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SUFFICIENT CONDITIONS FOR ASYMPTOTIC STABILITY OF LINEAR AUTONOMOUS IMPULSIVE SYSTEMS

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ABSTRACT. This paper studies the asymptotic stability of the linear impulsive system with constant coefficients $x' = Ax$, $x(t_k^+) = Cx(t_k)$. It proves that, if $i(s,t)$, the counting function of impulsive times on the interval (s,t), satisfies $|i(s,t) - p(t-s)| \leq K$, where p and K are positive constants and the matrices A and C are diagonalizable, then there exists a positive number ρ such that this linear system is asymptotically stable if the eigenvalues of the diagonal matrix $\Lambda + p(\rho I + \operatorname{Ln} N)$ have negative real parts, where Λ and N are matrices of eigenvalues of A and C, respectively. For general matrices A and C, an algorithm to find a bound for parameter p is given. These theorems extend the known result of asymptotic stability when A and C commute.

1. Introduction. The theory of differential equations is a formidable tool in the study of the evolution of phenomena in ecology, social behavior, economy, electronics, etc. Frequently, the modelling of these phenomena is accomplished by an ordinary differential equation

$$x' = f(t,x). \tag{1}$$

The qualitative study of this equation allows one to predict the evolution of the observed phenomenon, and from this prediction, the possibility of designing appropriate control policies is obtained. These controls can be of a distinct nature: a massive vaccination of the population, a prohibition of certain fishing practices, an abrupt growth of prices in the market, the administration of specific medicinal drugs, etc. These controls have a common feature: they carry out in a very short period

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of time, so short (compared with the time scale being used) that it is possible to consider that the effect of these controls is instantaneous and causes an abrupt collapse in the evolution of the phenomenon under equation (1); at the time t_k of application of a control, the position $x(t_k)$ of a solution $x(t)$ undergoes a jump and is transferred to a new position $g(t_k, x(t_k))$ from where the observed phenomenon continues its course. Mathematically, these situations can be described by differential equations with impulsive effect

$$\begin{aligned} x' &= f(t,x), \qquad t \neq t_k, \quad t \geq t_0, \\ x(t_k^+) &= g(t_k, x(t_k)), \end{aligned} \tag{2}$$

a new branch of research in differential equations, whose foundations are partially exposed in [**1, 4, 5, 10**].

The present paper concerns the problem of asymptotic stability of a linear impulsive system

$$\begin{aligned} x' &= Ax, \qquad t \neq t_k, \quad t \geq t_0 \\ x(t_k^+) &= Cx(t_k), \end{aligned} \tag{3}$$

where A and C are constant matrices with complex coefficients and the invertibility of the matrix C is assumed

Definition 1. We say that the linear system (3) is *asymptotically stable* if and only if

$$\lim_{t\to\infty} X(t) = 0,$$

where $X(t)$ is the fundamental matrix of system (3), that is,

$$X(t) = e^{A(t-t_k)}Ce^{A(t_k-t_{k-1})}\cdots e^{A(t_2-t_1)}Ce^{A(t_1-t_0)}, \quad \text{if } t \in [t_0, t_{k+1}]. \tag{4}$$

Since the solutions of Equation (3) define a finite dimensional vector space, then (4) implies the stability of the solutions of this equation [**1**].

For the ordinary differential equation

$$x' = Ax, \quad A = \text{constant}, \tag{5}$$

the question of asymptotic stability has been completely solved. A similar situation we find in the theory of a linear system of difference equation

$$x(n+1) = Cx(n). \tag{6}$$

This is not the case of the impulsive system (3), for which this problem has interesting features; it is known that the ordinary stability of matrix A (we mean that all eigenvalues of matrix A have negative real parts) and the discrete stability of matrix C (all eigenvalues of matrix C are contained in the disk $|z| < 1$) are not enough to assure the asymptotic stability of (3); in this problem the location of the impulsive times $\{t_k\}_{k=0}^{\infty}$ will play an important role [**8**]. An algebraic criterion involving the eigenvalues of matrices A and C and the sequence of impulsive times $\{t_k\}_{k=0}^{\infty}$, giving necessary and sufficient conditions for the asymptotic stability of system (3), has not been obtained yet. Maybe the closest results to this aim are given by Theorem 4.4 in [**1**], see also [**10**], and [**8**, Theorem 2]. Let us recall these results. We make precise some previous terminology. The sequence of impulsive times $\{t_k\}_{k=0}^{\infty}$ (we purposely define the first number of this sequence as t_0) is assumed to be strictly increasing; let $i(s,t)$ denote the number of impulses in the interval (s,t). We assume that this function satisfies the following condition

