the star-invariant subspace.

Introduction. Let U be the open unit disc in the complex plane, and denote by H^p, $0 < p \leq \infty$, the usual classes of analytic functions on U [**9, 13, 12**]. Let φ be an inner function and write $\varphi = cBS_\sigma$ where $|c| = 1$, B is a Blaschke product with zero sequence $\{z_k\}$, and S_σ is a singular inner function with positive measure σ which is singular with respect to Lebesgue measure [**9, 13, 12**].

In [**2**, Lemma 3] Ahern and Clark gave the following generalization of a famous theorem of Frostman [**10**] concerning the existence of the radial limit of a Blaschke product at a given point in T, the unit circle.

Theorem A. *Let ζ_0 be on the unit circle T, and suppose $\varphi = BS_\sigma$ and $\sigma(\{\zeta_0\}) = 0$. Then the following conditions are equivalent*:

(i) *Every divisor of φ has a radial limit of modulus 1 at ζ_0.*

(ii) *Every divisor of φ has a radial limit at ζ_0.*

(iii) $\sum_{k=1}^{\infty}(1-|z_k|)/|\zeta_0 - z_k| + \int_T d\sigma(u)/|u-\zeta_0| < \infty$.

We say that f is a divisor of φ if $\varphi = fg$ where both f and g lie in the unit ball of H^∞.

In [**6**, Theorem 3.1], Cohn noticed that condition (iii) implies a stronger result than (ii). Let φ be an inner function, and let

$$K_2 \equiv H^2 \ominus \varphi H^2$$

be the star-invariant subspace generated by φ. Let $BMOA$ denote the space of analytic functions of bounded mean oscillation and define $K_* \equiv K_2 \cap BMOA$. Then we have the following result.

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Theorem B. *Let ζ_0 be on T, and suppose $\varphi = BS_\sigma$. A necessary and sufficient condition that* $\lim_{r\to 1^-} f(r\zeta_0)$ *exist for all f in K_* is that*

$$\sum \frac{1-|z_k|}{|\zeta_0 - z_k|} + \int_T \frac{d\sigma(u)}{|\zeta_0 - u|} < \infty. \tag{1}$$

Thus, condition (iii) actually shows that the radial limit at ζ_0 holds for a larger class of functions than the class of functions that are divisors of φ.

In [**6**, Theorem 4.1], Cohn also studied the case where $1 < p < \infty$ and $p \neq 2$. The analogue of K_2 in this case is defined as follows.

Let K_p be the subspace of H^p defined by $K_p = \varphi \bar{H}_0^p \cap H^p$. Here, $\bar{H}_0^p \equiv \{e^{-it}\bar{f}(e^{it}) : f \in H^p\}$. For K_p, $1 < p < \infty$, the following theorem was proved in [**6**, Theorem 4.1].

Theorem C. *Let $\zeta_0 \in T$ and suppose $\varphi = BS_\sigma$. Suppose $p > 1$ and q is the exponent conjugate to p. A necessary and sufficient condition that* $\lim_{r\to 1^-} f(r\zeta_0)$ *exist for all $f \in K_p$ is that*:

$$\sum \frac{1-|z_k|}{|\zeta_0 - z_k|^q} + \int_T \frac{d\sigma(u)}{|\zeta_0 - u|^q} < \infty. \tag{2}$$

In the case $p = 2$, Theorem C is a result of Ahern and Clark, [**1**, p. 333].

It is the purpose of this paper to generalize Theorem C to the setting of a finitely connected bounded domain G in the plane with C^∞ boundary curves. In Section 0 we give the definition of $H^p(G)$ spaces, along with some basic results analogous to those for the theory of H^p spaces in the unit circle. In Section 1 we define and discuss the Szegö kernel and some of its properties in G. In that section, we also prove some estimates on the Szegö kernel with continuous positive weights. In Section 2 we formulate and prove an analogue of Theorem C in G. Theorem B will be generalized in a separate paper.

0. Basic elements of $H^p(G)$. In what follows G will be a bounded domain in the plane with connectivity n such that its boundary consists of simple closed analytic curves. bG will denote the boundary of G. We write

$$bG = \bigcup_{i=1}^{n+1} s_i,$$

where the s_i's are the boundary curves. For z and $a \in G$ let $g(z,a)$ be the Green's function of G with pole at a. Precisely,

$$g(z,a) = h(z,a) - \log|z-a|,$$

where $h(z,a)$ is the harmonic function on G whose boundary function equals $\log|z-a|$, $z \in bG$, cf. [**11**, p. 16], [**14**, p. 11].

