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AN INTRODUCTION TO ZARISKI SPACES OVER ZARISKI TOPOLOGIES

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ABSTRACT. Given a topology Ω on a set X, we consider a structure (Y, Γ) such that the relationship between (Y, Γ) and (X, Ω) is similar to the relationship between a module and its ring of scalars. Indeed, this structure is a module analogue of the Zariski topology on the prime spectrum of a ring R in that its construction uses the prime submodules of an R-module M in essentially the same way that the construction of the Zariski topology uses the prime ideals of R. It is shown that an R-module homomorphism f between two R-modules induces in a natural way a homomorphism between their associated structures, and in case f is an epimorphism, the induced homomorphism is continuous in nontrivial cases.

1. Zariski spaces. Throughout this paper R denotes a commutative ring with identity and M a unital R-module. If I is an ideal of R, we write $I \lhd R$, and $A \leq M$ means that A is a submodule of M. If $A \leq M$, then $(A : M)$ represents the ideal $\{r \in R : rM \subseteq A\}$.

A submodule P of M is called *prime* if P is proper, and whenever $rm \in P$, $r \in R$ and $m \in M$, then $m \in P$ or $r \in (P : M)$. The collection of all prime submodules of M is denoted by $\operatorname{spec} M$. If A is a submodule of M, then the *radical* of A, denoted $\operatorname{rad} A$, is the intersection of all prime submodules of M which contain A, unless no such primes exist, in which case $\operatorname{rad} A = M$. In fact, there exist modules M with no prime submodules at all, though any such module M could not be finitely generated. Such modules are called *primeless*. Studies of prime submodules can be found in [**1, 3, 5**] and [**7–12**], among others. In particular, one can find the following, easily proven but useful, result in [**5**] or [**7**].

Lemma 1. *Let P be a* (*proper*) *submodule of M. Then P is prime in M if and only if $(P : M)$ is prime in R and M/P is a torsion-free $R/(P : M)$-module.*

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As prime submodules are the analogue of prime ideals, an obvious question is whether there is a topology on $\operatorname{spec} M$ similar to the Zariski topology on $\operatorname{spec} R$. For a study on $\operatorname{spec} R$, see [**6**], for example.

Definition. Let A be any subset of the R-module M. The *variety* of A is $V(A) = \{P \in \operatorname{spec} M : A \subseteq P\}$ and $\zeta(M)$ denotes the set of all varieties of subsets of M, i.e., $\zeta(M) = \{V(A) : A \subseteq M\}$. Similarly, $\zeta(R)$ denotes the collection of all varieties of subsets of R, i.e., the closed sets of the Zariski topology on $\operatorname{spec} R$.

Clearly $V(A) = V(\operatorname{rad} RA)$ and $B \subseteq A$ implies $V(B) \supseteq V(A)$, similar to the case for $\zeta(R)$. In addition, for any collection of submodules N_λ, $\lambda \in \Lambda$, of M, $\cap_{\lambda\in\Lambda} V(N_\lambda) = V(\sum_{\lambda\in\Lambda} N_\lambda)$. However, it turns out that $(\operatorname{spec} M, \zeta(M))$ is not a topological space in general. The problem lies in the fact that a prime submodule can contain an intersection of two submodules without containing either submodule. This usually prevents the union of two varieties from being another variety. In fact, if M is finitely generated, then $\zeta(M)$ is a topology if and only if M is a multiplication module. (For this fact and a further study of $\zeta(M)$ as a topology, see [**11**].) What can be said then about the structure of $\zeta(M)$? Before answering this question we turn our attention briefly to semi-rings and semi-modules.

Let (X, Ω) be a topological space, where for our purposes topology means the collection of closed sets. If one essentially ignores the set X, then the set Ω is seen to be a commutative semi-ring under the operations intersection and union, which we will think of as addition and multiplication, respectively. Clearly we have $0_\Omega = X$ and $1_\Omega = \varnothing$.