Hypothesis (C). *There exist positive numbers p and K such that*

$$|i(s,t) - p(t-s)| \le K, \quad s < t.$$

We will assume that the solutions of Equation (3) are C^1 functions on each interval $(t_k, t_{k+1}]$, are left continuous at the impulse times t_k with righthand side limit at each point t_k denoted by $x(t_k^+)$.

Theorem A. *Let us assume that matrices A and C commute. Then, under Hypothesis* (C)*, a necessary and sufficient condition for the asymptotic stability of system* (3) *is that* $\operatorname{Re}\lambda < 0$ *where λ is any eigenvalue of*

$$\Lambda = A + p\operatorname{Ln} C. \tag{7}$$

The condition of commutativity of this theorem restricts the class of impulsive systems that can be considered in applications. In [**8**], a fair generalization of this theorem is given by considering that matrices A and C can be reduced simultaneously to an upper triangular form (commuting matrices certainly satisfy this condition):

Theorem B. *If matrices A and C can simultaneously be reduced to an upper triangular form, then, under condition* (C)*, a necessary and sufficient condition for the asymptotic stability of system* (3) *is that* $\operatorname{Re}\lambda < 0$*, where λ is any eigenvalue of* (7)*.*

In this paper we consider the problem for general matrices. Our results show that the asymptotic stability of system (3) cannot be established by the eigenvalues of matrix (7). If matrices A and C can be reduced to diagonal matrices Λ and N, respectively, by nonsingular matrices S and T, then we show that system (3) is asymptotically stable if the real parts of the eigenvalues of the matrix

$$\Lambda + p(2|\operatorname{Ln}(S^{-1}T)|I + \operatorname{Ln} N)$$

are negative. A similar result is true for general matrices A and C.

The present work relies on the method of quasidiagonalization; in a few words, it consists of reducing a matrix to an almost diagonal form (we refer to [**1**, Corollary of Theorem 6]). The development of this method started with [**9**] in connection with the problem of existence for periodic solutions of ordinary systems with a small parameter in the derivative; later it was applied in [**6**] to the problem of existence of exponential dichotomies for linear singularly perturbed systems. Recently the method of quasidiagonalization has been adapted for the nonautonomous linear systems [**7**],

$$\begin{aligned} x' &= A(t)x, \\ x(t_k^+) &= C_k x(t_k). \end{aligned}$$

That paper does not include the method of quasidiagonalization for the impulsive system with constant coefficients, an important particular case where simpler and more precise estimates and conclusions can be obtained; in particular, [**7**] does not study the asymptotic stability of the nonautonomous impulsive system. In this paper we show that this method can successfully be applied to the problem of asymptotic stability for linear impulsive system (3).

2. Notation and preliminaries. Before we go ahead, we introduce some notation. In this paper V denotes the n-dimensional vector space $\mathbf{C}^n$ with norm $|x| = \max\{|x_j| : 1 \le j \le n\}$. For an $n \times n$ matrix A, $|A|$ is the corresponding matrix norm. J denotes the interval $[t_0, \infty)$ and N is the set of integers $\{0, 1, 2, \dots\}$. For a function $x : J \to V$ we denote $|x|_\infty = \sup\{|x(t)|; t \in J\}$. The function $[t]$ is established on the set of real numbers $\mathbf{R}$ by $[t] = k$ if $t \in [k, k+1)$ where k is an integer.

