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SIMULTANEOUS APPROXIMATION BY BIRKHOFF INTERPOLATORS

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Dedicated to the late Arun Kumar VARMA

1. Introduction and problem description. In the present paper we give general inequalities for simultaneous approximation by Birkhoff interpolators (provided the underlying problem is regular).

Let

$$\Delta_n : -1 \le x_n < x_{n-1} < \cdots < x_1 \le 1$$

be a sequence of arbitrary points. With this sequence of points we associate an incidence matrix

$$E = (e_{i,j})_{i=1,\dots,n;j=0,\dots,R}$$

where R is a positive integer. Such matrices have as entries $|E| \ge n$ ones and $n(R+1) - |E|$ zeros and are such that in each row there is at least one entry equal to one. We also assume that the last column contains at least one entry equal to one. The Birkhoff interpolation problem consists of finding a polynomial P of degree $|E|-1$ such that the following $|E|$ interpolation conditions are fulfilled:

$$P^{(j)}(x_i) = a_i^{(j)} \quad \text{if } e_{i,j} = 1.$$

Here the $a_i^{(j)}$ are arbitrary real numbers.

The pair (E, Δ_n) is called *regular* if, for each choice of the $a_i^{(j)}$, such a polynomial exists and is uniquely determined. In this case there exist uniquely determined fundamental functions $A_{i,j} \in \prod_{|E|-1}$ such that the interpolating polynomial can be written as

$$P(x) = \sum_{e_{i,j}=1} a_i^{(j)} \cdot A_{i,j}(x).$$

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Moreover, the scheme

$$a_i^{(j)} \longrightarrow \sum_{e_{i,j}=1} a_i^{(j)} \cdot A_{i,j}(x)$$

describes a linear mapping from $\mathbf{R}^{|E|}$ into $\prod_{|E|-1}$.

In the sequel we will assume that the pairs (E, Δ_n) we are considering are regular. Now let R', $0 \le R' \le R$ be fixed, and let $f \in C^{R'}[-1,1]$. Then we may choose $a_i^{(j)}$ of the following form

$$a_i^{(j)} = \begin{cases} f^{(j)}(x_i) & \text{if } 0 \le j \le R' \text{ and } e_{i,j} = 1, \\ 0 & \text{if } R'+1 \le j \le R \text{ and } e_{i,j} = 1. \end{cases}$$

Note that the number R' indicates that all derivatives of order $R'+1, \dots, R$ of the interpolating polynomial P are forced to be equal to zero, i.e.,

$$P^{(j)}(x_i) = 0 \quad \text{if} \quad R'+1 \le j \le R \quad \text{and} \quad e_{i,j} = 1.$$

In this case the polynomial P from above is the result of applying a linear operator

$$L_{R'} := L_{R',E,\Delta_n} : C^{R'}[-1,1] \longrightarrow \prod\nolimits_{|E|-1}$$

to the given function f and can thus be written as

$$P(x) = L_{R'}(f;x) = \sum_{0 \le j \le R'; e_{i,j}=1} f^{(j)}(x_i) \cdot A_{i,j}(x).$$

Note that this description includes classical Hermite-Fejér interpolation, in particular.

In the present note we will consider such interpolation processes from a quantitative point of view. Our work is motivated by a number of recent papers in which special Birkhoff interpolation problems were investigated, namely, the nonmodified ("pure") and modified lacunary type; see, for example, [**1, 2, 4, 5**] and [**7**]. For more comprehensive information concerning Birkhoff interpolation, see [**3**].

The structure of an incidence matrix E describing a pure lacunary problem is, roughly speaking, such that—going through the matrix from the left to the right—a rectangular block of ones is followed by a rectangular block of zeros, which is then followed by another rectangular block of ones. The adjective "modified" refers to the fact that in the incidence matrix E describing the pure case, certain ones are either replaced by zeros or moved to positions that were equal to zero before. Details will become clear from the special cases considered below.

