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ON THE K-THEORY OF C^*-ALGEBRAS OF PRINCIPAL GROUPOIDS

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ABSTRACT. We consider the K-theory of C^*-algebras of principal r-discrete groupoids. We describe two basic situations in which three groupoids are related; they can very loosely be described as "factor groupoids" and "subgroupoids." For each, we show that there is a six-term exact sequence of associated K-groups. We present examples which arise from dynamical systems and from problems in the study of the orbit structure of topological systems. We also obtain the usual Mayer-Vietoris sequence in topological K-theory as a corollary.

1. Introduction. This paper is concerned with principal topological groupoids, their C^*-algebras and the K-theory of such C^*-algebras. For those not familiar with the terminology, *principal groupoid* is just a fancy way of saying *equivalence relation*; our objects of study are certain topological equivalence relations. We view these as objects in the theory of topological dynamics. Indeed, most important examples arise from well-known situations in topological dynamics, e.g., equivalence classes are the orbits of a free action of a countable group acting as homeomorphisms of a topological space or tail equivalence for a one-sided shift of finite type [**18**].

To an equivalence relation, G (satisfying certain conditions), one can construct a C^*-algebra, $C^*_r(G)$, the reduced groupoid C^*-algebra, as in [**16**]. An important tool in the study of C^*-algebras is K-theory. To any C^*-algebra, A, one may associate a pair of abelian groups, $K_0(A)$ and $K_1(A)$ [**1**]. The former also carries additional structure in the form of a pre-order. Combining these two constructions, we have a way of assigning to a topological equivalence relation, G, a pair of abelian groups, $K_0(C^*_r(G))$ and $K_1(C^*_r(G))$. One can view this as a kind of dynamical homology theory. It extends the usual notion of topological K-theory in that, if one restricts to the case that the

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equivalence relation is equality, the result is the usual topological K-theory of the underlying space. Of course, in specific situations, it is desirable to have a purely dynamical interpretation of these invariants.

In the case that the equivalence classes are the orbits of a single transformation, the K-theory was computed by Pimsner-Voiculescu [**13, 1**]. In the particular case of a minimal homeomorphism of the Cantor set, the K-theoretic invariants (including the order structure) have been useful in classifying the system up to orbit equivalence [**7**]. The case of more general homeomorphisms of the Cantor set has also been considered in [**2, 3**].

The present paper is concerned with some basic properties of this dynamical homology theory. We describe two situations where a trio of groupoids G, G' and H are related in some specific way and then describe the relation between their K-theories. These results are given in Theorem 2.1 and Theorem 2.4, although the hypotheses are described before the statements.

In spirit, these are much like the long exact sequence in homotopy theory related to a fibration.

Let us describe the first situation which we refer to as *factor groupoids*. Suppose X and Y are locally compact Hausdorff spaces and we have two inclusions of Y in X with disjoint images. We can then form the quotient space X' by identifying the two images. We suppose that X' is again Hausdorff and that the quotient map is proper. If we also suppose that X and Y carry equivalence relations G and H and that the inclusions map each H-equivalence class bijectively to a G-equivalence class, then X' naturally obtains an equivalence G' which we describe as a *factor groupoid*. In the case G, H and hence G' are equality, Theorem 2.1 (which relates the K-theories of X, Y and X') can be obtained by standard topological methods. Our Theorem 2.1 can be viewed as an *equivariant* extension of this result.

The second situation is more essentially an equivalence relation phenomenon (and hence probably more interesting). We refer to it as the *subgroupoid* situation. We have two equivalence relations $G' \subseteq G$ on the same space X. There are some strict conditions on the set-up which imply, in particular, that each G-equivalence class is either equal to a G'-equivalence class or is the union of two G'-equivalence classes. (Moreover, in most interesting situations, the former will happen on a dense G_δ-set in X.) The third equivalence relation H is basically that part of G where G and G' differ. (This statement sweeps a lot of topological difficulties under the rug for the moment.) Theorem 2.4 relates the K-theoretic invariants of G, G' and H. In Example 2.5, we show that the usual Mayer-Vietoris sequence for topological K-theory can be obtained from Theorem 2.4 and a result of Kumjian. Another motivation in the study of subgroupoids was [**14**] and [**7**] where various AF-relations (see Appendix) were obtained as subequivalences of relations associated with minimal homeomorphisms of the Cantor set. This is described in Example 2.6.

There appears to be a duality between the factor groupoid situation and the subgroupoid situation. In particular, Examples 2.3 and 2.7 look reminiscent of the beginning of a Jones tower construction [**8, 20**].

In order to make our results as accessible as possible to readers in dynamical systems, we try to describe the set-ups in the language of equivalence relations as well as groupoids. Also we present examples from familiar dynamical situations and an appendix which discusses AF-equivalence relations. The remainder of this section describes basic notions and notation. In Section 2 we present the statements of the two main theorems and several examples of each. Finally, Section 3 contains the proofs of the main results. In both cases, the proof relies critically on a result, a kind of excision theorem for C^*-algebra K-theory, in [**15**]. In the proofs, it is of course necessary to use a lot of C^*-algebra machinery including the theory of groupoid C^*-algebras, K-theory and even the Kasparov KK-theory (which manages to creep into the statements of the main results in Section 2, though in a minor role).

If $G \subseteq X \times X$ is an equivalence relation, then G has the algebraic structure of a groupoid [**16**]: a partially defined product

$$(x, y)(y', z) = (x, z) \quad \text{if } y = y',$$

and an inverse $(x, y)^{-1} = (y, x)$. The space of units [**16**], denoted G^0, is equal to $\{(x, x) \mid x \in X\}$ which we denote by Δ and which may be identified with X in the obvious way. The range and source maps, $r, s : G \to G^0$ can then be identified with the two canonical projections onto X. Such a groupoid is principal [**16**] and every principal groupoid occurs in this way.

Henceforth, we adopt the notation of [**16**]. The elements of G will be denoted x and y, and their product is xy. As in [**16**], G^2 denotes the set of all composable pairs in G. Note that (x, y) is in G^2 if and only if $r(y) = s(x)$. For sets $A, B \subseteq G$, we let

$$AB = \{xy \mid x \in A,\ y \in B,\ (x, y) \in G^2\}.$$

All of our groupoids will have a topology as described in [**16**]. First, we assume that G is locally compact, Hausdorff and second countable. Secondly, we assume that G is r-discrete, i.e., G^0 is open in G. Finally we assume that counting measure is a Haar system for G. It follows that the maps $r, s : G \to G^0$ are local homeomorphisms. Conversely, for an r-discrete groupoid it is easy to see that this condition implies that counting measure is a Haar system. From a dynamical point of view, this means that the equivalence relation G is made up of the graphs of local homeomorphisms γ from $X = G^0$ to itself; explicitly, $\gamma = r \circ s^{-1}$. Let us also mention that, for such groupoids, a straightforward compactness argument shows that, for fixed x in G^0 and $K \subseteq G$ compact, $r^{-1}\{x\} \cap K$ is finite. Hence, $r^{-1}\{x\}$ is countable.

We state for emphasis: *throughout this paper, our groupoids will all be second countable, locally compact, Hausdorff, principal r-discrete groupoids with counting measure as Haar system.*

Also note that locally compact and second countable imply σ-compact.

Let us give a specific example. Let X be a locally compact Hausdorff space and Γ be a countable (discrete) group acting freely on X by homeomorphisms. Let $G = X \times \Gamma$ with the product and inverse

$$\begin{aligned}(x, \gamma)(x', \gamma') &= (x, \gamma\gamma') \quad \text{if} \quad x' = \gamma(x)\\ (x, \gamma)^{-1} &= (\gamma(x), \gamma^{-1}),\end{aligned}$$

for x in X, γ in Γ. We give G the product topology. As an equivalence relation, the equivalence classes are just the Γ-orbits.

For any locally compact, Hausdorff space X, we use $C_c(X)$ and $C_0(X)$ to denote the spaces of continuous complex-valued functions which are compactly supported and vanish at infinity respectively. If X is compact, we denote both by $C(X)$.

If G is a groupoid satisfying our conditions above, we regard $C_c(G)$ as a linear space and define a product and involution by

$$(fg)(x) = \sum_{\substack{z \in G \\ r(z)=r(x)}} f(z)g(z^{-1}x)$$
$$(f^*)(x) = \overline{f(x^{-1})},$$

for f, g in $C_c(G)$ and x in G. We describe the regular representation of $C_c(G)$ as follows. Let $l^2(G)$ denote the (usually inseparable) Hilbert space of square summable functions on G. For each f in $C_c(G)$ we define an operator $\lambda(f)$ on $l^2(G)$ by

$$[\lambda(f)\xi](x) = \sum_{\substack{z \in G \\ r(z)=r(x)}} f(z)\xi(z^{-1}x)$$

for ξ in $l^2(G)$ and x in X. The completion of $\lambda(C_c(G))$ in the operator norm in $\mathcal{B}(l^2(G))$, the bounded linear operators on $l^2(G)$, is the reduced C^*-algebra of G and is denoted $C_r^*(G)$.

We refer the reader to [**1**] for a treatment of K-theory for C^*-algebras and also KK-theory. We use $[\ ,\ ]$ to denote the commutator; for a, b in an algebra A, $[a, b] = ab - ba$. We use $\mathcal{K}(\mathcal{H})$ to denote the C^*-algebra of compact operators on the Hilbert space $\mathcal{H}$ and M_n to denote the $n \times n$ complex matrices. If A is any C^*-algebra and A' is any C^*-subalgebra, we let $C(A'; A)$ denote the mapping cone of the inclusion $A' \subseteq A$, that is,

$$C(A'; A) = \{f : [0,1] \longrightarrow A \mid f \text{ continuous}, f(0) = 0, f(1) \in A'\}.$$

It is a C^*-algebra with pointwise product and $\|f\| = \sup \|f(t)\|$. For a full discussion we refer the reader to [**19, 15**].