For an $n \times n$ matrix A, there exists a nonsingular matrix L such that $L^{-1}AL$ has a Jordan form. This means that

$$L^{-1}AL = \operatorname{diag}\{A_1, A_2, \dots, A_m\},$$

where each matrix A_j is a Jordan block of dimensions $n_j \times n_j$. Let us define $\alpha(A) = \max\{n_j : 1 \le j \le m\}$. Moreover, for each matrix A_j we define the $n_j \times n_j$ matrix $S(n_j, \sigma) = \operatorname{diag}\{1, \sigma, \dots, \sigma^{n_j - 1}\}$. Finally, let

$$S(A, \sigma) = \operatorname{diag}\{S(n_1, \sigma), S(n_2, \sigma), \dots, S(n_m, \sigma)\}.$$

We emphasize the following norm formulas and notations

$$p(A, \sigma) := |S(A, \sigma)| = \max\{1, \sigma^{\alpha(A)-1}\}, \tag{8}$$

$$q(A, \sigma) := |S^{-1}(A, \sigma)| = \max\{1, \sigma^{-\alpha(A)+1}\}. \tag{9}$$

It is clear that

$$(LS(A, \sigma))^{-1} ALS(A, \sigma) = \Lambda + G(A, \sigma), \tag{10}$$

where $|G(A, \sigma)| = r(A, \sigma)$ and $r(A, \sigma)$ is defined as

$$r(A, \sigma) = \begin{cases} \sigma & \text{if } \alpha(A) > 1, \\ 0 & \text{if } \alpha(A) = 1. \end{cases}$$

The following assertion is a known result [**3**].

Lemma 1. *There exist nonsingular matrices L and M such that*

$$(LS(A,\sigma))^{-1}ALS(A,\sigma) = \Lambda + G(A,\sigma), \tag{11}$$

$$(MS(C,\sigma))^{-1}CMS(C,\sigma) = N + G(C,\sigma), \tag{12}$$

where Λ is a diagonal matrix of the eigenvalues of A and N is a diagonal matrix of the eigenvalues of C.

In the statement of Lemma 1, an important fact is omitted, namely, in what order are the eigenvalues of matrices A and C along the main diagonal of Λ and N written, respectively. This question may become crucial in the analysis of asymptotic stability of system (3). We discuss this situation in Example 2 of our paper.

3. Quasidiagonalization. In what follows, we denote $I = [0,\infty)$. Let us define the function $a : I \to J$ by

$$a(t) = t_k + (t-k)(t_{k+1} - t_k), \quad t \in [k, k+1], \quad k \in N.$$

It is easy to see that this function is continuous on I and is C^1 on each interval $(k, k+1)$ where $a'(t) = t_{k+1} - t_k$. Under condition (C) we obtain that the distance between impulsive times is less than $p^{-1}(K+1)$. Therefore, for a' we obtain the estimate

$$0 < a'(t) \le p^{-1}(K+1), \quad \forall\, t \in I \backslash \{N\}. \tag{13}$$

Since $a(t)$ has a continuous inverse, for $x(t)$, a solution of Equation (3), the function $y(t) = x(a(t))$ satisfies

$$\begin{aligned} y' &= a'(t)Ay, \qquad t \ne k, \quad t \ge 0, \\ y(k^+) &= Cy(k). \end{aligned} \tag{14}$$

Let us consider the C^1-function

$$\delta(t) = \begin{cases} 0 & |t| \ge 1, \\ (1+t)^2 & t \in (-1,0), \\ (1-t)^2 & t \in [0,1). \end{cases}$$

For $\sigma > 0$, according to Lemma 1, there exist nonsingular matrices L and M, reducing matrices A and C to the form (11) and (12), respectively. Let us define on the interval I the function

$$\psi_k(t,\sigma) = \delta(2^k(t-k))\mathrm{Ln}\,((LS(A,\sigma))^{-1}MS(C,\sigma)).$$

Outside of the interval $I_k = [k-2^{-k}, k+2^{-k}]$ this function equal zero. Let us consider the C^1 matrix function

$$T(t,\sigma) = LS(A,\sigma)\exp\left\{\sum_{k=1}^{\infty}\psi_k(t,\sigma)\right\}.$$