In this note we will give a general inequality for the degree of simultaneous approximation by Birkhoff interpolators of the type described above, provided they exist, and apply our general result to several problems of the $(0,1,\dots,R-2,R)$, the $(0,1,\dots,R-3,R)$ and of the $(0,1,\dots,R-3,R-1,R)$ type. These applications generalize and either reproduce or improve all the convergence results that were obtained in the papers mentioned above.

2. A general inequality on simultaneous approximation by Birkhoff interpolators. In the sequel all the norms will be Chebyshev. Furthermore, we will use the convention that all empty sums are equal to zero. In order to derive the general estimate, the following two results will be crucial.

Lemma 2.1 (see [**1**, Lemma 3.1]). *Let $f \in C^{R'}[-1,1]$, $R' \in \mathbf{N}_0$. Then, for $0 < h \le 2$ and $s \in \mathbf{N}$, there exists a function $f_{h,R'+s} \in C^{2R'+s}[-1,1]$ such that*

(i) $\|f^{(j)} - f^{(j)}_{h,R'+s}\| \le c \cdot \omega_{R'+s}(f^{(j)}; h)$ *for* $0 \le j \le R'$,

(ii) $\|f^{(j)}_{h,R'+s}\| \le c \cdot h^{-j} \cdot \omega_j(f,h)$ *for* $0 \le j \le R'+s$,

(iii) $\|f^{(j)}_{h,R'+s}\| \le c \cdot h^{-R'-s} \cdot \omega_{R'+s}(f^{(j-R'-s)}, h)$ *for* $R'+s \le j \le 2R'+s$.

Here the constant c depends only on R' and s.

Theorem 2.2 (see [**6**, Lemma 1]). *Let $r \ge 0$ and $n \ge r$. Then there exists a linear operator $Q_n = Q_{n,r} : C^r[-1,1] \to \prod_n$ such that, for all*

$f \in C^r[-1,1]$, all $|x| \le 1$ and $0 \le k \le r$, one has

$$|(Q_n f - f)^{(k)}(x)| \le c_r \cdot \Delta_n(x)^{r-k} \cdot \|f^{(r)}\|.$$

Here $\Delta_n(x) = \sqrt{1-x^2} \cdot n^{-1} + n^{-2}$, and the constant c_r depends only on r.

The main result now reads as follows.

Theorem 2.3. *Let $f \in C^{R'}[-1,1]$ and $L_{R'}$ be given as above. Then we have, for $x \in [-1,1]$, $0 < h \le 2$, $s \ge \max\{R-R', 1\}$ and $n \ge R'+s$, and $0 \le k \le R'$, that*

$$\begin{aligned}\|(L_{R'} f - f)^{(k)}\| \le c \cdot \omega_s(f^{(R')}; h) \\ \cdot \Big\{ h^{R'-k} + n^{-(R'+s-k)} \cdot h^{-s} \\ + \sum_{0 \le j \le R'} [h^{R'-j} + n^{-(R'+s-j)} \cdot h^{-s}] \\ \cdot \Big\| \sum_{e_{i,j}=1} |A_{i,j}^{(k)}| \Big\| \Big\} \\ + c \cdot \sum_{R'+1 \le j \le R} h^{R'-j} \cdot \omega_{j-R'}(f^{(R')}; h) \cdot \Big\| \sum_{e_{i,j}} |A_{i,j}^{(k)}| \Big\|.\end{aligned}$$

Proof. For an arbitrary polynomial, $\Phi \in \prod_{|E|-1}$ and $0 \le k \le R'$, we have

$$\begin{aligned}|(L_{R'} f - f)^{(k)}(x)| &\le \sum_{\substack{e_{i,j}=1;\\ 0 \le j \le R'}} |f^{(j)}(x_i) - \Phi^{(j)}(x_i)| \cdot |A_{i,j}^{(k)}(x)| \\ &\quad + \sum_{\substack{e_{i,j}=1;\\ R'+1 \le j \le R}} |\Phi^{(j)}(x_i)| \cdot |A_{i,j}^{(k)}(x)| \\ &\quad + |(\Phi - f)^{(k)}(x)| \\ &\le \Big\| \sum_{\substack{e_{i,j}=1;\\ 0 \le j \le R'}} \|f^{(j)} - \Phi^{(j)}\| \cdot |A_{i,j}^{(k)}| \Big\|\end{aligned}$$

$$+\left\| \sum_{\substack{e_{i,j}=1;\\ R'+1\le j\le R}} \|\Phi^{(j)}\| \cdot |A_{i,j}^{(k)}| \right\| + \|(\Phi - f)^{(k)}\|.$$