By a *harmonic measure* on G we mean a harmonic function h whose boundary values are constant on each component of bG [**15**, p. 153].

We recall that a holomorphic function f on G belongs to $H^p(G)$, $0 < p < \infty$, if $|f|^p$ has a *harmonic majorant* on G, i.e., a harmonic function h on G such that $|f(z)|^p \leq h(z)$ for all z in G [**11**, p. 51].

The function f is in H^∞ if it is both bounded and holomorphic on G. If $f \in H^p(G)$, then f has nontangential boundary values almost everywhere with respect to arc length ds on bG. We can identify $H^p(bG)$ with a closed subspace of $L^p(bG)$, and in what follows we will use this fact without further comment [**11**, p. 88].

Let $R(G)$ denote those rational functions on G whose poles are off $G \cup bG$. Then $R(G)$ is dense in $H^p(G)$ if $1 \leq p < \infty$ [**11**, p. 86].

Now we give the definitions of Blaschke products and inner functions in G.

A bounded analytic function B in G is called a (generalized) *Blaschke product* if

$$\log|B(z)| = \sum_i g(z, a_i) + h(z)$$

where $g(z,a)$ is the Green's function for G and h is a harmonic measure, cf., [**15**, p. 153]. It is well known [**15**, Lemma 21] that if $\{a_n\}$ is the sequence of the zeros of a function in $H^p(G)$, repeated according to multiplicity, then

$$\sum d(a_n, bG) < \infty,$$

where $d(a_n, bG)$ denotes the distance from a_n to the boundary bG. If $\{a_n\}$ satisfies this condition, then there is a Blaschke product B whose zeros are a_n.

A bounded analytic function f, $f \not\equiv 0$, can be factored into $f = Bg$, where B is a Blaschke product with the same zeros as f and g is zero free. The factorization is unique apart from bounded analytic functions U, such that $|U|$ is constant on each boundary contour, cf., [**15**, Lemma 3].

A bounded analytic function ϕ in G is called an *inner function* if the nontangential boundary values of $|\phi|$ on each boundary contour of G are equal almost everywhere to a constant [**15**, p. 154].

An inner function with no zeros is called a *singular function*. Singular functions are those functions ϕ in $H^\infty(G)$ for which

$$\log|\phi(z)| = -\int_{bG} \frac{\partial g(t,z)}{\partial n}\, d\mu(t) + h(z),$$

where μ is a positive measure on bG which is singular with respect to the measure given by arc length, $\partial/\partial n$ denotes differentiation along the outward normal, and h is a harmonic measure. The measure μ in this representation is unique [**15**, p. 154].

A function f in H^p is called an *outer function* if

$$\log|f(z)| = \frac{1}{2\pi}\int_{bG} \log|f(t)| \frac{\partial g(t,z)}{\partial n}\, ds, \quad z \in G,$$

where g is the Green's function of G and ds is the arc length measure on bG [**15**, p. 155].

Every function f in H^p may be factored into $f = \phi F$, where ϕ is an inner function and F is an outer function in H^p [**15**, Lemma 11].

Throughout this paper, c will be a constant which does not necessarily have the same value at each occurrence.

1. Some estimates for the Szegö kernel and auxiliary theorems. For $f, g \in L^2(bG)$, we let

$$\langle f, g\rangle_h = \int_{bG} f\bar{g}h\, ds$$

be the inner product of weight h, where h is a positive continuous function. For $z \in bG$, $dz = z'(t)\,dt$ and $ds = |z'(t)|\,dt$ where $z(t)$ is a parametrization of bG in the standard sense, i.e. counter-clockwise on the outer boundary curve, and clockwise on the curves inside the bounded component. For a in G and f in $H^2(bG)$, the evaluation map at $a \in G$ is a continuous linear functional on $H^2(bG)$. Thus, by the Riesz representation theorem, there is a unique function $S(z, a, hds)$ in $H^2(bG)$ which represents this functional in the sense that

$$f(a) = \langle f, S(\cdot, a, hds)\rangle_h$$

for all $f \in H^2(bG)$. $S(z, a, hds)$ is called the weighted Szegö kernel for $H^2(bG)$ at a. When $h \equiv 1$ we will write

$$S(z, a, ds) = S(z, a).$$

It is well known, cf. [**3**], [**14**, p. 389] and [**5**, p. 2], that $S(z, a)$ is holomorphic in $z \in G$ for fixed $a \in G$ and that $S(z, a) = \overline{S(a, z)}$. Also, $\overline{S(z, a)}$ is holomorphic in $a \in G$ for fixed $z \in G$, and $S(z, a) \in C^\infty(\bar{G} \times \bar{G} - \Delta)$, where Δ is the diagonal of $\bar{G} \times \bar{G}$.