2. Statements of the results and examples. Here we state our two main results and provide some examples of each. In each case we begin with an account of the hypotheses in the language of equivalence relations. The proofs of both results will be left until Section 3.

Situation 1. *Factor groupoids.* Suppose X and Y are spaces with equivalence relations G and H, respectively. Suppose $i_0, i_1 : Y \to X$

are two continuous, injective maps with disjoint images. We assume that, for $j = 0, 1$, i_j maps each H-equivalence class in Y in a bijective way to a G-equivalence class in X. (This is hypothesis (1) and the statement that i_j is a groupoid morphism which follow.)

We define X' to be the set X identifying $i_0(y)$ and $i_1(y)$ for all y in Y. More precisely, let $X' = (X - i_0(Y) - i_1(Y)) \cup Y$. There is a canonical projection map $\pi : X \to X'$, $\pi(i_j(y)) = y$, $y \in Y$, $j = 0, 1$, and we endow X' with the quotient topology. We must make the hypothesis that X' is Hausdorff and that the map π is proper. Then X' is locally compact and metrizable.

Now X' has a natural equivalence relation G' because of our hypotheses on i_0, i_1, i.e., $G' = \pi \times \pi(G)$. To simplify notation, we let π denote the natural map from G to G'. We give G' the quotient topology. We must verify that this makes G' into a groupoid in the sense described in Section 1 and that $\pi : G \to G'$ is proper.

Since $\pi : G \to G'$ is continuous and proper, it induces an inclusion of $C_c(G')$ in $C_c(G)$. It is easy to verify that this is actually a $*$-homomorphism between these algebras using the basic properties of i_0 and i_1. We will show that it extends to a $*$-homomorphism $\alpha : C_r^*(G') \to C_r^*(G)$ which is also injective.

Let us give our hypotheses in a more precise way using the language of groupoids. Suppose G and H are two groupoids satisfying the conditions of Section 1 and i_0, i_1 are two continuous injective groupoid morphisms from H to G with disjoint images. We also assume that

(1) for any x in G and $j = 0, 1$, the following are equivalent:

(i) x is in $i_j(H)$,

(ii) $r(x)$ is in $i_j(H^0)$

(iii) $s(x)$ is in $i_j(H^0)$.

(2) The space,

$$G^0 \Big/ \{i_0(x) \sim i_1(x) \mid x \in H^0\}$$

with the quotient topology, is Hausdorff and the natural quotient map from G^0 is proper.

Given this, we define

$$G' = G \Big/ \{i_0(x) \sim i_1(x) \mid x \in H\},$$

and let $\pi : G \to G'$ be the natural quotient map. In a natural way, G' is a groupoid in the purely algebraic sense.

The main result for this situation is the following.

Theorem 2.1. *For r-discrete principal groupoids H, G, G' satisfying the conditions of Section 1 and maps i_0, i_1 as in the "factor groupoid" situation described above, there is a six-term exact sequence*

$$\begin{tikzcd}
K_1(C_r^*(H)) \arrow[r] & K_0(C_r^*(G')) \arrow[r, "\alpha_*"] & K_0(C_r^*(G)) \arrow[d, "{[i_0,i_1]_*}"] \\
K_1(C_r^*(G)) \arrow[u, "{[i_0,i_1]_*}"] & K_1(C_r^*(G')) \arrow[l, "\alpha_*"'] & K_0(C_r^*(H)) \arrow[l]
\end{tikzcd}$$

where the maps $[i_0, i_1]_*$ are induced by a natural element $[i_0, i_1]$ in $KK(C_r^*(G), C_r^*(H))$ described in the proof.

Example 2.2. Let (X, d) be a compact metric space, and let Γ be a countable group acting freely on X. Suppose x_0 and x_1 are points of X such that, for any $\varepsilon > 0$, the set $\{\gamma \in \Gamma \mid d(\gamma x_0, \gamma x_1) \geq \varepsilon\}$ is finite. Then the space X' obtained by identifying γx_0 and γx_1 for all γ in Γ, is Hausdorff. Moreover, Γ acts on X' in a natural way.

To apply Theorem 2.1, let $G = X \times \Gamma$ as in Section 1. Let $H^0 = \Gamma$ and $H = \Gamma \times \Gamma$ be the trivial equivalence relation. (Here Γ is given the discrete topology.) The maps i_0, i_1 are given by $i_j(\gamma_1, \gamma_2) = (\gamma_1(x_j), \gamma_2\gamma_1^{-1})$ from H to G, $j = 0, 1$. In this case $C_r^*(H) \cong \mathcal{K}(l^2(\Gamma))$. Hence, we have $K_0(C_r^*(H)) \cong \mathbf{Z}$ and $K_1(C_r^*(H)) = 0$. In this situation $G' = X' \times \Gamma$.

Example 2.3. This example refers to the following three Bratteli diagrams.

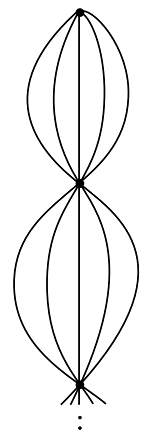


FIGURE B_1.

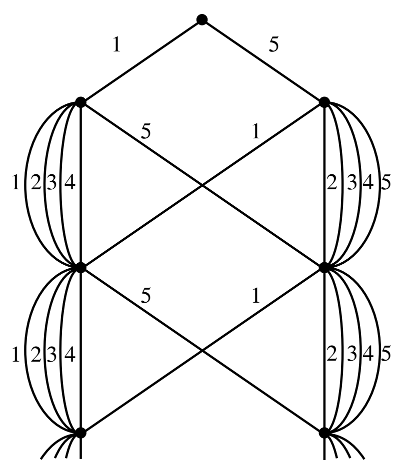


FIGURE B_2.

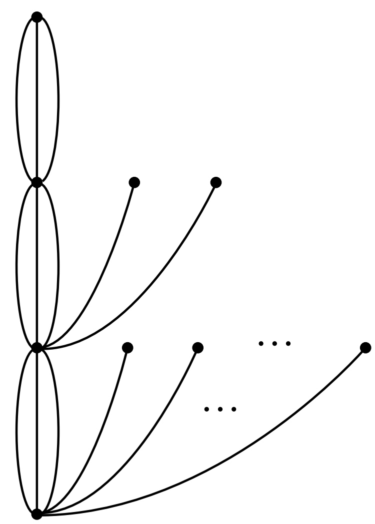


FIGURE B_3.

The number of sources in V_n, the nth vertex set in B_3, is $2 \cdot 5^{n-1}$ and so the edges from sources in V_n are indexed by $\{1,5\} \times \{1,2,3,4,5\}^{n-1}$, for $n \geq 1$.

Let G be the groupoid associated with B_2, see Appendix. The process of reading the edge labels defines a homeomorphism between the unit space of $G = X$, i.e., the infinite path space of B_2, and the set $\{1,5\} \times \{1,2,3,4,5\}^{n-1}$ with its usual product topology. For convenience, we identify these two.

Let H be the groupoid associated with B_3. The maps i_0, i_1 are described as follows. If $(e_{n+1}, e_{n+2}, \dots)$ is a path in B_3 starting at some source in V_n, $n \geq 2$, then to the edge e_{n+1} we associate an element of $\{1,5\} \times \{1,2,3,4,5\}^{n-1}$, say $(f_1, f_2, \dots, f_n)$. To the path $(e_{n+2}, e_{n+3}, \dots)$, we read edge labels to associate to it a sequence $(f_{n+2}, f_{n+3}, \dots)$ in $\prod_{n+2}^{\infty}\{2,3,4\}$. We define $i_0(e_{n+1}, e_{n+2}, \dots)$ to be $(f_1, f_2, \dots, f_n, 1, f_{n+2}, f_{n+3}, \dots)$ in X. We also define $i_1(e_{n+1}, e_{n+2}, \dots)$ to be $(f_1, f_2, \dots, f_n, 5, f_{n+2}, f_{n+3}, \dots)$. To identify the quotient $X/i_0 \sim i_1$ with the path space of B_1, we proceed as follows. First we identify the path space of B_1 with $\{1,2,3,4,5\}^{\mathbf{N}}$. For x in $\{1,5\} \times \{1,2,3,4,5\}^{\mathbf{N}}$

$(= X)$ we define $\alpha(x)$ in $\{1,2,3,4,5\}^{\mathbf{N}}$ by

$$\begin{aligned}\alpha(x)_n &= x_{n+1} \quad \text{if} \quad x_{n+1} \in \{2,3,4\}\\ \alpha(x)_n &= x_k \quad\quad \text{if} \quad x_{n+1} \in \{1,5\},\end{aligned}$$

where k is the largest integer less than or equal to n with $x_k \in \{1,5\}$, for $n \geq 1$. That is, α leaves fixed the entries of x which are 2, 3 or 4 and shifts the 1 and 5 entries to the right. We leave it to the reader to verify that α is a homeomorphism from $X/i_0 \sim i_1$ to the path space of B_1 and identifies the two equivalence relations. That is, $G'$is identified with the groupoid of diagram B_1. In this case we have

$$K_0(C_r^*(H)) \cong \mathbf{Z}\left[\frac{1}{3}\right]$$
$$K_0(C_r^*(G')) \cong \mathbf{Z}\left[\frac{1}{5}\right]$$

with usual orderings from $\mathbf{R}$ and

$$K_1(C_r^*(H)) \equiv K_1(C_r^*(G)) \cong K_1(C_r^*(G')) \cong 0.$$

Situation 2. *Subgroupoids.* We begin our discussion in a very heuristic fashion. Let X be a space with an equivalence relation G. Our second equivalence relation G' will again be on the space X (unlike Situation 1). In fact, it will be a subequivalence relation, i.e., $G' \subseteq G$. This should be so that each G-equivalence class is a G'-equivalence class or is the union of two distinct G'-equivalence classes. In order for this to happen in some sort of "topologically regular" fashion, we proceed more precisely as follows.