Using (8) and (9), we easily obtain that $T(t,\sigma)$ and $T^{-1}(t,\sigma)$ are bounded on I:

$$\begin{aligned} |T(\cdot,\sigma)|_\infty &\le p(A,\sigma)|L|\exp\{|\mathrm{Ln}\,((LS(A,\sigma))^{-1}MS(C,\sigma)|\}, \\ |T^{-1}(\cdot,\sigma)|_\infty &\le q(A,\sigma)|L^{-1}|\exp\{|\mathrm{Ln}\,((LS(A,\sigma))^{-1}MS(C,\sigma))|\}. \end{aligned} \tag{15}$$

The change of variable $y = T(t,\sigma)z$ in (14) reduces this equation to

$$\begin{aligned} z' &= (a'(t)T^{-1}(t,\sigma)AT(t,\sigma) \\ &\quad - T^{-1}(t,\sigma)T'(t,\sigma))z, \qquad t \ne k, \\ z(k^+) &= T^{-1}(k,\sigma)CT(k,\sigma)z(k). \end{aligned} \tag{16}$$

From the definition of $T(\cdot,\sigma)$, we obtain

$$\begin{aligned} T^{-1}(t,\sigma)T'(t,\sigma) = \sum_{k=1}^{\infty} & 2^k\delta'(2^k(t-k)) \\ & \cdot \mathrm{Ln}\,((LS(A,\sigma))^{-1}MS(C,\sigma)). \end{aligned} \tag{17}$$

Lemma 2.

$$\int_0^t |T^{-1}(s,\sigma)T'(s,\sigma)|\,ds \le 2[t+1]|\mathrm{Ln}\,((LS(A,\sigma))^{-1}MS(C,\sigma))|.$$

Proof. From (8) and (17), we obtain

$$\int_0^t |T^{-1}(s,\sigma)T'(s,\sigma)|\,ds \le |\mathrm{Ln}\,((LS(A,\sigma))^{-1}MS(C,\sigma))| \cdot \sum_{k=1}^{[t+1]} \int_{I_k} |2^k\delta'(2^k(s-t_k))|\,ds.$$

The symmetrical definition of function δ implies

$$\int_{I_k} |2^k\delta'(2^k(s-t_k))|\,ds = 2.$$

Now the result sought easily follows. □

System (16) can be written in the form

$$\begin{aligned} z' &= a'(t)(\Lambda + G(A,\sigma) + F(t,\sigma))z \\ &\qquad - T^{-1}(t,\sigma)T'(t,\sigma)z, \qquad t \ne k, \\ z(k^+) &= (N + G(C,\sigma))z(k), \end{aligned} \tag{18}$$

where

$$F(t,\sigma) = T^{-1}(t,\sigma)AT(t,\sigma) - \Lambda - G(A,\sigma).$$

We emphasize that $F(\cdot,\sigma)$ is zero outside $\cup_{k=1}^{\infty} I_k$; therefore, we obtain

Lemma 3. *The function $F(t,\sigma)$ has the following integral estimate*:

$$\int_0^\infty |F(s,\sigma)|\,ds \le \mu(\sigma), \tag{19}$$

where $\mu(\sigma)$ is a constant.

Proof. Using (15), it follows from the definition of the function $F(\cdot,\sigma)$ and the interval I_k that

$$\begin{aligned} \int_0^\infty |F(s,\sigma)|\,ds &\le \sum_{k=1}^{\infty} \int_{I_k} |T^{-1}(s,\sigma)AT(s,\sigma) - \Lambda - G(A,\sigma)|\,ds \\ &\le \sum_{k=1}^{\infty} \int_{I_k} (p(A,\sigma)q(A,\sigma)|L||L^{-1}| \\ &\qquad \cdot |\mathrm{Ln}\,(L^{-1}M)|^2|A| + |\Lambda| + r(A,\sigma))\,ds \\ &= \mu(\sigma), \end{aligned}$$

where

$$\mu(\sigma) = 2p(A,\sigma)q(A,\sigma)|L||L^{-1}||\mathrm{Ln}\,(L^{-1}M)|^2|A| + |\Lambda| + r(A,\sigma),$$