We need estimates on the Szegö kernel $S(z, a)$ when z and a are "fairly close" to one another, and at the same time they are "fairly close" to a boundary curve $s \subset bG$. In order to state these estimates as simply as possible, we will assume that s shares a common arc C with the unit circle, and that $|a| < 1$ when a is "near" C. These estimates combined with conformal mapping of a general domain onto one of this type will suffice for our applications.

Theorem 1.1. *Let $\zeta_0 \in bG$. Then there exists a simply connected domain N in G with C^∞ boundary bN such that $bG \cap bN$ is an arc in C containing ζ_0, and the following inequality holds for all z and a in N:*

$$\frac{A}{|1 - \bar{a}z|} \leq |S(z, a)| \leq \frac{B}{|1 - \bar{a}z|}. \tag{3}$$

Here, A and B are constants independent of z and a.

Note 1.1. Theorem 1.1 seems to be a known result. A proof is given in [**16**, p. 12].

Note 1.2. The same type of estimate has been proved in a more general domain and for several complex variables, cf. [**8**, Theorem C].

We now consider the problem of proving the same type of estimate for the weighted Szegö kernel $S(z, a, hds)$. We will need the following theorem. For a proof see [**17**].

Theorem 1.3. *Let $h \geq 0$ be continuous on bG. Then there is a function $F \in H^\infty(G)$ such that $|F|^2 = h^2$ on bG, $F(\zeta) = 0$ for a preassigned ζ and F has at most n zeros on G.*

We now state and prove our result.

Theorem 1.4. $S(z, a, hds) = \overline{F(a)}F(z)S(z, a) + k(z, a)$; *where $|k(z, a)| \leq M$ with M independent of z and a, and F is a bounded holomorphic function on G such that $|F(u)|^2 = 1/h(u)$ on bG.*

Proof. Let F be the holomorphic function on G with boundary values $|F(\zeta)|^2 = 1/h(\zeta)$, and let a_k, $k = 0, \cdots, m$, be its zeros. Let X be the space $H^2_{\{a_1,\ldots,a_m\}} = \{f \in H^2 : f(a_k) = 0; k = 1, \cdots, m\}$. We note that $H^2 \ominus X$ is of dimension $\leq m$, and the Szegö kernel there is given as a finite sum of bounded functions. For the definitions of those functions, see [**7**, p. 294]. We claim that

$$\overline{F(a)}F(z)S(z, a, ds) = S_X(z, a, hds),$$

where the right hand side is the weighted Szegö kernel for X. To prove our claim, let $f \in X$. Then $f = Fg$, $g \in H^2$, and

$$\begin{aligned}\int_{bG} f(z)F(a)&\overline{F(z)S(z, a, ds)}h\, ds \\ &= \int_{bG} F(z)g(z)F(a)\overline{F(z)S(z, a, ds)}h\, ds \\ &= F(a)\int_{bG} g(z)|F(z)|^2 h\overline{S(z, a, ds)}\, ds \\ &= F(a)\int_{bG} g(z)\overline{S(z, a, ds)}\, ds\end{aligned}$$

$$= F(a)g(a)$$
$$= f(a).$$

Since the Szegö kernel is the unique function in X that satisfies this reproducing property, the claim is proved and the theorem follows. □

Corollary 1.5. *Let $\zeta_0 \in bG$. Then there exists a simply connected domain N in G with C^∞ boundary bN such that $bG \cap bN$ is an arc in C containing ζ_0, and the following inequality holds for all z and a in W:*

$$\frac{c_1}{|1-\bar{a}z|} \leq |S(z,a,hds)| \leq \frac{c_2}{|1-\bar{a}z|}.$$

c_1 and c_2 are independent of z and a.

We now briefly discuss some properties of what we are going to call the " Carleson curve." Suppose $f(z)$ is a bounded analytic function on U, the unit disc, satisfying $||f||_\infty \leq 1$. Let $0 < \delta < 1$. In constructing the "Carleson region" R in U to prove the Corona theorem, a curve Γ associated with f, which we will call the "Carleson curve" in U, was constructed with the following properties, cf., [**12**, p. 342] and [**6**, p. 727].