Suppose we have L contained in G such that

(1) L is closed,

(2) $r(L) \cap s(L)$ is empty,

(3) $G' = G - L - L^{-1}$ is such that $G'G' \subseteq G'$,

(4) LG', $G'L \subseteq L$.

That is, G' is obtained by removing L and L^{-1} from G so that for x in L, $r(x)$ and $s(x)$ are in the same G-equivalence classes but distinct

G'-equivalence classes. The simplest example is the following: let $X = \{1, \ldots, n\}$, $G = X \times X$ and $L = \{(i,j) \mid 1 \leq i \leq k, k < j \leq n\}$ for some fixed $1 \leq k < n$.

We then define

$$\begin{aligned} H_0 &= L^{-1}L \\ H_1 &= LL^{-1} \\ H' &= H_0 \cup H_1 \\ H &= H' \cup L \cup L^{-1} \end{aligned}$$

which are all groupoids in the purely algebraic sense. In fact, H_0, H_1 and H are the reductions of G onto the sets $s(L)$, $r(L)$ and $s(L) \cup r(L)$, respectively. It is important to note, however, at a topological level, $s(L)$ and $r(L)$ may not be closed in G^0 even though L is closed in G.

As suggested by the last remark, the relative topologies on H_0, etc., are not necessarily so nice. However, we introduce a new topology on each which is better. We give three (equivalent) descriptions of the topology on H_0.

(A) Regard the product in the groupoid as a map from $G^2 \cap (L^{-1} \times L)$ onto H_0. The domain is given the relative topology of G^2, i.e., $G \times G$, and then H_0 receives the quotient topology.

(B) Let $\{x_n\}_1^\infty$ be a sequence in H_0. The sequence converges to x in H_0 if and only if there are sequences $\{y_n\}$ and $\{z_n\}$ in L converging to y and z in G (and hence in L since it is closed) such that $x_n = y_n^{-1} z_n$ for all n, and hence $x = y^{-1}z$.

(C) Choose a sequence $\{U_n\}$ of open subsets of G whose union is G and, for each n, $\overline{U}_n$ is compact and contained in U_{n+1}. For each n, let $H_{0,n} = (U_n \cap L^{-1})(U_n \cap L)$ with the relative topology of G. Then H_0, which is the union of the $H_{0,n}$, is given the inductive limit topology.

The set H_1 is treated in a similar way, while H' is given the disjoint union topology. Using the usual topology on L and L^{-1}, H is given the disjoint union topology. In Section 3, we will show that all of H_0, H_1, H' and H are groupoids with these topologies. It is also easy to see that the inclusions of these in G are continuous.

Since G' is an open subset of G, we have $C_c(G')$ included in $C_c(G)$. This is actually a $*$-homomorphism of algebras and the inclusion extends to a $*$-homomorphism $\alpha : C_r^*(G') \to C_r^*(G)$. The main result is then the following.

Theorem 2.4. *For r-discrete principal groupoids, H, G, G' satisfying the conditions of Section 1 and L as in the "subgroupoid" situation described above, there is a six-term exact sequence*

$$\begin{tikzcd}
K_0(C_r^*(H)) \arrow[r] & K_0(C_r^*(G')) \arrow[r, "\alpha_*"] & K_0(C_r^*(G)) \arrow[d, "{[L]_*}"] \\
K_1(C_r^*(G)) \arrow[u, "{[L]_*}"] & K_1(C_r^*(G')) \arrow[l, "\alpha_*"'] & K_1(C_r^*(H)) \arrow[l]
\end{tikzcd}$$

where the maps $[L]_*$ are induced by a natural element $[L]$ in $KK^1(C_r^*(G), C_r^*(H))$ described in the proof.

Example 2.5. Let M be a compact metrizable space, and let U_1 and U_2 be two open subsets of M which cover M. For emphasis, we let i_1 and i_2 denote the two inclusion maps of U_1 and U_2 in M. Also, $U_1 \dot\cup U_2 = X$ denotes the disjoint union of U_1 and U_2. We define an equivalence relation on $U_1 \dot\cup U_2$ by

$$\begin{aligned}
G = &\{(x,x) \mid x \in U_1 \dot\cup U_2\} \\
&\cup \{(x,y),(y,x) \mid x \in U_1, y \in U_2, i_1(x) = i_2(y)\}
\end{aligned}$$

which is endowed with the relative topology of $(U_1 \dot\cup U_2) \times (U_1 \dot\cup U_2)$. This groupoid was considered by Kumjian [**9**] who showed that $C_r^*(G)$ is strongly Morita equivalent to $C(M)$ [**17**].

Let

$$L = \{(x,y) \mid x \in U_1, y \in U_2, i_1(x) = i_2(y)\}.$$

It is easy to verify that $H_0 \cong U_1 \cap U_2 \cong H_1$, both with the cotrivial groupoid structure, i.e., equivalence is equality. Moreover, we have

$$H \cong (U_1 \cap U_2) \times (\{1,2\} \times \{1,2\})$$

and $G' \cong U_1 \dot\cup U_2$.

Therefore, we have

$$\begin{aligned}
C_r^*(H) &\cong M_2(C_0(U_1 \cap U_2)) \\
C_r^*(G') &\cong C_0(U_1 \dot\cup U_2) \\
&\cong C_0(U_1) \oplus C_0(U_2),
\end{aligned}$$

and

$$K_i(C_r^*(G)) \cong K_i(C(M)),$$

by using Kumjian's result for the last part. Then the sequence of Theorem 2.4 becomes the Mayer-Vietoris sequence in K-theory for the cover $\{U_1, U_2\}$ of X.

Example 2.6. Let X be a compact metric space, and let ϕ be a homeomorphism of X with no periodic orbits. Let $G = X \times \mathbf{Z}$ be the associated groupoid as in Section 1. Suppose that Y is a closed, nonempty subset of X which meets each ϕ-orbit at most once. That is, $\phi^n(Y) \cap Y$ is empty for $n \neq 0$. Let $\mathbf{Z}^+$ and $\mathbf{Z}^-$ denote $\{1, 2, 3, \dots\}$ and $\{0, -1, -2, \dots\}$, respectively. Define

$$L = \{(\phi^l(y), k) \mid y \in Y, l \in \mathbf{Z}^+, k + l \in \mathbf{Z}^-\}.$$

In this example, the G-equivalence classes are the orbits of ϕ and such a class is also a G'-equivalence class if it does not meet Y. For a ϕ-orbit which meets Y, say at y, it is the union of two G'-equivalence classes; namely, the forward ϕ-orbit of $\phi(y)$ and the backward ϕ-orbit of y. The C^*-algebra of G is the crossed-product $C(X) \times_\phi \mathbf{Z}$ while the C^*-algebra of G' may also be described as the C^*-subalgebra of $C(X) \times_\phi \mathbf{Z}$ generated by $C(X)$ and $uC_0(X - Y)$, see [**14**] for details. The main result of [**14**] is that, when X is totally disconnected and ϕ is minimal, G' is an AF-equivalence relation.

In the general situation above, one can show that

$$\begin{aligned} H_0 &\cong Y \times \mathbf{Z}^- \times \mathbf{Z}^- \\ H_1 &\cong Y \times \mathbf{Z}^+ \times \mathbf{Z}^+ \\ H &\cong Y \times \mathbf{Z} \times \mathbf{Z}, \end{aligned}$$

i.e., cotrivial equivalence on Y, trivial equivalence on $\mathbf{Z}^-, \mathbf{Z}^+$ and $\mathbf{Z}$, respectively, so that we have

$$C_r^*(H) \cong C(Y) \otimes \mathcal{K}(l^2(\mathbf{Z})).$$

The exact sequence of Theorem 2.4, in the case of X totally disconnected and ϕ minimal, is then the same as that appearing in [**14**, 4.1]. Note that we have

$$K_1(C_r^*(G')) = 0$$

since G' is AF while

$$K_0(C_r^*(H)) \cong K_0(C(Y)) \cong C(Y, \mathbf{Z}),$$
$$K_1(C_r^*(H)) \cong K_1(C(Y)) = 0,$$

since Y is also totally disconnected in this case.

The result of Theorem 2.4, in conjunction with the Pimsner-Voiculescu sequence [**13, 1**] will also give the exact sequence of [**6**]. (The results of [**6**] are more general since the partial homeomorphism need not extend to a homeomorphism.)

Example 2.7. We refer to the Bratteli diagrams B_1, B_2 and B_3 of Example 2.2. We let B_4 be the diagram shown below:

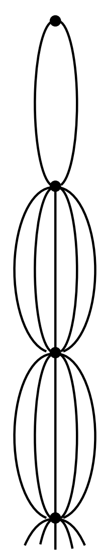


FIGURE B_4.