Now we choose the polynomial $\Phi \in \prod_{|E|-1}$ as follows:

$$\Phi = Q_{|E|-1,R'+s}(f_{h,R'+s})$$

where $f_{h,R'+s} \in C^{2R'+s}[-1,1]$.

For brevity, we write $f_h := f_{h,R'+s}$ and $Q := Q_{|E|-1,R'+s}$. Then from Theorem 2.2, we obtain

$$\|(Qf_h - f_h)^{(j)}\| \le c \cdot n^{-(R'+s-j)} \cdot \|f_h^{(R'+s)}\|, \quad 0 \le j \le R' + s.$$

It hence follows that

$$\begin{aligned}
&\left\| \sum_{\substack{e_{i,j}=1;\\ 0\le j\le R'}} \|f^{(j)} - \Phi^{(j)}\| \cdot |A_{i,j}^{(k)}| \right\| \\
&\qquad \le \left\| \sum_{\substack{e_{i,j}=1;\\ 0\le j\le R'}} \{\|f^{(j)} - f_h^{(j)}\| + \|f_h^{(j)} - (Qf_h)^{(j)}\|\} |A_{i,j}^{(k)}| \right\| \\
&\qquad \le \left\| \sum_{\substack{e_{i,j}=1;\\ 0\le j\le R'}} \{c \cdot \omega_{R'+s}(f^{(j)}; h) \right. \\
&\qquad\qquad\qquad \left. + c \cdot n^{-(R'+s-j)} \cdot \|f_h^{(R'+s)}\|\} \cdot |A_{i,j}^{(k)}| \right\| \\
&\qquad \le \left\| \sum_{\substack{e_{i,j}=1;\\ 0\le j\le R'}} \{c \cdot \omega_{R'+s}(f^{(j)}; h) \right. \\
&\qquad\qquad\qquad \left. + c \cdot n^{-(R'+s-j)} \cdot h^{-R'-s} \cdot \omega_{R'+s}(f; h)\} \cdot |A_{i,j}^{(k)}| \right\| \\
&\qquad \le \left\| \sum_{\substack{e_{i,j}=1;\\ 0\le j\le R'}} \{c \cdot h^{R'-j} + \omega_{s+j}(f^{(R')}; h) \right.
\end{aligned}$$

$$+ c \cdot n^{-(R'+s-j)} \cdot h^{-s} \cdot \omega_s(f^{(R')}; h)\} \cdot |A_{i,j}^{(k)}|\Big\|$$

$$\leq c \cdot \Big\| \sum_{\substack{e_{i,j}=1;\\ 0\leq j\leq R'}} \{h^{R'-j} + n^{-(R'+s-j)} \cdot h^{-s} \cdot \omega_s(f^{(R')}; h)\} \cdot |A_{i,j}^{(k)}|\Big\|$$

$$\leq c \cdot \omega_s(f^{(R')}; h) \cdot \sum_{0\leq j\leq R'} \{h^{R'-j} + n^{-(R'+s-j)} \cdot h^{-s}\}$$

$$\cdot \Big\| \sum_{e_{i,j}=1} |A_{i,j}^{(k)}|\Big\|.$$

For $j = k$, we have that

$$\|(\Phi - f)^{(k)}\| \leq c \cdot \omega_s(f^{(R')}; h) \cdot \{h^{R'-k} + n^{-(R'+s-k)} \cdot h^{-s}\}.$$

We also need estimates for

$$\Big\| \sum_{\substack{e_{i,j}=1;\\ R'+1\leq j\leq R}} \|\Phi^{(j)}\| |A_{i,j}^{(k)}|\Big\|.$$