(1) $\Gamma = U \cap bR$ separates $\{z : |f(z)| > \varepsilon\}$ from $\{z : |f(z)| < \varepsilon\}$ where $\varepsilon = \varepsilon(\delta) < \delta$.

(2) $\{z : |f(z)| < \varepsilon\} \subseteq R$.

(3) Arc length on Γ is a Carleson measure.

(4) There is a constant $\varepsilon_0 > 0$ such that $0 < \varepsilon_0 \leq |f(z)| \leq \delta$ for $z \in \Gamma$.

(5) Γ is a countable union of arcs or radial segments, $\Gamma = \cup\gamma_n$, where $\gamma_n = [a_n, b_n]$ denotes either a radial segment or an arc, and there are constants $0 < c_1 < c_2 < 1$ such that $c_1 \leq |(a_n - b_n)/(1 - \bar{a_n}b_n)| \leq c_2$.

The following theorem was proved in [**6**, Theorem 2].

Theorem 1.6. *Let $\zeta_0 \in T$ and suppose $\phi = BI_\sigma$. Let $\{w_n\}$ be the set of midpoints of the segments γ_n given in property* (5) *above. Then*:

(i) *The condition*

$$\sum \frac{1-|a_n|}{|\zeta_0 - a_n|} + \int_T \frac{d\sigma(t)}{|\zeta_0 - t|} < \infty,$$

where $\{a_n\}$ *is the zero set of* B*, holds if and only if*

$$\sum \frac{1-|w_n|}{|\zeta_0 - w_n|} < \infty.$$

(ii) *For* $q > 1$*, the condition*

$$\sum \frac{1-|a_n|}{|\zeta_0 - a_n|^q} + \int_T \frac{d\sigma(t)}{|\zeta_0 - t|^q} < \infty$$

holds if and only if

$$\sum \frac{1-|w_n|}{|\zeta_0 - w_n|^q} < \infty.$$

Let N be a set in G of the type described in the statement of Theorem 1.1. Then, mapping U onto N conformally by, say, ψ, we can prove properties in N similar to (1)–(5) above, and a theorem similar to Theorem 1.6 for G. For details see [**16**, p. 24]. Let φ be an inner function in G with zero set $\{a_n\}$, and denote by ϕ the inner part of its restriction to N φ_N. Let $\psi(\Gamma) = \mathcal{K}$, where Γ is the Carleson curve associated with $\phi \circ \psi$ in U. Let $u_k = \psi(w_k)$ and $c_n = \psi(\gamma_n)$. For $u_0 \in bG$ the condition corresponding to (i) in Theorem 1.6 in the more general domain G is

$$\sum \frac{d(a_k, bG)}{d(u_0, a_k)} + \int_{bG} \frac{d\lambda(u)}{|u_0 - u|} < \infty, \tag{4}$$

and condition (ii) in Theorem 1.2 becomes

$$\sum \frac{d(a_k, bG)}{|u_0 - a_k|^q} + \int_{bG} \frac{d\lambda(u)}{|u_0 - u|^q} < \infty, \tag{5}$$

where $d(x, bG)$ is the distance from the point x to the boundary bG.

Theorem 1.7. *Let u_0, φ and $\{u_k\}$ be as above. Then*

(i′) *Condition* (4) *holds for φ if and only if*

$$\sum \frac{d(u_k, bG)}{|u_0 - u_k|} < \infty.$$

(ii′) *Condition* (5) *holds for φ and $q > 1$ if and only if*

$$\sum \frac{d(u_k, bG)}{|u_0 - u_k|^q} < \infty.$$

2. Normal limits in K_p, $p > 1$. Suppose φ is an inner function on G. Let K_p be the set of functions in H^p which are orthogonal to φH^q with respect to the inner product $\langle f, g\rangle$ defined by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{bG} f(z)\overline{g(z)}\,ds, \quad f \in H^p(G), \quad g \in H^q(G),$$

so that $K_p = (\varphi H^q)^{\perp} \cap H^p$, $1/p + 1/q = 1$.