(fortunately this complicated constant will not play an essential role). □

4. Asymptotic stability. We will consider that system (18) is a perturbation of system

$$\begin{aligned} w' &= a'(t)\Lambda w, \qquad t \neq k, \quad t \geq 0 \\ w(k^+) &= Nw(k), \end{aligned} \tag{20}$$

whose fundamental matrix is

$$W(t) = \exp\{(a(t) - a(0))\Lambda + j(0,t)\mathrm{Ln}\, N\},$$

where $j(0,t)$ is the counting function on the interval $(0,t)$ of the impulsive times of system (20) (we point out that, for a positive integer k, the formula $j(0,k) = k-1$ is valid); the branch of function Ln is appropriately defined according to the location of eigenvalues of C. From the variation of constants formula [**1**], we obtain that the solutions of (18) satisfy the integral equation

$$\begin{aligned} z(t) = W(t)z_0 + \int_0^t & W(t)W^{-1}(s)(a'(s)G(A,\sigma) \\ & + a'(s)F(s,\sigma) - T^{-1}(s)T'(s))z(s)\,ds \\ + \sum_{k\in(0,t)} & W(t)W^{-1}(k^+)G(C,\sigma)z(k^+). \end{aligned} \tag{21}$$

Let

$$\nu(t) = \max\{\mathrm{Re}\,((a(t)-a(0))\lambda_j + [t]\ln\mu_j) : 1 \leq j \leq n\},$$

and

$$\theta(t) = |\mathrm{e}^{-\nu(t)}z(t)|.$$

From (21) and the estimate $j(0,t) \leq [t]$, we obtain

$$\begin{aligned} \theta(t) \leq |z_0| + \int_0^t & (a'(s)|G(A,\sigma)| \\ & + a'(s)|F(s,\sigma)| + |T^{-1}(s,\sigma)T'(s,\sigma)|)|\theta(s)|\,ds \\ + \sum_{k\in(0,t)} & |G(C,\sigma)|\theta(k^+). \end{aligned}$$

The Gronwall inequality for piecewise continuous functions [**2**] and (21) imply

$$\theta(t) \le |z_0|(1+r(C,\sigma))^{[t]} \exp\left\{ \int_0^t [a'(s)(|G(A,\sigma)| + |F(s,\sigma)|) + |T^{-1}T'|]\, ds \right\}.$$

Applying the estimate given by Lemma 3, we obtain

$$\theta(t) \le K|z_0|(1+r(C,\sigma))^{[t]} \exp\left\{ \int_0^t (a'(s)r(A,\sigma) + |T^{-1}(s,\sigma)T'(s,\sigma)|)\, ds \right\},$$

where $K = \exp\{(K+1)\mu(\sigma)p^{-1}\}$. Lemma 2 allows us to write the inequality

$$\theta(t) \le K|z_0|(1+r(C,\sigma))^{[t]} \exp\{r(A,\sigma)(a(t)-a(0)) + 2[t+1]\mathrm{Ln}\,((LS(A,\sigma))^{-1}MS(C,\sigma))|\}.$$

Modifying the definition of constant K, we can write

$$\theta(t) \le K|z_0|(1+r(C,\sigma))^{[t]} \exp\{r(A,\sigma)(a(t)-a(0)) + 2[t]\mathrm{Ln}\,((LS(A,\sigma))^{-1}MS(C,\sigma))|\}.$$

The definition of function $\theta(t)$ implies

$$|z(t)| \le K|z_0| \exp\{\nu(t) + r(A,\sigma)(a(t)-a(0)) + [t]\ln(1+r(C,\sigma)) + 2[t]\mathrm{Ln}\,((LS(A,\sigma))^{-1}MS(C,\sigma))|\}.$$