Definition. Let (X, Ω) be a topological space, and let Γ be a collection of subsets of a (possibly empty) set Y such that $Y \in \Gamma$ and Γ is closed with respect to finite intersections. Further suppose that there exists a mapping $* : \Omega \times \Gamma \to \Gamma$ such that $(\Gamma, \cap)$ is an Ω-semi-module. That is to say, for all $\tau, \tau' \in \Omega$ and for all $\gamma, \gamma' \in \Gamma$, the following properties hold:

(i) $\tau * (\gamma \cap \gamma') = (\tau * \gamma) \cap (\tau * \gamma')$;

(ii) $(\tau \cap \tau') * \gamma = (\tau * \gamma) \cap (\tau' * \gamma)$;

(iii) $(\tau \cup \tau') * \gamma = \tau * (\tau' * \gamma)$;

(iv) $\varnothing * \gamma = \gamma$;

(v) $\tau * Y = Y = X * \gamma$.

Then (Y, Γ) is called an Ω-*space.*

The reader might note that objects of a somewhat more general nature were studied by Fofanova [**2**] and others, under the name polygons.

Theorem 2. *Let M be an R-module, and let the $\zeta(R)$-action on $\zeta(M)$ be given by $V(I) * V(A) = V(RIA)$, where $I \subseteq R$ and $A \subseteq M$. Then* $(\operatorname{spec} M, \zeta(M))$ *is a $\zeta(R)$-space.*

Proof. It is clear that $(\zeta(M), \cap)$ is a commutative monoid with identity $\operatorname{spec} M = V(0)$. Now suppose that $V(I) = V(J)$, $I, J \subseteq R$, and suppose $V(A) = V(B)$, $A, B \subseteq M$. We must show that $V(RIA) = V(RJB)$. If $P \in V(RIA)$, then either $P \supseteq A$, thus $P \supseteq RJB$, or $(P : M) \supseteq RI$, and thus $P \supseteq (P : M)B \supseteq RJB$, by Lemma 1. By symmetry we have $V(RIA) = V(RJB)$. The rest of the proof is now a routine check, particularly so since we may without loss of generality assume that $I, J \lhd R$ and $A, B \leq M$. For example, we have $V(I) * (V(J) * V(A)) = V(I) * V(JA) = V(I(JA)) = V(IJ) * V(A) = (V(I) \cup V(J)) * V(A)$. The other properties follow similarly. $\square$

We call $\zeta(M)$ the *Zariski space* on M.

2. Continuous mappings and homomorphisms. Let (X, Ω) be a topological space, and let (Y, Γ) and (Y', Γ') be Ω-spaces. The motivation for the following terminology is perfectly obvious. A mapping $\vartheta : Y \to Y'$ is said to be *continuous* if $\vartheta^{-1}(\gamma') \in \Gamma$ for every $\gamma' \in \Gamma'$. A mapping $\varphi : \Gamma \to \Gamma'$ is said to be an Ω-*homomorphism* if, for every $\tau \in \Omega$ and all $\gamma, \delta \in \Gamma$, $\varphi(\gamma \cap \delta) = \varphi(\gamma) \cap \varphi(\delta)$ and $\varphi(\tau * \gamma) = \tau * \varphi(\gamma)$, that is, φ is a homomorphism of semi-modules. Now, given a continuous map $\vartheta : Y \to Y'$, we observe that ϑ^{-1} determines a map $\hat{\vartheta} : \Gamma' \to \Gamma$, given by $\hat{\vartheta}(\gamma') = \vartheta^{-1}(\gamma')$. In the event $\hat{\vartheta}$ is an Ω-homomorphism, we say that ϑ is a *continuous homomorphism.*

Consider now an R-module homomorphism $f : M \to N$. The main result of this section says that if f is an epimorphism, and N is not primeless, then there exists a continuous homomorphism $\vartheta : \operatorname{spec} N \to \operatorname{spec} M$ such that, for all $P \in \operatorname{spec}\ N$, $\vartheta(P) = f^{-1}(P)$. We will actually go a bit further. If f is not surjective, then there is no longer necessarily a corresponding function ϑ. Even so, we will define a relation φ between $\zeta(M)$ and $\zeta(N)$ such that, whenever f is surjective, then $\varphi(V(A)) = \vartheta^{-1}(V(A))$ for all $V(A) \in \zeta(M)$. In other words, $\varphi = \hat{\vartheta}$. It is perhaps surprising that this relation φ turns out to be a function, indeed a $\zeta(R)$-homomorphism, even if f is not surjective.