We start by giving a boundary representation for $f \in K_p$. Let R be a rational function in $H^p(G)$. Then

$$\begin{aligned} 0 &= \int_{bG} f(z)\overline{\varphi(z)R(z)}\,ds \\ &= \int_{bG} \overline{f(z)}\varphi(z)R(z)\,ds, \end{aligned}$$

for all $f \in K_p$. By the multiply connected domain version of the F. and M. Riesz theorem [**11**, p. 85], one obtains

$$\bar{f}\varphi\,ds = H\,dz \quad \text{a.e. } [ds],$$

for some $H \in H^1(G)$. This implies

$$\bar{f} = \left(\frac{H}{\varphi}\right)\frac{dz}{ds} \quad \text{a.e } [ds],$$

i.e.,

$$f(z) = \overline{\left(\frac{H}{\varphi}\right)} \frac{d\bar{z}}{ds} \quad \text{a.e. } [ds], \quad z \in bG. \tag{6}$$

Denote by $K_q(1/|\varphi|^2)$ the set of functions in H^q orthogonal to φH^p with respect to the weighted inner product with weight $1/|\varphi|^2$. By an argument similar to the one we used to get a boundary representation for functions in K_p, we obtain

$$f = \overline{\left(\frac{H}{\varphi}\right)} |\varphi|^2 \overline{\left(\frac{dz}{ds}\right)}.$$

From now on, if $u_0 \in bG$ and $u \in G$, then $\lim_{u \to u_0}$ means the limit when u is approaching u_0 along the line in G normal to u_0, which we will call the normal limit at u_0.

The following two results are needed.

Theorem 2.1 [**4**, Theorem 2]. *Let G be a bounded finitely connected domain with smooth boundary curves, and let $1 < p < \infty$. Then, for $F \in L^p(G)$, we have*

$$F = f + g\overline{\frac{dz}{ds}},$$

where $f \in H^p(G)$ and $\overline{g(dz/ds)}$ is orthogonal to H^q, q is the exponent conjugate to p, ds is arclength on bG, and

$$\left\| g\frac{dz}{ds} \right\|_p \leq c\|F\|_p,$$

where c depends only on p.

Lemma 2.2. *Let $1 < p < \infty$ and q conjugate to p. Then $K_p^* = K_q(1/|\varphi|^2)$ in the sense that if $k \in K_q(1/|\varphi|^2)$ then the map*

$$L_k(f) = \langle f, k \rangle$$

gives a bounded linear functional on K_p with $\|L_k\| \doteq \|k\|_q$, and every such functional on K_p is of this form.

Proof. If $f \in K_q(1/|\varphi|^2)$ then the map

$$L_k(f) = \langle f, k \rangle$$

defines a bounded linear functional on K_p. We now consider a bounded linear functional L on K_p and show that $L = L_k$ for some unique $k \in K_q(1/|\varphi|^2)$ and also establish that $\|L\| \doteq \|k\|_q$, where the notation $A \doteq B$ means there are positive constants m and M such that $mA \le B \le MA$. Let $L \in K_p^*$; by the Hahn-Banach theorem we can extend L to a bounded linear functional on $H^p(G)$. Using duality between $H^p(G)$ and $H^q(G)$ [**4**, Corollary 2], [**11**, p. 63], we can find a function $F \in H^q$ with $\|F\|_q \doteq \|L\|$ such that

$$L(f) = \langle f, F \rangle$$

for all $f \in K_p$. By the theorem of M. Riesz,

$$\frac{F}{\varphi} = k_1 + \overline{h\frac{dz}{ds}},$$

where $\overline{h(dz/ds)}$ orthogonal to H^q, $k_1 \in H^p$ and

$$\left\| h\frac{dz}{ds} \right\|_q \le c\|F\|_q$$

where c depends only on q. Thus

$$F = \varphi k_1 + \varphi\overline{h\frac{dz}{ds}}.$$

If we let $k = \varphi\overline{h(dz/ds)}$, we see that $k = F - \varphi k_1$ is holomorphic. Also, k can be written as $|\varphi|^2(\overline{h(dz/\varphi ds)})$. It follows from the representation formula above that $k \in K_q(1/|\varphi|^2)$. Thus,

$$L(f) = \langle f, \varphi k_1 + k \rangle = \langle f, k \rangle$$

for $f \in K_p$. Since $\|k\|_q \doteq \|h(dz/ds)\|$ we see that

$$\|k\|_q \le c\|L\|.$$

We now establish the uniqueness of k. Let $l \in K_q(1/|\varphi|^2)$ be such that

$$\langle f, k\frac{1}{|\varphi|^2}\rangle = \langle f, l\frac{1}{|\varphi|^2}\rangle, \quad f \in K_p.$$

For $F \in H^p(G)$ we have

$$\begin{aligned}\int_{bG}(k-l)\bar{F}\frac{1}{|\varphi|^2}\,ds &= \int_{bG}(k-l)(\overline{\varphi k_1 + k_2})\frac{1}{|\varphi|^2}\,ds\\ &= \int_{bG}(k-l)(\overline{k_2})\frac{1}{|\varphi|^2}\,ds\\ &= 0.\end{aligned}$$

To finish the proof of the lemma we need an inequality of the reverse type.