The C^*-algebra associated with B_4 is $*$-isomorphic with the 2×2 matrices over the C^*-algebra associated with B_1. Here we let G be the groupoid associated with B_4. As in Example 2.2, we identify the path space with $\{1,5\} \times \{1,2,3,4,5\}^{\mathbf{N}}$. Define

$$\begin{aligned} L = \{(x,y) \in G \mid \ & \text{there exist } K, M, N \text{ such that} \\ & x_K = 1, x_i \in \{2,3,4\}, \quad i > K, \\ & y_M = 5, y_i \in \{2,3,4\}, \quad i > M, \\ & \text{and } x_i = y_i \text{ for } i > N\}. \end{aligned}$$

It is an easy exercise to see that L satisfies the appropriate hypotheses. One can also identify G' with the groupoid associated with B_2 and H with the groupoid of B_3. We leave the details of this to the reader.

3. Proofs of the results. In both cases the proofs of the main theorem follow from an application of the main result of [**15**]. However, it still requires some work to see that hypotheses of [**15**] hold.

Before getting into the two specific situations, we will need the following technical result.

Lemma 3.1. *Let G and H be r-discrete principal groupoids satisfying the conditions of Section 1, and let i be a groupoid morphism from H to G which satisfies*

(1) *for x in G, the following are equivalent*:

(i) *x is in $i(H)$*,

(ii) *$r(x)$ is in $i(H^0)$*,

(iii) $s(x)$ is in $i(H^0)$;

(2) *i is injective*;

(3) *i is continuous.*

Let K be any compact set in H^0. If $\{x_n\}_{n\geq 1}$ is a sequence and x is any point in $r^{-1}(K)$, or $s^{-1}(K)$, then x_n converges to x if and only if $i(x_n)$ converges to $i(x)$ in G. That is,

$$i : r^{-1}(K) \longrightarrow i(r^{-1}(K)) = r^{-1}(i(K))$$

is a homeomorphism.

Proof. The "only if" statement follows from continuity of i. For the "if" part, choose a compact neighborhood U of $i(x)$ such that r is a homeomorphism from U to $r(U)$, a neighborhood of $r(i(x)) = i(r(x))$ in G^0. Choose a compact neighborhood V of x in H such that $i(V) \subseteq U$ using the continuity of i and such that $r \mid V$ is a homeomorphism. The sequence $r(x_n)$ is in K and, since K is compact, it has an accumulation point. Now if $r(x_n)$ converges to z in H, $i(r(x_n)) = r(i(x_n))$ converges to $i(z)$. Since i is injective, $z = r(x)$. We conclude that $r(x_n)$ has at most one accumulation point $r(x)$ and $r(x_n)$ converges to $r(x)$. So, for sufficiently large n, $r(x_n)$ is in $r(V)$ so we may find y_n in Y such that $r(x_n) = r(y_n)$. Since $r|V$ is a homeomorphism and y_n, x are all in V, y_n converges to x. Now consider $i(y_n^{-1}x_n) = i(y_n)^{-1}i(x_n)$ which converges to $i(x)^{-1}i(x)$ which is in G^0. Since G is r-discrete, we must have $i(y_n^{-1}x_n)$ in G^0, for n large. Thus, $i(y_n) = i(x_n)$ and hence $x_n = y_n$ for n large, since i is injective. Since y_n converges to x in H, we are done. $\square$

Situation 1. *Factor groupoids.* The first step is to check that G' satisfies the conditions of Section 1. That is, we must see that G' is Hausdorff and the natural maps $r, s : G' \to X'$ are local homeomorphisms. Both of these facts follow almost at once from the facts that the quotient X' is Hausdorff and that in G and H the r, s maps are local homeomorphisms. One also needs the first property of i_0, i_1; that they map equivalence classes onto equivalence classes. We leave the details to the reader.

For technical reasons we will need the following description of G. Fix a sequence of open sets in $H, H_1, H_2, H_3, \ldots$, whose union is H and so that, for each k, $\overline{H}_k$ is compact and contained in H_{k+1}. For each k, we define

$$G_k = G/\{i_0(x) \sim i_1(x) \mid x \in H - H_k\},$$

and we let $\pi_k : G \to G_k$, $\pi'_k : G_k \to G'$ be the obvious quotient maps. Then each G_k is locally compact and Hausdorff and G is the inverse limit (in the category of locally compact spaces with proper, continuous maps) of the system

$$G' \longleftarrow G_1 \longleftarrow G_2 \longleftarrow \cdots .$$

At the level of functions, we have

$$C_c(G') \subseteq C_c(G_k) \subseteq C_c(G),$$

and the union of the $C_c(G_k)$ is dense in $C_c(G)$ (in the inductive limit topology: a sequence f_n in $C_c(G)$ converges to f if there is a compact set $K \subseteq G$ with $\operatorname{supp}(f_n) \subseteq K$ for all n and $f_n \to f$ uniformly.)

We now define representations of our groupoids and their C^*-algebras. Let λ denote the regular representation of $C_c(G)$ on $l^2(G)$. The completion of $\lambda(C_c(G))$ is $C_r^*(G)$. If we restrict λ to $C_c(G')$, we obtain the regular representation of G' except some irreducible factors (corresponding to points of G' where π is two-to-one) appear twice as many times. We conclude that the $\lambda(C_c(G'))^-$ is $*$-isomorphic to $C_r^*(G')$. (We will simply identify the two.)

We define a self-adjoint unitary z on $l^2(G)$ by

$$(z\xi)(x) = \begin{cases} \xi(i_1(y)) & \text{if } x = i_0(y) \text{ for some } y \text{ in } H \\ \xi(i_0(y)) & \text{if } x = i_1(y) \text{ for some } y \text{ in } H \\ \xi(x) & \text{otherwise,} \end{cases}$$

for ξ in $l^2(G)$, x in X. Let us remark that every element of $C_r^*(G')$ commutes with z (see Lemma 3.2), and later we will show that $C_r^*(G')$ is exactly the commutant of z in $C_r^*(G)$.

We define a representation of $C_r^*(H) \oplus C_r^*(H)$ on $l^2(G)$ as follows. For h_0, h_1 in $C_c(H)$, define, for ξ in $l^2(G)$

$$(\mu(h_0 \oplus h_1)\xi)(i_j(x)) = \sum_{\substack{s(y)=s(x)\\ y \in H}} h_j(xy^{-1})\xi(i_j(y))$$

for $j = 0, 1, x$ in H,

$$(\mu(h_0 \oplus h_1)\xi)(x) = 0,$$

for x in $G - i_0(H) - i_1(H)$. That is, μ is just the direct sum of the regular representation of $C_r^*(H)$ with itself and with the zero representation. The closure, then, of $\mu(C_c(H) \oplus C_c(H))$ is $C_r^*(H) \oplus C_r^*(H)$.

Lemma 3.2. *Suppose $k \geq 1$, and let f be in $C_c(G_k)$. Define $\tilde{f} : H \to \mathbf{C}$ by*

$$\tilde{f}(x) = f(\pi_k \circ i_0(x)) - f(\pi_k \circ i_1(x)), \quad x \in H.$$

Then $\tilde{f}$ is in $C_c(H)$ and

$$z\lambda(f)z - \lambda(f) = \mu(\tilde{f} \oplus -\tilde{f}).$$

Proof. It is clear that $\tilde{f}$ is continuous. Moreover, the support of $\tilde{f}$ is contained in $\overline{H}_k$, which is compact. The final formula is a straightforward computation which we omit. ◻

Next we want to show that $C_r^*(G)$ lies in the multiplier algebra of $C_r^*(H) \oplus C_r^*(H)$ [**12**]. This fact follows easily from the following.

Lemma 3.3. *Let f be in $C_c(G)$ and g_0, g_1 be in $C_c(H)$. Define $h_0, h_1 : H \to \mathbf{C}$ by*

$$h_j(k) = \sum_{\substack{y \in H \\ s(y)=s(x)}} f(i_j(xy^{-1}))g_j(y),$$

$j = 0, 1, x$ in H. Then h_0 and h_1 are in $C_c(H)$ and

$$\lambda(f)\mu(g_0 \oplus g_1) = \mu(h_0 \oplus h_1).$$

Proof. As noted in Section 1, the set $\{y \mid s(y) = s(x), g_j(y) \neq 0\}$ is finite for any x in H, so h_j is well-defined, $j = 0, 1$. By observing that the formula for h_j is bilinear in f and $g_0 \oplus g_1$, we may restrict our attention to the case where $\operatorname{supp}(f) \subseteq K$, $\operatorname{supp}(g_j) \subseteq K_j$, $j = 0, 1$, where K, K_0 and K_1 are compact sets in G and H, respectively, where $r, s : K \to G^0$, $r, s : K_0 \to H^0$, $r, s : K_1 \to H^0$ are all homeomorphisms onto their images. In this case, the support of h_j is contained in $i_0^{-1}(K)K_0 \cup i_1^{-1}(K)K_1$ in H. We must show that $i_0^{-1}(K)K_0$ is compact in H; the argument for the other set is the same. Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in $i_0^{-1}(K)$ and K_0, respectively, such that $x_n y_n$ is defined for all n. By compactness of K and K_0, we may pass to subsequences such that $i_0(x_{n_k})$ converges to z in K and y_{n_k} converges to y in K_0. Since G^2 is closed in $G \times G$, $zi_0(y)$ is defined. By the first hypothesis on i_0, z is in the image of i_0; say $z = i_0(x)$. By Lemma 3.1, we conclude that x_{n_k} converges to x in H. Therefore, $x_{n_k}y_{n_k}$ converges to xy in H and xy is in $i_0^{-1}(K)K_0$. Hence, $i_0^{-1}(K)K_0$ is compact and so h_0 is in $C_c(H)$. (The continuity of h_0 is clear.) The final formula is an easy computation. ◻

Let $E = C_r^*(H) \oplus C_r^*(H)$ be considered as a graded right Hilbert module over $C_r^*(H)$ in the usual way [**1**]. Lemma 2.3 asserts that the products defined there extend to define a $*$-homomorphism $\bar{\lambda} : C_r^*(G) \to \mathcal{B}(E)$. Let F be the natural grading operator on E. Lemma 2.2 then implies that $(E, \bar{\lambda}, F)$ determines an element of the Kasparov group $KK(C_r^*(G), C_r^*(H))$ [**1**] which we denote by $[i_0, i_1]$. As described in [**1**, 23.1], this induces group homomorphisms

$$[i_0, i_1]_* : K_i(C_r^*(G)) \to K_i(C_r^*(H)).$$

To prove Theorem 2.1, we will use the main result of [**15**] which we state now. We suppose that A and B are separable C^*-algebras acting on the Hilbert space $\mathcal{H}$. We suppose that z is a self-adjoint unitary operator on $\mathcal{H}$. We assume:

(1) for a in A, b in B, ab is in A; that is, B acts as multipliers of A,

(2) $zAz = A$,

(3) for b in B, $zbz - b$ is in A.