The proof of the following lemma is an easy exercise, and as such is left to the reader.

Lemma 3. *If M is an R-module and P is prime in M, then for any submodule B of M, either $B \subseteq P$ or $P \cap B$ is prime in B.*

The next result appears in [**9**, Results 1.1 and 1.2].

Lemma 4. *Let $f : M \to N$ be an R-module epimorphism. Then there exists a bijection between* $\operatorname{spec} N$ *and the set of all prime submodules of M containing* $\ker f$*.*

Corollary 5. *Let $f : M \to N$ be an R-module epimorphism, and let A be a submodule of M such that* $\ker f \subseteq A$*. Then $V(A) \to V(f(A))$, given by $P \to f(P)$ is a bijection, unless $V(A)$ is the empty set, in which case so is $V(f(A))$.*

Theorem 6. *Let $f : M \to N$ be an R-module homomorphism. Define $\varphi : \zeta(M) \to \zeta(N)$ by $\varphi(V(A)) = V(f(A))$. Then φ is a $\zeta(R)$-homomorphism.*

Proof. To see that φ is well-defined, suppose that $V(A) = V(B)$ for some $A, B \subseteq M$, and let $P \in V(f(A))$. If $P \supseteq f(M)$, then $P \supseteq f(B)$ and so $P \in V(f(B))$. On the other hand, if $P \not\supseteq f(M)$, then $P \cap f(M)$ is a prime submodule of $f(M)$ by Lemma 3, so that by Corollary 5 we have $f^{-1}(P \cap f(M)) \in V(A + \ker f) \subseteq V(A) = V(B)$. Thus $P \cap f(M) \supseteq f(B)$, which clearly implies $P \in V(f(B))$. This argument is symmetrical, hence we have $V(f(A)) = V(f(B))$. The two homomorphism properties of φ follow easily from the homomorphism properties of f and from the operations defined on the varieties. $\square$

We also note that the above theorem holds even in the case when either M or N is primeless.

Corollary 7. *Let $f : M \to N$ be an R-module epimorphism, such that N is not primeless. Define $\vartheta : \operatorname{spec} N \to \operatorname{spec} M$ by $\vartheta(P) = f^{-1}(P)$. Then ϑ is a continuous homomorphism.*

Proof. That ϑ is well-defined follows from Corollary 5. It likewise follows that, for any $V(A) \in \zeta(M)$, $\vartheta^{-1}(V(A)) = V(f(A + \ker f)) = V(f(A)) = \varphi(V(A))$, using the notation of Theorem 6, and the proof is complete. $\square$

In the remainder of this section, we compare and contrast some of the properties of the functions f and φ given in Theorem 6.

Lemma 8. *Given the functions f and φ as described in Theorem* 6, *the following hold*:

(i) *if f is surjective, then φ is surjective*;

(ii) *if φ is surjective and N is finitely generated, then f is surjective*;

(iii) *if f is bijective, then φ is bijective.*

Proof. (i) If f is surjective, then for any $V(B) \in \zeta(N)$, B a submodule of N, we have $\varphi(V(f^{-1}(B))) = V(B)$.

(ii) If f is not surjective then, for every subset A of M, $f(A)$ is contained in some prime submodule of N because N is finitely generated. Hence there does not exist a subset A of M such that $\varphi(V(A)) = V(N) = \varnothing$.

(iii) Suppose $\varphi(V(A)) = \varphi(V(B))$, i.e., $V(f(A)) = V(f(B))$. If $P \in V(A)$, then since $P \supseteq \ker f = 0$, $f(P)$ is prime in N and $f(P) \supseteq f(A)$. Hence $f(P) \supseteq f(B)$, so that $P \supseteq B$. It follows that $V(A) = V(B)$. $\square$

The next two examples each demonstrate that f being injective need not imply that φ is injective, unless of course f is also surjective. Indeed, the second example shows that φ need not be injective even if $\ker\varphi = \{V(0)\}$, where $\ker\varphi$ is defined as the set of all $V(A) \in \zeta(M)$ such that $\varphi(V(A)) = V(0)$.