Let H^p_b and $(K_p)_b$ be the unit balls in H^p and K_p, respectively. By duality between H^p and H^q we get

$$\begin{aligned}\|k\|_q &\doteq \sup\{|\langle F, k\rangle| : F \in H^p_b\}\\ &\geq \sup\{|\langle F, k\rangle| : F \in (K_p)_b\}\\ &= \|L\|.\end{aligned}$$

This completes the proof of the lemma. □

The following theorem is the main result in this section. It is a generalization of Theorem 3.1 in [**6**].

Theorem 2.3. *Let $u_0 \in bG$ and suppose $\varphi = BS_\sigma$. Let $p > 1$ and q be the exponent conjugate to p. For $\lim_{u\to u_0} f(u)$ to exist for all $f \in K_p$, it is necessary and sufficient that*

$$\sum_k^\infty \frac{d(a_k, bG)}{|a_k - u_0)|^q} + \int_{bG}\frac{d\sigma(u)}{|u_0 - u|^q} < \infty.$$

Proof of sufficiency. For $f \in K_p(G)$ and $u \in G$, the reproducing property of the Szegö kernel together with (6) yield the following:

$$\begin{aligned}f(u) &= \int_{bG} f(w)\overline{S(w,u)}\,ds\\ &= \overline{\int_{bG}\frac{H(w)}{\varphi(w)}S(w,u)\,dw}.\end{aligned}$$

Thus,

$$\overline{f(u)} = \int_{bG} \frac{H(w)}{\varphi(w)} S(w,u)\, dw. \tag{7}$$

Let N be the set in Theorem 1.1, and write

$$bN = I \cup C,$$

where I is inside G and $u_0 \in C$ (see the discussion preceding Theorem 1.1 for the definition of C). Assume at first that all the zeros of φ are inside N. Then the righthand side of (7) equals

$$\int_C \frac{H}{\varphi} S(w,u)\, dw + \int_{bG-C} \frac{H}{\varphi} S(w,u)\, dw. \tag{8}$$

The second integral in (8) has a finite limit as u tends to u_0. To deal with the first integral, we write it as

$$\int_{bN} \frac{H}{\varphi} S(w,u)\, dw - \int_I \frac{H}{\varphi} S(w,u)\, dw. \tag{9}$$

Since the integral over I also has a finite limit as u tends to u_0, we need only deal with the integral over bN. To do this we use a technique developed by W. Cohn in [**6**, Theorem 3.1], which involves taking the integral from bN to $\mathcal{K}$. Write $\varphi_N = \phi O$, where O is its outer part, and note that ϕ must have the same zeros as φ_N and the same singular measure on C.

Case 1. *φ is a Blaschke product.* Let φ_k be the product of the first k factors of φ. If we let

$$\overline{f_k(u)} = \int_{bN} \frac{H}{\varphi_k} S(w,u)\, dw,$$

we obtain

$$\lim_{k\to\infty} f_k(u) = f(u)$$

by the dominated convergence theorem. Now let R' be the Carleson region for ϕ in N, i.e., $R' = \psi(R)$, where R is the Carleson region for

$\phi\circ\psi$ in U. There are finitely many zeros of φ_k, so they lie inside finitely many of the closed curves $c_1, c_2, \dots$, which constitute the curve $\mathcal{K}$. It follows from the fact that ϕ is bounded away from 0 on $N - R'$ and O is continuous on $\bar{N}$ that

$$|\varphi_k| \geq |\varphi| > \delta > 0,$$

on $N - R'$ for all k. Since $S(u, w)$ is holomorphic in w, Cauchy's theorem now yields

$$\int_{bN} \frac{H}{\varphi_k} S(w, u)\, dw = \sum_{n=1}^{M_k} \int_{c_n} \frac{H}{\varphi_k} S(w, u)\, dw$$

where the sum is taken over all curves c_n whose interiors contain zeros of φ_k. If c_m does not contain zeros of φ_k in its interior, then by Cauchy's theorem again

$$\int_{c_m} \frac{H}{\varphi_k} S(w, u)\, dw = 0.$$

Thus

$$\overline{f_k(u)} = \int_{\mathcal{K}} \frac{H}{\varphi_k} S(w, u)\, dw.$$

Let $k \to \infty$ and apply the dominated convergence theorem to get

$$\overline{f(u)} = \int_{\mathcal{K}} \frac{H}{\varphi} S(w, u)\, dw. \tag{10}$$

Case 2. *φ is not a Blaschke product.* If φ is not a Blaschke product, we can find a sequence of Blaschke products $\{B_n\}$ converging uniformly to φ on $\bar{G}$ [**11**, Chapter 5]. We then repeat the argument in Case 1 to write $\overline{f(u)}$ as an integral over the same system of curves $\mathcal{K}$, and then we take the limit when n tends to infinity to conclude that (10) holds for a general inner function.