Finally, from the definition of $\nu(t)$, we obtain

$$|z(t)| \le K|z_0| \exp\{g(t,\sigma)\} \tag{22}$$

where the function $g(t,\sigma)$ is defined by

$$\begin{aligned} g(t,\sigma) = \max\{\mathrm{Re}\,((a(t)-a(0))(\lambda_j + r(A,\sigma)) &\\ + [t](2|\mathrm{Ln}\,((LS(A,\sigma))^{-1}MS(C,\sigma))| &\\ + \ln(1+r(C,\sigma)) + \ln\mu_j)) : 1 \le j \le n\}. & \end{aligned} \tag{23}$$

Now the problem of the asymptotic stability of system (3) is equivalent to the requirement

$$\lim_{t\to\infty} g(t,\sigma) = -\infty, \tag{24}$$

for some value of σ. Let us define the number

$$\begin{aligned}\lambda = \max\{&\mathrm{Re}\,(\lambda_j + r(A,\sigma)\\ &+ p(2|\mathrm{Ln}\,((LS(A,\sigma))^{-1}MS(C,\sigma))|\\ &+ \ln(1+r(C,\sigma)) + \ln\mu_j)) : 1 \le j \le n\}.\end{aligned} \tag{25}$$

Theorem 1. *Let ε be a positive number. Then, under condition* (C)*, if L and M satisfy* (11) *and* (12)*, then the fundamental matrix $X(t)$ of system* (3) *has the exponential estimate*

$$|X(t)| \le Ke^{(\lambda+\varepsilon)(t-t_0)}, \qquad t \ge t_0 \tag{26}$$

where the constant $K = K(\varepsilon)$ does not depend on t_0.

Proof. From the identity

$$[t] = i(0, a(t)),$$

condition (C) gives

$$|[t] - pa(t)| \le K,$$

implying

$$[t] = (p + o(1))(a(t) - a(0)),$$

where $o(1)$ denotes a function satisfying $\lim_{t\to\infty} o(1) = 0$. Therefore,

$$\begin{aligned}g(t,\sigma) = (a(t)-a(0))\max\{&\mathrm{Re}\,(\lambda_j + r(A,\sigma)\\ &+ (p+o(1))(2|\mathrm{Ln}\,((LS(A,\sigma))^{-1}MS(C,\sigma))|\\ &+ \ln(1+r(C,\sigma)) + \ln\mu_j)); 1 \le j \le n\}\\ = (a(t)-a(0))&(\lambda + o(1)).\end{aligned}$$

This last identity implies the estimate

$$|z(t)| \le K|z_0| \exp^{(\lambda+\varepsilon)(a(t)-a(0))},$$

where the constant $K = K(\varepsilon)$ is independent of t_0. Since solutions $z(t)$ of system (16) and solutions $y(t)$ of system (14) are related by the formula $y = T(t,\sigma)z$, then (15) implies for $Y(t)$, the fundamental matrix of (14) (with a new constant K) the estimate (with some new constant K)

$$|Y(t)| \le K(\varepsilon)|y_0| \exp^{(\lambda+\varepsilon)(a(t)-a(0))}.$$

The identity $Y(t) = X(a(t))$ and $a(0) = t_0$ implies (26). $\square$

From (26), follows

Theorem 2. *Let condition* (C)*,* (11) *and* (12) *be satisfied; if* $\lambda < 0$*, then system* (3) *is asymptotically stable.*

Example 1. Let us consider (3), where matrices A and C are defined by

$$A = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.$$

In this case we have

$$L = I, \qquad M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S(A,\sigma) = S(C,\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix};$$

therefore,

$$S^{-1}(A,\sigma)\mathrm{Ln}\,(L^{-1}M)S(C,\sigma) = 2^{-1}\begin{pmatrix} 1 & -\sigma \\ -\sigma^{-1} & 1 \end{pmatrix},$$

implying

$$|S^{-1}(\sigma)\mathrm{Ln}\,(L^{-1}M)S(\sigma)| = 2^{-1}\pi \max\{1+\sigma, 1+\sigma^{-1}\}.$$

The number λ defined by (25) is

$$\lambda = -3 + \sigma + p\{\pi \max\{1+\sigma, 1+\sigma^{-1}\} + \ln(1+\sigma) + \ln 2\}.$$

For any σ contained in the interval $(0,3)$, from the condition $\lambda < 0$, we obtain a bound for the parameter p; by a computational calculation, a larger bound for p, given by the method of quasidiagonalization with this particular choice of matrices L and M, is $p = 0.2639\ldots$ obtained for $\sigma = 0.8516\ldots$; the system considered in the present example is asymptotically stable for any $p \in (0, 0.2639\ldots)$.