We also assume that there is a continuous path $\{e_t\}_{t\geq 1}$ in A satisfying:

(4) (i) $0 \leq e_t \leq e_s \leq 1$, for all $t \leq s$,

(ii) $e_s e_t = e_t$, for all $s \geq t+2$,

(iii) for all a in A,

$$\|e_t a - a\|, \ \|a e_t - a\| \longrightarrow 0, \quad t \to \infty,$$

(iv) $[e_t, z] = 0$, for all $t \geq 1$.

We then define the C^*-algebras A', B' by

$$\begin{aligned} A' &= \{a \in A \mid [a, z] = 0\} \\ B' &= \{b \in B \mid [b, z] = 0\}. \end{aligned}$$

We further assume that

(5) for all b in B, there is a b' in B' such that $\|b - b'\| \leq 2\|[b, z]\|$; that is, almost commuting with z implies nearly commuting with z.

(6) There is a dense $*$-subalgebra $\mathcal{A}$ in A such that, for any a in $\mathcal{A}$ and $t_0 \geq 1$, there is a $t \geq t_0$ such that

(i) $a e_s = a = e_s a$, for $s \geq t$, and, for any such t, there is a b in B such that

(ii) $be_s = a = e_s b$, for $t+2 \geq s \geq t$,

(iii) $[b-a, z] = 0$,

(iv) $\|b\| \leq \|a\|$.

Under these hypotheses, we have

$$K_i(C(B'; B)) \cong K_i(C(A'; A)),$$

for $i = 0, 1$. We let

$$\begin{aligned}
&A = \mu(C_c(H) \oplus C_c(H))^- \cong C_r^*(H) \oplus C_r^*(H),\\
&B = \lambda(C_c(G))^- \cong C_r^*(G),\\
&\mathcal{H} = l^2(G)
\end{aligned}$$

and z be as above. To define our approximate unit for A, we choose a sequence of functions χ_n in $C_c(H)$ such that

(i) $\operatorname{supp}(\chi_n) \subseteq H^0$,

(ii) $0 \leq \chi_n \leq \chi_{n+1} \leq 1$,

(iii) $\lim_{n\to\infty} \chi_n = 1$ on H^0 (pointwise).

For each n, we let $e_n = \mu(\chi_n \oplus \chi_n)$ and, for $n \leq t \leq n+1$, we let

$$e_t = (t-n)e_{n+1} + (n+1-t)e_n.$$

In hypothesis (6), we let $\mathcal{A}$ be $\mu(C_c(H) \oplus C_c(H))$. It is easy to see that (1) follows from Lemma 3.3, (3) from Lemma 3.2 and (4) from the construction of e_t. Also, for g_0 and g_1 in $C_c(H)$, it is easily seen that

$$z\mu(g_0 \oplus g_1)z = \mu(g_1 \oplus g_0)$$

from which (2) follows.

Lemma 3.4. *Let h_0 and h_1 be in $C_c(H)$. There are positive integers k and g in $C_c(G)$ such that*

(i) *$e_l\mu(h_0 \oplus h_1) = \mu(h_0 \oplus h_1) = \mu(h_0 \oplus h_1)e_l$, for all $l \geq k$ and, for all $k \leq l \leq k+2$,*

(ii) $\lambda(g)e_l = \mu(h_0 \oplus h_1)$,

(iii) $[\lambda(g), e_l] = 0$,

(iv) $\lambda(g)(e_l - e_l^2) = 0$,

(v) $[\lambda(g)(1 - e_l), z] = 0$.

Proof. Since the supports of h_0 and h_1 are compact, they are contained in some H_k. Also, choose k sufficiently large so that $\chi_k = 1$ on the ranges and sources of the supports of h_0 and h_1. Property (i) then follows by an easy computation. Let

$$K = r(\overline{H}_{k+3}) \cup s(\overline{H}_{k+3})$$

which is compact in H^0. Then, as r and s are continuous,

$$\begin{aligned} X &= i_0(r^{-1}(K) \cup s^{-1}(K)) \cup i_1(r^{-1}(K) \cup s^{-1}(K)) \\ &= r^{-1}(i_0(K) \cup i_1(K)) \cup s^{-1}(i_0(K) \cup i_1(K)) \end{aligned}$$

is closed in G and, by Lemma 3.1, the relative topology on X is the same as that on $r^{-1}(K) \cup s^{-1}(K)$ in H. Let $X' = \pi_k(X)$ which is a closed set in G_k. If we define $g' : X' \to \mathbf{C}$ by

$$g'(\pi_k(i_j(x))) = h_j(x)$$

we see that g' is well-defined since, if $\pi_k(i_0(x)) = \pi_k(i_1(x))$, then x is in $H - H_k$ and h_0 and h_1 are both supported in H_k. Also, the support of g' is contained in $\pi_k(i_0(\overline{H}_k) \cup i_1(\overline{H}_k))$ which is compact in X. Moreover, since the relative topology of X is the same as that of $r^{-1}(K) \cup s^{-1}(K)$, g' is continuous. Let g be any element of $C_c(G_k)$ such that $g|X' = g'$. The proof that g satisfies (i)–(v) is similar to the proof of Lemma 3.11. We omit the details. □

Lemma 3.5. *Let h_0 and h_1 be in $C_c(H)$. There is a positive integer k and a b in $C_r^*(G)$ such that*

(i) $e_l\mu(h_0 \oplus h_1) = \mu(h_0 \oplus h_1) = \mu(h_0 \oplus h_1)e_l$, *for all $l \geq k$ and, for all $k \leq l \leq k+2$,*

(ii) $be_l = \mu(h_0 \oplus h_1)$

(iii) $[b, e_l] = 0$

(iv) $b(e_l - e_l^2) = 0$

(v) $[b(1-e_l), z] = 0$

(vi) $\|b\| \leq \|\mu(h_0 \oplus h_1)\|$.

Proof. Apply Lemma 3.4 to obtain k and g. Let $\delta = \|\mu(h_0 \oplus h_1)\|$ and define

$$\zeta(t) = \begin{cases} 1 & 0 \leq t \leq \delta^2 \\ t^{-1/2}\delta & \delta^2 \leq t, \end{cases}$$

for t in $[0, \infty)$. Define $b = \lambda(g)\zeta(\lambda(g^*g))$. Since, for $k \leq l \leq k+2$, e_l commutes with $\lambda(g)$ and so does b. Moreover, we have

$$\lambda(g)\zeta(\lambda(g^*g))e_l = \lambda(g)e_l = \mu(h_0 \oplus h_1).$$

The other properties are easily verified. $\square$

Lemma 3.6. *For A and B as above, hypotheses* (5) *and* (6) *of* [**15**] *hold.*

Proof. First we consider (5). Let b be in B. If $[b, z] = 0$, then we let $b' = b$ and we are done. Otherwise, find a positive integer n and f in $C_c(G_n)$ so that

$$\|\lambda(f) - b\| \leq \frac{1}{2}\|[b, z]\|.$$

Let $\tilde{f}$ be as in Lemma 3.2, and let $h_0 = \tilde{f}$ and $h_1 = -\tilde{f}$ be in $C_c(H)$. That is,

$$z\lambda(f)z - \lambda(f) = \mu(h_0 \oplus h_1).$$

Now let k and b_0 be as in Lemma 3.5 for h_0, h_1. Let $b' = \sigma(f) + b_0/2$. It is then routine to verify b' satisfies the conclusion of (5). (See the proof of Lemma 3.13.)

Hypothesis (6) follows immediately from Lemma 3.5. $\square$

There is one more issue; recall from [**15**] that

$$B' = \{b \in B \mid [b, z] = 0\}.$$

We must check that B' coincides with $C_r^*(G')$.

Proposition 3.7. $C_r^*(G') \cong B'$.

Proof. As noted earlier, the map sending f in $C_c(G')$ to $\alpha(f) = \pi \circ f$ in $C_c(G)$ is a $*$-homomorphism. Moreover, $\lambda \circ \alpha$ is just the regular representation of $C_c(G')$, with some summands appearing with doubled multiplicity. By Lemma 3.2, $\lambda \circ \alpha(C_c(G'))$, and hence $C_r^*(G')$ is contained in B'. We must show the reverse inclusion.