Thus we may write

$$\overline{f(u)} = \sum F_k(u),$$

where

$$F_k(u) = \int_{c_k} \frac{H(w)}{\varphi(w)} S(w, u)\, dw. \tag{11}$$

Using Holder's inequality and the fact that $|\varphi| > \varepsilon_0$ on $\mathcal{K}$, together with the inequality

$$|S(u,w)| \leq \frac{c}{|1-u\bar{w}|},$$

which was proved in Theorem 1.1, one obtains

$$|F_k(u)| \leq c\left(\int_{c_k} |H(w)|^p|\,dw|\right)^{1/p}\left(\int_{c_k} \frac{|dw|}{|1-u\bar{w}|^q}\right)^{1/q}. \tag{12}$$

An elementary estimate on the second factor in (12) using the properties of the curves c_k yields

$$|F_k(u)| \leq c\left(\int_{c_k} |H(w)|^p|dw|\right)^{1/p}\left(\frac{d(u_k, bG)}{|u_k-u_0|^q}\right)^{1/q}.$$

Thus,

$$\|F_k\|_\infty \leq c\left(\int_{c_k} |H(w)|^p|dw|\right)^{1/p}\left(\frac{d(u_k, bG)}{|u_k-u_0|^q}\right)^{1/q}.$$

Applying Holder's inequality a second time, and using the fact that arc length is a Carleson measure on $\mathcal{K}$, we obtain that

$$\sum \|F_k\|_\infty \leq c\left[\int_C |H(w)|^p|dw|\right]^{1/p}\left[\sum \frac{d(u_k, bG)}{|u_k-u_0|^q}\right]^{1/q} \tag{13}$$

$$\leq c\|H\|_p^p \tag{14}$$

$$< \infty,$$

where we have used the fact that $|dw|$ is a Carleson measure on $\mathcal{K}$ to go from (13) to (14). By Theorem 1.7, (ii$'$), the last factor in (13) is finite. It follows from the Weirstrass M-test that $\bar{f} = \sum F_k$ is continuous at u_0 and $\lim_{u\to u_0} \overline{f(u)}$ exists. This completes the proof of sufficiency. $\square$

Proof of necessity. Suppose $\lim_{u\to u_0} f(u)$ exists for all $f \in K_p$. Set

$$K_u(z) = S\left(z, u, \frac{1}{|\varphi|^2}\,ds\right) - \overline{\varphi(u)}\varphi(z)S(z,u); \quad u \in G.$$

Then it is straightforward to verify that $K_u \in K_q(1/|\varphi|^2)$. On the other hand, for $f \in K_p$ we have

$$\begin{aligned}
\int_{bG} f\overline{K_u(z)}\,ds &= \int_{bG} f\overline{S(z,u,\frac{1}{|\varphi|^2}ds)}\,ds \\
&= \int_{bG} f\overline{\overline{F(u)}F(z)S(z,u)}\,ds \\
&\quad + \int_{bG} f\overline{k(z,u)}\,ds,
\end{aligned} \tag{15}$$

where F and k are as in Theorem 1.4. The second term in (15) stays bounded. With X as in Theorem 1.4 and $u_0 \in \gamma$, for $f \in X$ we write $f = Fg$, $g \in H^2$. The first term in (15) can be written as

$$\begin{aligned}
F(u)\int_{bG} gF(z)\overline{F(z)S(z,u)}\,ds &= F(u)\int_{bG} g|F(z)|^2\overline{S(z,u)}\,ds \\
&= \sum_{i=1}^{n} k_i F(u)\int_{bG-\gamma} g\overline{S(z,u)}\,ds \\
&\quad + cF(u)\int_{\gamma} g\overline{S(z,u)}\,ds \\
&= T_1 + T_2,
\end{aligned}$$

where k_i is the modulus of F on γ_i. Now T_1 stays bounded as $u \to u_0$. For T_2, we write