Example. Let $R = \mathbf{Z}$, $M = \mathbf{Z}/\mathbf{Z}2$ and $N = \mathbf{Z}/\mathbf{Z}4$. Now $\operatorname{spec} M = \{R\bar{0}\}$ and $\operatorname{spec} N = \{R\bar{2}\}$. Take f to be the mapping $\bar{a} \mapsto 2\bar{a}$. Then f is clearly injective, but $V(M) \in \ker\varphi$, since $V(f(M)) = V(R\bar{2}) = V(\bar{0})$.

Example. Let $M = R = \mathbf{Z}$, and let $f : M \to M$ be given by $r \mapsto 2r$. Then $\ker\varphi = \{V(0)\}$, but φ is not injective, since $\varphi(V(M)) = \varphi(V(2M))$.

Another example shows that the converse of Lemma 8 (iii) does not hold in general, though the succeeding lemma provides a partial converse in the case M is *semi-prime*, that is to say, $\operatorname{rad} 0 = 0$.

Example. Let R and N be as in the first example above, and let $K = R\bar{2}$. Consider $f : N \to N/K$, the usual epimorphism. Clearly f is not injective, whereas the induced φ is both injective and surjective.

Lemma 9. *Let f and φ be as described in Theorem 6, and suppose that f is surjective. If, in addition, M is semi-prime and $\ker\varphi = \{V(0)\}$, then f and φ are both injective.*

Proof. Let $K = \ker f$. Then $\varphi(V(K)) = V(f(K)) = V(0)$, so $V(K) \in \ker\varphi = \{V(0)\}$. But $V(K) = V(0)$ implies that $K = 0$, since M is semi-prime. Now φ is injective by Lemma 8 (iii). $\square$

3. Subtractive subspaces and quotient semi-modules. Before reaching our next goal of presenting an isomorphism theorem for $\zeta(R)$-spaces, it seems useful, if not necessary, to provide some basic facts about quotient semi-modules and subtractive subspaces, all of which can be found in [**4**], albeit in a more general setting. In particular, by quotient semi-module we mean a Bourne factor semi-module.

Let (Y, Γ) be an Ω-space, and let $\Delta \subseteq \Gamma$. If $(\Delta, \cap)$ is an Ω-sub-semi-module of $(\Gamma, \cap)$, we say that Δ is a *subspace* of Γ. If Δ has the additional property that $\gamma \in \Gamma$, $\delta \in \Delta$, and $\gamma \cap \delta \in \Delta$ imply $\gamma \in \Delta$, then Δ is said to be *subtractive*. Given any subspace Δ of Γ, the *subtractive closure* of Δ, obtained by intersecting all subtractive subspaces of Γ which contain Δ, is denoted $\Im(\Delta)$. The next result is provided here for convenience, see [**4**].

Lemma 10. *Let Δ be a subspace of the Ω-space Γ. The following are equivalent*:

(i) Δ *is subtractive*;

(ii) Δ *is the kernel of some Ω-homomorphism* $\varphi : \Gamma \to \Gamma'$;

(iii) $\delta \in \Delta$ *and* $\delta \subseteq \gamma$, $\gamma \in \Gamma$, *imply* $\gamma \in \Delta$.

We now record a fact which, though it is easily shown, nevertheless proves to be quite useful. In the Ω-space Γ, let $\tau, \tau' \in \Omega$ and $\gamma, \gamma' \in \Gamma$. If $\tau \subseteq \tau'$ and $\gamma \subseteq \gamma'$, then $\tau * \gamma \subseteq \tau' * \gamma'$. Just note that $\tau * \gamma = (\tau \cap \tau') * (\gamma \cap \gamma') = (\tau * \gamma) \cap (\tau * \gamma') \cap (\tau' * \gamma) \cap (\tau' * \gamma') \subseteq \tau' * \gamma'$.