5. Matrices A and C are diagonalizable. An important case in applications occurs if matrices A and C can be reduced to a diagonal form

$$L^{-1}AL = \Lambda, \quad M^{-1}CM = N. \tag{27}$$

Under this condition $r(A,\sigma) = r(C,\sigma) = 0$ and $S(A,\sigma) = S(C,\sigma) = I$; the respective calculation of number λ in (25) yields the following

Theorem 3. *Under condition* (C) *and* (27), *the system* (3) *is asymptotically stable if the real parts of the eigenvalues of matrix*

$$\Lambda + p(2|\mathrm{Ln}\,(L^{-1}M)|I + \mathrm{Ln}\,N), \tag{28}$$

are negative.

If $M = L$, then $\mathrm{Ln}\,(L^{-1}M) = 0$; henceforth, for simultaneously diagonalizable matrices, we infer

Corollary 1. *If* (27) *holds with $L = M$, then system* (3) *is asymptotically stable if the real parts of the eigenvalues of matrix* (7) *are negative.*

This corollary is also a particular case of Theorem B.

Example 2. Let us consider (3), where matrices A and C are defined by

$$A = \begin{pmatrix} -2 & 4 \\ 3 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad 0 < a < 1.$$

Matrix A has the eigenvalues $\lambda_1 = -4$ and $\lambda_2 = 4$. In this example the matrix A can be reduced to a diagonal form $\Lambda = \mathrm{diag}\,\{-4, 4\}$ by means of

$$L = \begin{pmatrix} 2 & 2 \\ -1 & 3 \end{pmatrix}.$$

We first assume that $M = I$. Let $\rho = 2|\mathrm{Ln}\, L| > 0$. The eigenvalues of the matrix (28) are $\lambda_1 = -4 + p\rho$ and $\lambda_2 = 4 + p(\rho + \ln a)$. If $a < e^{-2\rho}$ the condition on p

$$-\frac{4}{\rho + \ln a} < p < \frac{4}{\rho}$$

can be satisfied. Under this condition both eigenvalues λ_1 and λ_2 are negative and the asymptotic stability for system (3) is obtained. But the relevance of this example consists of the following remark. Let us calculate $\mu_\pm$ the eigenvalues of matrix (7). Denoting $\alpha = \ln a$, we obtain

$$\mu_\pm = \frac{p\alpha \pm \sqrt{(p\alpha)^2 + 4(16 + 2p\alpha)}}{2}.$$

We observe that, if $p < -8/\alpha$, then one of the eigenvalues is positive. But $a < e^{-2\rho}$ implies

$$-\frac{4}{\rho + \ln a} < -\frac{8}{\ln a} < \frac{4}{\rho}.$$

Thus, for $p \in (-4/(\rho + \ln a), -8/\ln a)$, system (3) is asymptotically stable, but this cannot be inferred from the eigenvalues of the matrix (7), in other words, the negativity of the eigenvalues of the matrix (7) is not a necessary condition for the asymptotic stability of system (3).

Let us assume now that matrices A and C are reduced to the diagonal forms $\Lambda = \mathrm{diag}\,\{-4, 4\}$ and $N = \mathrm{diag}\,\{a, 1\}$ by means of matrices L and

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Under these circumstances, the eigenvalues of matrix (28) are

$$\lambda_1 = -4 + p(2|\ln(L^{-1}M)|) + \ln a),$$
$$\lambda_2 = 4 + 2p|\mathrm{Ln}\,(L^{-1}M)| > 0.$$

The eigenvalue λ_2 implies that the requirements of Theorem 2 cannot be accomplished. Thus, in Theorems 1 and 2, it is relevant the order we write the eigenvalues of A and C along the main diagonal of matrices Λ and N.

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