Let C denote the C^*-algebra generated by A and B. This contains B' and $C_r^*(G')$. Suppose there exists b_0 in B', not in $C_r^*(G')$. As in Lemma 3.9 of [**11**], we may find a representation ρ of C on a Hilbert space $\mathcal{H}$ and vectors ξ_0, η_0 in $\mathcal{H}$ such that

$$\begin{gathered}\langle \rho(b_0)\xi_0, \eta_0\rangle \neq 0\\ \langle \rho(b')\xi_0, \eta_0\rangle = 0,\\ \text{for all } b' \text{ in } C_r^*(G').\end{gathered}$$

We apply the disintegration theorem, [**16**, Theorem 1.21], to the restrictions of ρ to B and to A, the latter viewed as the groupoid C^*-algebra of the disjoint union of two copies of H. Some caution is needed since $\rho|A$ may be degenerate. Let $\mathcal{H}_0$ denote the subspace of $\mathcal{H}$ on which $\rho(A)$ acts nondegenerately. Using the fact that B acts as multipliers of A, it can easily be seen that $\mathcal{H}_0$ is $\rho(B)$-invariant. As in [**16**], let (μ, K, L) and (μ^0, K^0, L^0) be the representations of G and $H \uplus H$ obtained from $\rho|B$ and $\rho|A$ on $\mathcal{H}_0$ via the disintegration theorem. Define another representation of G by decomposing

$$G = (i_0(H) \uplus i_1(H)) \cup (G - i_0(H) - i_1(H))$$

as Borel groupoids, using (μ^0, K^0, L^0) on $i_0(H) \cup i_1(H)$ and zero on its complement. Denote the representation of B by $(\rho', \mathcal{H}')$; notice that we may identify $\mathcal{H}'$ with $\mathcal{H}_0$ in an obvious way. It is easy to check that, for any b in B and a in A, we have

$$\rho(b)\rho(a) = \rho'(b)\rho(a),$$

from which we conclude $\rho' = \rho|B$ on $\mathcal{H}_0$. Therefore, we conclude that the measures μ^0 and $\mu|i_0(H^0) \cup i_1(H^0)$ are equivalent and the Hilbert bundles K^0 and $K|i_0(H^0) \cup i_1(H^0)$ are isomorphic by a bundle map intertwining the representations L^0 and $L \mid i_0(H) \cup i_1(H)$. For simplicity of notation, we assume that they are equal.

Returning to b_0 in B', let $\delta = |\langle \rho(b_0)\xi_0, \eta_0\rangle|$. There exists an f in $C_c(G)$ such that $\|\lambda(f) - b_0\| < \delta/4$. In fact, we may assume that f in $C_c(G_k)$ for some k. Let $\tilde{f}$ be as in Lemma 3.2, so we have

$$\|\mu(\tilde{f} \oplus -\tilde{f})\| = \|z\lambda(f)z - \lambda(f)\| = \|[z, \lambda(f)]\| < \delta/4$$

since $[b_0, z] = 0$. Let Y denote the closure of $i_0(\mathrm{supp}\,(\tilde{f})) \cup i_1(\mathrm{supp}\,(\tilde{f}))$ which is compact in G. Define f_0 on G by

$$\begin{aligned} f_0(i_0(x)) &= \tilde{f}(x) \\ f_0(i_1(x)) &= -\tilde{f}(x) \\ f_0(y) &= 0, \quad y \notin Y. \end{aligned}$$

Note that f_0 is a compactly supported Borel function on G. Choose a continuous compactly supported function g on G such that $g \mid Y = f_0 \mid Y$. Now choose a sequence $\{h_n\}_1^\infty$ of continuous compactly supported functions of G such that

(i) $0 \le h_n \le h_{n-1} \le 1$, for all n,

(ii) $h_n \mid Y = 1$

(iii) $h_n \to \chi_Y$ pointwise.

The pointwise product gh_n is in $C_c(G)$ and

$$\begin{aligned} \left(f - \frac{1}{2}gh_n\right)(i_0(x)) &= \left(f - \frac{1}{2}g\right)(i_0(x)) \\ &= f(i_0(x)) - \frac{1}{2}\tilde{f}(x) \\ &= \frac{1}{2}f(i_0(x)) + \frac{1}{2}f(i_1(x)) \\ &= \left(f - \frac{1}{2}gh_n\right)(i_1(x)) \end{aligned}$$

so that $f - gh_n/2$ is in $\alpha(C_c(G'))$. We conclude that, for every n,

$$\begin{aligned} 0 &= \left\langle \rho\left(f - \frac{1}{2}gh_n\right)\xi_0, \eta_n \right\rangle \\ &= \langle \rho(f)\xi_0, \eta_0\rangle - \frac{1}{2}\langle \rho(gh_n)\xi_0, \eta_0\rangle. \end{aligned}$$

For the first term, we have

$$\begin{aligned}|\langle \rho(f)\xi_0, \eta_0\rangle| &\geq \|\langle \rho(b_0)\xi_0, \eta_0\rangle\| - \|b_0 - \lambda(f)\| \\ &\geq 3\delta/4.\end{aligned}$$

On the other hand, we compute

$$\langle \rho(gh_n)\xi_0, \eta_0\rangle = \int_G g(x)h_n(x)\langle L(x)\xi_0(s),(x), \eta_0(r(x))\rangle \, d\mu(x)$$

since ρ is the integrated form of (μ, K, L). The integrand above is in $L^1(G, \mu)$, and we may apply the dominated convergence theorem ($|gh_n| \leq |gh_1|$) to conclude that

$$\begin{aligned}\lim_{n\to\infty} &|\langle \rho(gh_n)\xi_0, \eta_0\rangle| \\ &= \left| \int_g f_0(x)\langle L(x)\xi_0(s(x)), \eta_0(r(x))\rangle \, d\mu(x)\right| \\ &= \left| \int_{i_0(H)\cup i_1(H)} f_0(x)\langle L(x)\xi_0(s(x)), \eta_0(r(x))\rangle \, d\mu(x)\right| \\ &= |\langle \rho(\tilde{f} \oplus -\tilde{f}))\xi_0, \eta_0\rangle| \\ &\leq \|\mu(\tilde{f} \oplus -\tilde{f})\| < \delta/4.\end{aligned}$$

Thus, we arrive at a contradiction and so we conclude $B' \subseteq C_r^*(G')$ as desired. □

Proof of Theorem 2.1. We begin with the short exact sequence [**19, 15**],

$$0 \longrightarrow C_0(0,1) \otimes B \longrightarrow C(B'; B) \longrightarrow B' \longrightarrow 0$$

and obtain from it an associated six-term exact sequence of K-groups. By Proposition 3.7, we have $K_i(B') \cong K_i(C_r^*(G'))$ and, more generally, $K_i(C_0(0,1) \otimes B) \cong K_{i+1}(B) \cong K_{i+1}(C_r^*(G))$. By the main result of [**15**] (the excision theorem), we also have $K_i(C(B'; B)) \cong K_i(C(A'; A))$ where

$$A \cong C_r^*(H) \oplus C_r^*(H)$$

as before and

$$\begin{aligned}A' &= \{a \in A \mid [a, z] = 0\} \\ &= \{(a, a) \mid a \in C_r^*(H)\} \\ &\cong C_r^*(H).\end{aligned}$$

Now, using the short exact sequence for $C(A';A)$ analogous to the one above for $C(B';B)$, its associated six-term sequence for K-groups and the computation of A' above, it can easily be shown that

$$K_i(C(A';A)) \cong K_{i+1}(C_r^*(H)).$$

This yields the exact sequence of Theorem 2.1; that the maps are as claimed there follows from results in [**19**] and [**1**]. We leave the details to the reader. $\square$

Situation 2. *Subgroupoids.* We begin by establishing some basic properties of H_0, H_1, H', H.

Lemma 3.8. (i) *L, L^{-1}, H_0 and H_1 are pairwise disjoint.*

(ii) *H_0, H_1, H' and H are all groupoids and satisfy the conditions of Section* 1.

(iii) *$C_r^*(H') \cong C_r^*(H_0) \oplus C_r^*(H_1)$.*

(iv) *$C_c(L)$ may be completed to be a $C_r^*(H_0) - C_r^*(H_1)$ equivalence bimodule* [**17, 10**].

(v) *$C_r^*(H_0)$, $C_r^*(H_1)$ and $C_r^*(H)$ are all strongly Morita equivalent* [**17, 10**].

Proof. (i) If w, x, y, z are all in L and $w^{-1}x = yz^{-1}$, then $s(x) = s(w^{-1}x) = s(yz^{-1}) = r(z)$ which contradicts hypothesis (1). By definition, then, we have $H_0 \cap H_1 = \varnothing$. The other cases are similar.

(ii) An argument similar to that in part (i) shows that $H_0 = L^{-1}L$ is disjoint from L and L^{-1}. Then

$$H_1H_1 = LL^{-1}LL^{-1} \subseteq LG'L^{-1} \subseteq LL^{-1} = H_1$$

and, in a similar way, $H_0H_0 \subseteq H_0$ and so H_0 and H_1 are groupoids in a purely algebraic sense. Similar arguments apply to H' and H.

It is easy to check that the products and inverse operations are continuous. It is also easy to see each of these groupoids is r-discrete. The product map $\pi(x,y) = xy$ from G^2 to G is a local homeomorphism since G is r-discrete [**16**]. Therefore, by definition, the restriction of π

to $L^{-1}\times L\cap G^2$ (which is closed in G^2) to H_0 is a local homeomorphism. From this, it follows that H_0 is locally compact and Hausdorff. Also, the map $\phi(x,y) = (x, x^{-1})$ is a local homeomorphism from G^2 to G^2 and maps $L^{-1}\times L\cap G^2$ to itself. Moreover, the diagram,

$$\begin{tikzcd} G^2 \arrow[r, "\phi"] \arrow[d, "\pi"'] & G^2 \arrow[d, "\pi"] \\ G \arrow[r, "r"'] & G \end{tikzcd}$$

commutes and so we conclude that the restriction of r to H_0 is also a local homeomorphism. Similarly, s is a local homeomorphism and so H_0 is a groupoid. The groupoids H_1 and H' are treated similarly. Since L is closed, H is locally compact, and since the map taking x in L to (x,x^{-1}) in $G^2\cap L\times L^{-1}$ is a local homeomorphism, $r: L\to H$ is again a local homeomorphism. Thus, H is a groupoid in the sense of Section 1.