Lemma 11. *Let Δ be a subspace of the Ω-space Γ, and let $\beta \in \Gamma$. Then*

(i) $\Im(\Delta)$ *is subtractive*;

(ii) $\Im(\Delta) = \{\gamma \in \Gamma : \gamma \supseteq \delta$ *for some* $\delta \in \Delta\}$;

(iii) $\Omega * \beta = \{\tau * \beta : \tau \in \Omega\}$ *is a subspace of* Γ *and* $\Im(\Omega * \beta) = \{\gamma \in \Gamma : \gamma \supseteq \beta\}$.

Proof. The proof of (i) is routine.

(ii) Let $\sum = \{\gamma \in \Gamma : \gamma \supseteq \delta$ for some $\delta \in \Delta\}$, and let $\gamma \in \sum$. Then there exists $\delta \in \Delta$ such that $\gamma \supseteq \delta$. Now for every subtractive subspace Φ which contains Δ, we have $\gamma \cap \delta = \delta \in \Delta \subseteq \Phi$, which implies $\gamma \in \Phi$. It now remains only to show that $\sum$ is itself a subtractive subspace. If $\gamma, \gamma' \in \sum$ and δ, δ' are the known elements of Δ such that $\delta \subseteq \gamma$ and $\delta' \subseteq \gamma'$, then $\gamma \cap \gamma' \supseteq \delta \cap \delta' \in \Delta$, so that $\gamma \cap \gamma' \in \sum$. Now if $\tau \in \Omega$, then by the remarks preceding this result, $\tau * \gamma \supseteq \tau * \delta \in \Delta$, which likewise gives $\tau * \gamma \in \sum$. Finally, if $\alpha \in \Gamma$ and $\alpha \cap \gamma \in \sum$, then for some $\hat{\delta} \in \Delta$ we have $\alpha \cap \gamma \supseteq \hat{\delta}$, and clearly $\alpha \supseteq \alpha \cap \gamma$.

The proof of (iii) follows in a like manner to the preceding proof. $\square$

In light of the last result, the notation $\Im(\beta)$ will be used interchangeably with $\Im(\Omega * \beta)$.

Now let Δ be a subspace of the Ω-space Γ. Then the relation $\equiv_\Delta$ defined on Γ by $\gamma \equiv_\Delta \gamma'$ if there exists $\delta, \delta' \in \Delta$ such that $\gamma \cap \delta = \gamma' \cap \delta'$ is in fact an equivalence relation. For each $\gamma \in \Gamma$, let $[\gamma]$ denote the equivalence class of γ, and let Γ/Δ denote the collection of all such equivalence classes. If we define $[\gamma] + [\gamma'] = [\gamma \cap \gamma']$ and $\tau[\gamma] = [\tau * \gamma]$ for all $\tau \in \Omega$, $\gamma \in \Gamma$, then Γ/Δ becomes a left Ω-semi-module. It is clear that this quotient semi-module is not actually an Ω-space, one reason being that the defined operation is not intersection. Nevertheless, we will show that if M satisfies ACC on semi-prime submodules, then every quotient semi-module of $\zeta(M)$ is isomorphic to a $\zeta(R)$-space. By a *semi-prime* submodule, we mean an intersection of prime submodules, and by an Ω-*isomorphism* we mean an Ω-homomorphism which is bijective. The reader should note that the latter definition differs somewhat from that found in [**4**].

One final comment before moving on to the last section. In the general case, if G and H are semi-modules over the semi-ring T, and $\varphi : G \to H$ is a surjective T-homomorphism, then the best that can be said is that there is a semi-isomorphism $\tilde{\varphi} : G/\ker\varphi \to H$ given by $[g] \mapsto \varphi(g)$. (See [**4**, Corollary 13.48] for details.) This means that $\tilde{\varphi}$ is a surjective homomorphism and has trivial kernel, but it need not be an injection.