$$\begin{aligned}
T_2 &= cF(u)\int_{bG} g\overline{S(z,u)} - cF(u)\int_{bG-\gamma} g\overline{S(z,u)}\,ds \\
&= cF(u)g(u) - T_3 \\
&= cf(u) - T_3,
\end{aligned}$$

where T_3 is also bounded as $u \to u_0$. It follows from the assumption that $\lim_{u\to u_0} f(u)$ exists for all $f \in K_p$ and the Banach-Steinhaus theorem that the continuous linear functional Λ_u defined on K_p by

$$\Lambda_u : f \longrightarrow \langle f, K_u\rangle$$

is bounded and satisfies

$$|\langle f, K_u\rangle| \le c\|f\|_p$$

where c is independent of u. From Lemma 2.3 above, we conclude that

$$\|K_u\|_q \le c$$

where c is independent of u.

A normal family argument shows that

$$\begin{aligned}\lim_{u\to u_0} K_u(z) &= S(z,u_0) - \bar{\lambda}\varphi(z)S\biggl(z,u_0,\frac{1}{|\varphi|^2}\,ds\biggr)\\ &= K(z), \quad z\in G,\end{aligned}$$

and $K \in K_q$.

If we restrict K to the neighborhood N, then we get a function in $H^q(N)$ which we will also denote by $K(z)$. As in the proof of the sufficiency part, consider the Carleson curve $\mathcal{K}$ in N for the function ϕ. Since arc length on $\mathcal{K}$ is a Carleson measure,

$$\int_{\mathcal{K}} |K(z)|^q |dz| \le c\|K\|_q^q,$$

i.e.,

$$\int_{\mathcal{K}} \biggl|S(z,u_0) - \bar{\lambda}S\biggl(z,u_0,\frac{1}{|\varphi|^2}\,ds\biggr)\biggr|^q |dz| \le c\|K\|_q^q.$$

It follows from Theorem 1.1, Corollary 1.5, and the fact that $|\varphi| \le \delta$ on $\mathcal{K}$, that, for δ small enough,

$$\begin{aligned}|K(z)| &= \biggl|S(z,u_0) - \bar{\lambda}\varphi(z)S\biggl(z,u_0,\frac{1}{|\varphi|^2}\,ds\biggr)\biggr|\\ &\ge \frac{c}{|1-\bar{u_0}z|}, \quad z\in G.\end{aligned}$$

Thus,

$$\int_{\mathcal{K}} \frac{|dz|}{|1-\bar{u_0}z|^q} < \infty.$$

Recall that $\mathcal{K}$ is $\psi(\Gamma)$ where ψ is the Riemann map from U onto N, that Γ is the Carleson curve for $\phi\circ\psi$ in U and is a union of segments

γ_n with midpoints w_n, and that $\psi(w_n) = u_n$. Thus the last inequality yields

$$\sum_n^{\infty} \frac{d(u_n, bG)}{|u_0 - u_n)|^q} < \infty.$$

One now uses Theorem 1.7 to show that condition (5) holds. This concludes the proof of necessity. $\square$

To complete the proof of the theorem we still have to consider the case in which there are some zeros of φ that accumulate outside of N. Write $\varphi = \phi_1\phi_2$, where ϕ_1 is the Blaschke product in G with these zeros. Let $K_p(I)$ denote $(IH^q)^\perp \cap H^p$, and let $K_p(I)(|\phi_1|^2)$ denote the same space with the inner product having weight $|\phi_1|^2$. Write

$$K_p(\varphi) = K_p(\phi_1) \oplus \phi_1 K_p(\phi_2)(|\phi_1|^2).$$

The normal limit at u_0 exists for all functions in $K_p(\varphi)$ if and only if the normal limit at u_0 exists for all functions in $K_p(\phi_2)(|\phi_1|^2)$. In fact, ϕ_1 is continuous at u_0 since its zeros are outside N. For $K_p(\phi_1)$, the integral over bN corresponding to the one in (9) is equal to zero. Therefore, the limit of the functions in that space exist.

The proof of sufficiency can be done the same as for the space $K_p(\phi_2)$ with only a few changes, making use of the estimates in Corollary 1.5 and the fact that $|\phi_1|^2$ is constant almost everywhere on bG.

For the necessity we need only notice that $K_p(\phi_2)$ is contained in $K_p(\varphi)$. In $K_p(\phi_2)$ the problem has already been solved. Therefore, condition (ii′) in Theorem 1.7 holds for the zeros of ϕ_2, which implies it also holds for the zeros of φ. $\square$

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