(iii) This is a direct result of the definition of H'.

(iv), (v). These both follow easily from the results and techniques of [**10**] and [**17**]. In fact, one can see that $C_r^*(H_0)$ and $C_r^*(H_1)$ are both full corners in $C_r^*(H)$. $\square$

We represent the algebra $C_c(G)$ on the Hilbert space $l^2(G)$ via the regular representation, which we denote by λ. We regard $l^2(H)$ as a subspace of $l^2(G)$ and define a degenerate representation μ of $C_c(H)$ on $l^2(G)$ by setting μ to be the regular representation on $l^2(H)$ and zero on the orthogonal complement. We also define the self-adjoint unitary operator z on $l^2(G)$ by

$$(z\xi)(x) = \begin{cases} -\xi(x) & \text{if } s(x)\in H_1,\\ \xi(x) & \text{otherwise,}\end{cases}$$

for ξ in $l^2(G)$, x in G. The completions of $\lambda(C_c(G))$ and $\mu(C_c(H))$ are $C_r^*(G)$ and $C_r^*(H)$, respectively.

Lemma 3.9. *Let f be in $C_c(G)$. Define $\tilde f: H\to \mathbf{C}$ by*

$$\tilde f(x) = \begin{cases} 0 & \textit{if } x \textit{ is in } H',\\ -f(x) & \textit{if } x \textit{ is in } L\cup L^{-1}.\end{cases}$$

Then $\tilde{f}$ is in $C_c(H)$ and

$$z\lambda(f)z - \lambda(f) = 2\mu(\tilde{f}).$$

Proof. From the definitions of the topology on H, it is clear that $\tilde{f}$ is in $C_c(H)$. The last part follows by direct computation, which we leave to the reader. $\square$

Lemma 3.10. *Let f be in $C_c(G)$, and let g be in $C_c(H)$. Define $h : H \to \mathbf{C}$ by*

$$h(x) = \sum_{\substack{y \in H \\ s(y)=s(x)}} f(xy^{-1})g(y), \quad x \in H.$$

Then h is in $C_c(H)$ and we have

$$\lambda(f)\mu(g) = \mu(h).$$

Proof. We first observe that h is well-defined; for a fixed x, the set

$$\{y \in H \mid s(y) = s(x), \quad g(y) \neq 0\}$$

is finite. It is clear that the formula above is bilinear in f and g and therefore we may, using a partition of the supports of f and g, restrict to the case $\operatorname{supp}(f) \subseteq K_1$, $\operatorname{supp}(g) \subseteq K_2$ where K_1 and K_2 are compact and $r, s : K_1 \to G^0$ and $f, r : K_2 \to H^0$ are injective. Then we have $h(x) = 0$ unless x is in K_1K_2. We claim that this set is compact in H. Let $\{x_n\}$ and $\{y_n\}$ be sequences in K_1 and K_2, respectively, such that $r(y_n) = s(x_n)$, i.e., x_ny_n is a sequence in K_1K_2. Since K_1 and K_2 are compact, we may pass to convergent subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$. Now x_{n_k} converges to x in G and y_{n_k} converges to y in H (hence also in G). Of course, x is in K_1 and y is in K_2 and $r(y) = s(x)$ so xy is in K_1K_2. Since $s(x_{n_k})$ and $s(x)$ are in H^0, x_{n_k} and x are in H. We apply Lemma 3.1 with $K = r(K_2)$ to assert that x_{n_k} also converges to x in the topology of H. Then $x_{n_k}y_{n_k}$ converges to xy in H. This establishes the claim. Thus, the support of h is compact in H. Since the inclusion of H in G is continuous, and since f and g are continuous, h is also continuous. The final formula is a straightforward computation which we leave to the reader. $\square$

Let $E = C_r^*(H)$ be considered as a trivially graded right $C_r^*(H)$-module. It follows from Lemma 3.10 that the product defined there extends to a $*$-homomorphism $\bar{\lambda} : C_r^*(G) \to \mathcal{B}(E)$. Then Lemmas 3.9 and 3.10 and [**1**, 17.5.2] show that $(E, \bar{\lambda}, z)$ determines an element of the Kasparov group $KK^1(C_r^*(G), C_r^*(H))$, which we denote by $[L]$. This induces homomorphism

$$[L]_* : K_i(C_r^*(G)) \longrightarrow K_{i+1}(C_r^*(H)),$$

see [**1**, 23.1].

To prove Theorem 2.4, we will again appeal to the main result of [**15**]. We let $A = \mu(C_c(H))^- \cong C_r^*(H)$, $B = \lambda(C_c(G))^- \cong C_r^*(G)$, $\mathcal{H} = l^2(G)$ and z be as above. Let χ_n be in $C_c(H)$ as in the factor groupoid situation; let $e_n = \mu(x_n)$ and $e_t = (t-n)_{e_{n+1}} + (n+1-t)e_n$. As before, we let $\mathcal{A} = \mu(C_c(H))$.

Hypotheses (1) and (3) of [**15**] follow from Lemmas 3.10 and 3.9, respectively. For any h in $C_c(H)$, let

$$h' = \begin{cases} h & \text{on } H', \\ -h & \text{on } L \cup L^{-1}. \end{cases}$$

It is easy to see that h' is in $C_c(H)$ and that $z\mu(h)z = \mu(h')$ from which (2) follows. As before, (4) follows easily from the construction of $\{e_t\}$.

Lemma 3.11. *Let h be in $C_c(H)$. There is a positive integer k and g in $C_c(G)$ such that*

(i) $e_l\mu(h) = \mu(h) = \mu(h)e_l$ *for all* $l \geq k$,

(ii) $\lambda(g)e_l = \mu(h)$,

(iii) $[\lambda(g), e_l] = 0$,

(iv) $\lambda(g)(e_l - e_l^2) = 0$,

(v) $[\lambda(g)(1 - e_l), z] = 0$.

Proof. Choose k such that $\chi_k = 1$ on the images of $\operatorname{supp}(h)$ under r and s. Property (i) follows. Let $K = \operatorname{supp}(\chi_{k+3})$, and let

$$X = r^{-1}(K) \cup s^{-1}(K) \cup L \cup L^{-1}.$$

Since K is compact, (in H and hence in G), r and s are continuous and L is closed, X is a closed set in G. Also note that $X \supseteq \operatorname{supp}(h)$. Moreover, by Lemma 3.1, the relative topologies from G and H agree on X. Therefore, we may find g in $C_c(G)$ such that $g \mid X = h \mid X$.

Let $k \leq l \leq k+2$. First we compute

$$\begin{aligned}(\chi_l g)(x) &= \chi_l(r(x))g(x)\\ (g\chi_l)(x) &= \chi_l(s(x))g(x),\end{aligned}$$

for any X in H. (The second is using the product of Lemma 3.9 since χ_l is in $C_c(H)$ and g is in $C_c(G)$. The first is defined analogously.) Note then that $(\chi_l g)(x) = 0$, unless $r(x)$ is in $\operatorname{supp}(\chi_l) \subseteq K$, and hence x is in X. So we have

$$(\chi_l g)(x) = \chi_l(r(x)) \cdot h(x).$$

Again, if this is nonzero, then x is in the support of h, hence $\chi_l(r(x)) = 1$. We have shown that, if $(\chi_l g)(x)$ is nonzero, then it equals $h(x)$. A similar argument shows that, if $(g \cdot \chi_l)(x)$ is nonzero, then it also equals $h(x)$. Moreover, it is clear that, if $h(x)$ is nonzero, it equals $(\chi_l g)(x)$. We conclude that $\chi_l g = g\chi_l = h$. Parts (ii), (iii) and (iv) follow at once. As for (v), we have

$$\begin{aligned}z(\lambda(g)(1-e_l))z - \lambda(g)(1-e_l) &= z\lambda(g)z - \lambda(g) - z\mu(h)z + \mu(h)\\ &= \mu(\tilde{g}) - \mu(h')\\ &= \mu(\tilde{g} - h'),\end{aligned}$$

where $\tilde{g}$ is as in Lemma 3.9 and h' is as above. It is easy to see that $\tilde{g} = h'$, since $g \mid L \cup L^{-1} = h \mid L \cup L^{-1}$. $\square$

Lemma 3.12. *Let h be in $C_c(H)$. There is a positive integer k and a b in $C_r^*(G)$ such that*

(i) $e_l\mu(h) = \mu(h) = \mu(h)e_l$, *for all* $l \geq k$, *and, for all* $k \leq l \leq k+2$,

(ii) $be_l = \mu(h)$

(iii) $[b, e_l] = 0$
(iv) $b(e_l - e_l^2) = 0$
(v) $[b(1 - e_l), z] = 0$
(vi) $\|b\| \leq \|\mu(h)\|$.