4. An isomorphism theorem. In contrast to our last comment in the previous section, we show that if $f : M \to N$ is any R-module epimorphism with kernel K, then the induced $\zeta(R)$-homomorphism $\varphi : \zeta(M) \to \zeta(N)$ gives rise to a $\zeta(R)$-isomorphism $\tilde{\varphi} : \zeta(M)/\Im(V(K)) \to \zeta(N)$. The following result follows immediately from Lemma 11.

Lemma 12. *Let A and B be submodules of M. Then the following are equivalent*:

(i) $V(B) \in \Im(V(A))$;

(ii) $V(B) \supseteq V(A)$;

(iii) $B \subseteq \operatorname{rad} A$.

Recall from Theorem 6 that if $f : M \to N$ is an R-module homomorphism, then $\varphi : \zeta(M) \to \zeta(N)$ defined by $\varphi(V(A)) = V(f(A))$ is a $\zeta(R)$-homomorphism. It is this mapping φ to which we refer in the next two results.

Lemma 13. *Let $f : M \to N$ be an R-module epimorphism, and let φ be the induced $\zeta(R)$-homomorphism. Then the following hold*:

(i) $\ker \varphi = \Im(V(\ker f))$, *and*

(ii) *if A is any submodule of M such that* $\ker \varphi \subseteq \Im(V(A))$, *then* $\varphi(\Im(V(A))) = \Im(\varphi(V(A)))$.

Proof. (i) This follows from Lemma 12 and Corollary 5.

(ii) First note that φ is surjective by Lemma 8 (i). Now Lemma 12 implies that $\ker f \subseteq \operatorname{rad} A$, hence it follows that $f(\operatorname{rad} A) = \operatorname{rad} f(A)$. Therefore, $\varphi(\Im(V(A))) = \{V(f(B)) : B \subseteq \operatorname{rad} A\} = \{V(f(B)) : f(B) \subseteq f(\operatorname{rad} A) = \operatorname{rad} f(A)\} = \Im(V(f(A))) = \Im(\varphi(V(A)))$. $\square$

We do not know if, given an arbitrary $\zeta(R)$-epimorphism, the image of every subtractive subspace which contains the kernel is subtractive, but Lemma 13 (ii) partially answers the question.

Theorem 14. *Let $f : M \to N$ be an R-module epimorphism with $K = \ker f$, and let $\varphi : \zeta(M) \to \zeta(N)$ be the induced $\zeta(R)$-homomorphism. Then φ induces a $\zeta(R)$-isomorphism $\tilde{\varphi} : \zeta(M)/\Im(V(K)) \to \zeta(N)$.*

Proof. By the remarks at the end of the previous section, all that remains to show is that $\tilde{\varphi}$ is injective. Suppose then that $V(f(A)) = V(f(B))$. First observe that, if $P \in V(A+K)$, then $f(P) \supseteq f(A+K) = f(A)$, and since $f(P)$ is prime in N, then $f(P) \supseteq f(B)$, so that $P \supseteq B + K$. A similar argument shows that $V(B + K) \subseteq V(A + K)$.

Hence we have $V(A) \cap V(K) = V(A+K) = V(B+K) = V(B) \cap V(K)$, and since $V(K) \in \Im(V(K))$,then $[V(A)] = [V(B)]$. $\square$

Corollary 15. *Let A be any submodule of M. Then the quotient semi-module $\zeta(M)/\Im(V(A))$ is $\zeta(R)$-isomorphic to the $\zeta(R)$-space $\zeta(M/A)$.*

Proof. Apply Theorem 14 to the canonical epimorphism $f : M \to M/A$. $\square$

Given a quotient semi-module $\zeta(M)/\Delta$, Δ being a subtractive subspace of $\zeta(M)$, then Corollary 15 shows that $\zeta(M)/\Delta$ is essentially a $\zeta(R)$-space, provided Δ is of the form $\Im(V(A))$ for some submodule A of M. In fact, Theorem 16 shows that every subtractive subspace of $\zeta(M)$ is of the form $\Im(V(A))$ for some submodule A of M if and only if M satisfies the ascending chain condition on semi-prime submodules.