Proof. The proof uses Lemma 3.11 and the same technique as in Lemma 3.5. We omit the details. □

Lemma 3.13. *For A and B as above, hypotheses* (5) *and* (6) *of* [**15**] *hold.*

Proof. First, we consider (5). Let b be in B. If $[b, z] = 0$, then let $b' = b$ and we are done. If not, find f in $C_c(G)$ such that

$$\|\lambda(f) - b\| \leq \frac{1}{4}\|[b, z]\|.$$

Let $\tilde{f}$ be as in Lemma 3.9. Letting $h = \tilde{f}$, apply Lemma 3.12 to obtain b_0 in B. Then we have

$$\begin{aligned} z(\lambda(f) + b_0)z - (\lambda(f) + b_0) &= z\lambda(f)z - \lambda(f) + z[b_0, z] \\ &= 2\mu(h) + z[\mu(h), z] \\ &= 0, \end{aligned}$$

since $z\mu(h)z = -\mu(h)$. So then we have $b' = \lambda(f) + b_0$ is in B', and

$$\begin{aligned} \|b - b'\| \leq \|b - \lambda(f)\| + \|b_0\| &\leq \frac{1}{4}\|[b, z\| + \|\mu(h)\| \\ &= \frac{1}{4}\|[b, z]\| + \frac{1}{2}\|[\lambda(f), z]\| \\ &\leq \frac{1}{4}\|[b, z]\| + \frac{1}{2}\|[b, z]\| + \|\lambda(f) - b\| \\ &\leq \|[b, z]\| \end{aligned}$$

using Lemmas 3.12 and 3.9.

Hypothesis (6) follows at once from Lemma 3.12. □

Again, one issue is left to resolve, which is identifying A' and B'.

Proposition 3.14. (i) $A' = \{a \in C_r^*(H) \mid [a, z] = 0\} \cong C_r^*(H')$.
(ii) $B' = \{b \in C_r^*(G) \mid [b, z] = 0\} \cong C_r^*(G')$.

Proof. (i) Consider h in $C_c(H)$. Define

$$\tilde{h}(x) = \begin{cases} 0 & x \text{ in } H' \\ -h(x) & x \text{ in } L. \end{cases}$$

Then, from the definitions of the topologies on H, $\tilde{h}$ is also in $C_c(H)$. Moreover, a straightforward computation shows that $z\mu(h)z - \mu(h) = 2\mu(\tilde{h})$. This immediately gives $\mu(C_c(H')) \subseteq A'$. Next we claim that the restriction of μ to $C_c(H')$ is unitarily equivalent to the direct sum of two copies of the regular representation of H'. For any x in $r(L)$, say, we have

$$\begin{aligned} \{y \in H \mid r(y) = x\} = &\{y \in H' \mid r(y) = x\} \\ &\cup z\{y \in H' \mid r(y) = s(z)\} \end{aligned}$$

where z is chosen as any element of L with $r(z) = x$. If one repeats this for each point x in H^0 (suitably modified for x in $s(L)$), one obtains a decomposition of H into $H' \cup \gamma(H')$, where γ is a map obtained by piecing together the different multiplications by z above. This defines a unitary operator between $l^2(H)$ and $l^2(H') \oplus l^2(H')$. One then checks directly that this intertwines $\mu \mid C_c(H')$ and the direct sum of the regular representation of $C_c(H')$ with itself. Therefore we conclude

$$C_r^*(H') \cong \mu(C_c(H'))^- \subseteq A'.$$

As for the reverse inclusion, suppose a is in $C_r^*(H)$ and $[a, z] = 0$. Then we may find $\{h_n\}$ a sequence in $C_c(H)$ such that $\mu(h_n)$ converges to a. Thus $[\mu(h_n), z]$ tends to zero. Consider $h_n' = h_n + \tilde{h}_n$. We have

$$\begin{aligned} z\mu(h_n')z - \mu(h_n') &= (z\mu(h_n)z - \mu(h_n)) + (z\mu(\tilde{h}_n)z - \mu(\tilde{h}_n)) \\ &= 2\mu(\tilde{h}_n) - \mu(\tilde{h}_n) - \mu(\tilde{h}_n) \\ &= 0, \end{aligned}$$

so h_n' is in $C_c(H')$. Moreover,

$$\|\mu(\tilde{h}_n)\| = \frac{1}{2}\|[\mu(h_n), z]\|,$$

which tends to zero, so $\mu(h'_n)$ converges to a. We conclude that a in $C^*_r(H')$.

(ii) The containment $\supseteq$ follows from Lemma 3.9, and the observation that $\lambda \mid C_c(G')$, is contained in the direct sum of two copies of the regular representation of $C_c(G')$. The argument is similar to that in (i). As for the reverse containment, we may regard B' as a $C_0(G^0)$-bimodule contained in $C^*_r(G)$. Theorem 3.10 [**11**] characterizes such bimodules. It is straightforward to calculate, in the notation of [**11**], $Q(B') = G'$ and hence $C^*_r(G') \cong B'$ by [**11**, 3.10]. □

Proof of Theorem 2.4. This is exactly the same as the proof of Theorem 2.1 except in the computation of $K_*(C(A'; A))$. One again uses the six-term exact sequence as in the proof of Theorem 2.1. Now we note that

$$C^*_r(H') \cong C^*_r(H_0) \oplus C^*_r(H_1).$$

By Lemma 3.8(v) and [**17** (1,2)], the groups, for $i = 0, 1$,

$$K_i(C^*_r(H)), \quad K_i(C^*_r(H_0)), \quad K_i(C^*_r(H_1))$$

may be identified. Under these identifications the map induced by the inclusion A' in A sends (x, y) to $x + y$, for x, y in $K_i(C^*_r(H))$. (Recall that $C^*_r(H_0)$ and $C^*_r(H_1)$ are full corners in $C^*_r(H)$. It is then easy to compute from the six-term exact sequence that

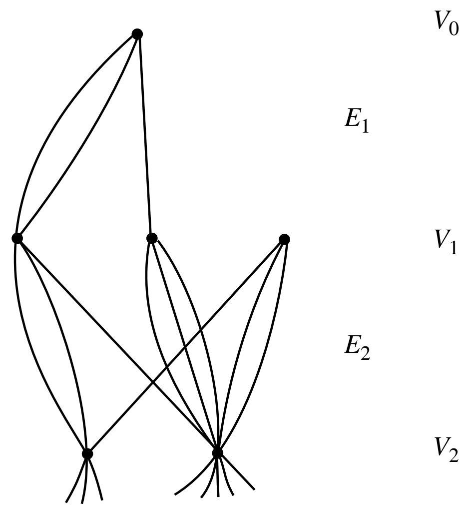
$$K_i(C(A'; A)) \cong K_i(C^*_r(H)).$$

This then yields the exact sequence of Theorem 2.4 except for the matter that the maps $[L]_*$ are as claimed. As in the proof of Theorem 2.1, we leave this calculation to the reader. □

APPENDIX

AF-**groupoids**. A discussion of AF-groupoids (or AF-equivalence relations) appears in [**16**]. This includes a description of their K-theory. We present a version here which is based on the ideal of a Bratteli diagram [**4, 5**].

A Bratteli diagram is an infinite graph consisting of a vertex set V and a set of (directed) edges E. Moreover, both are decomposed as



DIAGRAM

countable unions

$$V = V_0 \cup V_1 \cup \cdots$$
$$E = E_1 \cup E_2 \cup \cdots$$

where each V_n and E_n are finite and nonempty. There are range and source maps

$$r : E_n \longrightarrow V_n, \quad s : E_n \longrightarrow V_{n-1}.$$

We sketch such a diagram vertically as shown above.

We require that there are no "sinks;" i.e., $s^{-1}\{v\}$ is nonempty for all v in V.

Given such a diagram, we define a space X as follows. For each source v in V, $r^{-1}\{v\}$ is empty, say v in V_n, we let

$$X_v = \{(e_{n+1}, e_{n+2}, \dots) \mid s(e_{n+1}) = v,\ s(e_{k+1}) = r(e_k),\ k > n\}$$

which is given the relative topology of the product space $\prod_{k>n} E_k$ and is therefore compact, metrizable and totally disconnected. Now let X

be the disjoint union of the X_v with the inductive limit topology. The equivalence relation on X is tail equivalence or confinal equivalence; two elements $(e_{m+1}, e_{m+2}, \dots)$, $(f_{n+1}, f_{n+2}, \dots)$ are equivalent if, for some N, $e_k = f_k$ for all $k \geq N$. Let us be slightly more precise in order to topologize the relation.

For each $N = 1, 2, \dots$, let

$$G_N = \{((e_{m+1}, e_{m+2}, \dots), (f_{n+1}, f_{n+2}, \dots)) \in X \times X \mid \\ m, n \leq N \text{ and } e_k = f_k \text{ for all } k > N\}.$$

Give G_N the relative topology of $X \times X$. Notice that $G_N \subseteq G_{N+1}$ for all N, and is an open subset. Let G be the union of the G_N, $N = 1, 2, \dots$, and give G the inductive limit topology. Such a G is an AF-groupoid.

The C^*-algebra $C^*(G)$ is an AF-algebra. First of all $K_1(C^*(G)) = 0$. To compute $K_0(C^*(G))$, we let $\mathbf{F}(V_n)$ denote the free abelian group on the nth vertex set. Let

$$\mathbf{F}(V_n)^+ = \Big\{ \sum_{v \in V_n} k_v v \mid k_n \geq 0 \Big\}.$$

For each n, we have a group homomorphism $\alpha_n : \mathbf{F}(V_n) \to \mathbf{F}(V_{n+1})$ given by

$$\alpha_n(v) = \sum_{\substack{e \in E_{n+1} \\ s(e) = v}} r(e).$$

Note that $\alpha_n(\mathbf{F}(V_n)^+) \subseteq \mathbf{F}(V_{n+1})^+$. Then $K_0(C^*(G))$ is the inductive limit in the category of ordered abelian groups of the system

$$\mathbf{F}(V_0) \xrightarrow[\alpha_0]{} \mathbf{F}(V_1) \xrightarrow[\alpha_1]{} \mathbf{F}(V_2) \longrightarrow \cdots .$$

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