Theorem 16. *The following are equivalent*:

(i) *M satisfies* ACC *on semi-prime submodules*;

(ii) *for every subtractive subspace Δ of $\zeta(M)$ there exists a submodule N of M such that $\Delta = \Im(V(N))$*;

(iii) *for every submodule N of M there exists a finitely generated submodule L of N such that* $\operatorname{rad} N = \operatorname{rad} L$.

Proof. (i) $\Rightarrow$ (ii). Let Δ be any subtractive subspace of $\zeta(M)$. If $V(M) \in \Delta$, then by Lemmas 10 and 11 we have $\Delta = \zeta(M) = \Im(V(M))$. So suppose that $V(M) \notin \Delta$. Let D be the collection of all semi-prime submodules A of M such that $V(A) \in \Delta$, and note that $D \neq \varnothing$ since $V(A) = V(\operatorname{rad} A)$ for every $A \leq M$. Now choose N to be a maximal element of D. To see that $\Delta = \Im(V(N))$, let $V(B) \in \Delta$, where B is a submodule of M. If $S = \operatorname{rad}(B+N)$, then $V(S) = V(B+N) = V(B) \cap V(N) \in \Delta$. Since $S \neq M$, then S is a semi-prime submodule of M. It follows that $B \subseteq S = N = \operatorname{rad} N$, so that $V(B) \in \Im(V(N))$ by Lemma 12.

(ii) $\Rightarrow$ (iii). Let H be any submodule of M, and let $\Delta = \{V(G) : G \subseteq \operatorname{rad} L$ for some finitely generated submodule L of $H\}$. It is easy to check that Δ is a subtractive subspace of $\zeta(M)$. By hypothesis, there exists a submodule A of M such that $\Delta = \Im(V(A))$. Since $V(A) \in \Im(V(A)) = \Delta$, without loss of generality we can suppose that $A \subseteq \operatorname{rad} F$, for some finitely generated submodule F of H. For any m in H we have $V(Rm) \in \Delta = \Im(V(A))$, so that, by Lemma 12, $Rm \subseteq \operatorname{rad} A \subseteq \operatorname{rad} F$. Thus $H \subseteq \operatorname{rad} F$ and hence $\operatorname{rad} H = \operatorname{rad} F$.

(iii) $\Rightarrow$ (i). Let $S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots$ be any ascending chain of semi-prime submodules of M, and let $G = \cup_i S_i$. By hypothesis, there exists a finitely generated submodule F of G such that $\operatorname{rad} G = \operatorname{rad} F$. Hence there exists a positive integer n such that $F \subseteq S_n$. Then $\operatorname{rad} G = \operatorname{rad} F \subseteq S_n \subseteq G \subseteq \operatorname{rad} G$, so that $S_n = S_{n+1} = S_{n+2} = \cdots$. Thus M satisfies ACC on semi-prime submodules. □

Corollary 17. *If M satisfies* ACC *on semi-prime submodules, then every quotient semi-module of $\zeta(M)$ is isomorphic to $\zeta(M/A)$ for some submodule A of M.*

Proof. By Theorem 16 and Corollary 15. □

Clearly any Noetherian module satisfies ACC on semi-prime submodules. Finally, we shall show that any Artinian module satisfies ACC on semi-prime submodules.

Proposition 18. *Every Artinian module satisfies* ACC *on semi-prime submodules.*

Proof. Let M be an Artinian module. If M does not contain any prime submodules, then there is nothing to prove. Suppose that M contains a prime submodule. Let S be minimal in the collection $\sum$ of semi-prime submodules of M which are finite intersections of primes. If K is any prime submodule of M, then $S \cap K \in \sum$ and $S \cap K \subseteq S$, so that $S = S \cap K \subseteq K$. It follows that $S = \operatorname{rad} 0$.

Moreover, let U be any submodule of M containing the prime submodule K such that U/K is simple. Since K is prime, the modules M/K and U/K have the same annihilator, hence M/K is semi-simple. But M/K is Artinian, so that M/K is Noetherian. Thus, M/K is

Noetherian for any prime submodule K of M. Since S is a finite intersection of prime submodules, it follows that M/S is Noetherian and hence M satisfies ACC on semi-prime submodules. $\square$

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