Let $R' + 1 \leq j \leq R$. Then we have by Lemma 2.1

$$\begin{aligned}
\|\Phi^{(j)}\| &= \|(Qf_h)^{(j)}\| \leq \|Qf_h^{(j)} - f_h^{(j)}\| + \|f_h^{(j)}\| \\
&\leq c \cdot n^{-(R'+s-j)} \cdot \|f_h^{(R'+s)}\| + c \cdot h^{-j} \cdot \omega_j(f; h) \\
&\leq c \cdot n^{-(R'+s-j)} \cdot h^{-(R'+s)} \cdot \omega_{R'+s}(f; h) \\
&\quad + c \cdot h^{-j} \cdot \omega_j(f; h) \\
&\leq c \cdot n^{-(R'+s-j)} \cdot h^{-(R'+s)} \cdot h^{R'} \cdot \omega_{R'+s-R'}(f^{(R')}; h) \\
&\quad + c \cdot h^{-j} \cdot h^{R'} \cdot \omega_{j-R'}(f^{(R')}; h) \\
&\leq c \cdot \{n^{-(R'+s-j)} \cdot h^{-s} \cdot \omega_s(f^{(R')}; h) \\
&\qquad + h^{R'-j} \cdot \omega_{j-R'}(f^{(R')}; h)\}.
\end{aligned}$$

We finally obtain that

$$\Big\| \sum_{\substack{e_{i,j}=1;\\ R'+1\leq j\leq R}} \|\Phi^{(j)}\| |A_{i,j}^{(k)}|\Big\| \leq c \cdot \sum_{R'+1\leq j\leq R} \{n^{-(R'+s-j)} \cdot h^{-s} \cdot \omega_s(f^{(R')}; h)$$

$$+ h^{R'-j} \cdot \omega_{j-R'}(f^{(R')}; h)\} \cdot \Big\| \sum_{e_{i,j}=1} |A_{i,j}^{(k)}|\Big\|.$$

Combining all these gives, for all $s \geq \max\{1, R - R'\}$,

$$
\begin{aligned}
\|(L_{R'}f - f)^{(k)}\| \leq\ & c \cdot \omega_s(f^{(R')}; h) \\
& \cdot \sum_{0 \leq j \leq R'} \{h^{R'-j} + n^{-(R'+s-j)} \cdot h^{-s}\} \cdot \Big\| \sum_{e_{i,j}=1} |A_{i,j}^{(k)}| \Big\| \\
& + c \cdot \sum_{R'+1 \leq j \leq R} \{n^{-(R'+s-j)} \cdot h^{-s} \cdot \omega_s(f^{(R')}; h) \\
& \qquad + h^{R'-j} \cdot \omega_{j-R'}(f^{(R')}; h)\} \\
& \cdot \Big\| \sum_{e_{i,j}=1} |A_{i,j}^{(k)}| \Big\| \\
& + c \cdot \omega_s(f^{(R')}; h) \cdot \{h^{R'-k} + n^{-(R'+s-k)} \cdot h^{-s}\} \\
\leq\ & c \cdot \omega_s(f^{(R')}; h) \cdot \Big\{ h^{R'-k} + n^{-(R'+s-k)} \cdot h^{-s} \\
& + \sum_{0 \leq j \leq R'} [h^{R'-j} + n^{-(R'+s-j)} \cdot h^{-s}] \cdot \Big\| \sum_{e_{i,j}=1} |A_{i,j}^{(k)}| \Big\| \Big\} \\
& + c \cdot \sum_{R'+1 \leq j \leq R} h^{R'-j} \cdot \omega_{j-R'}(f^{(R')}; h) \cdot \Big\| \sum_{e_{i,j}=1} |A_{i,j}^{(k)}| \Big\|.
\end{aligned}
$$

□

Corollary 2.4. *For the special choice* $h = 1/n$ *in Theorem* 2.3*, we obtain that*

$$
\begin{aligned}
\|(L_{R'}f - f)^{(k)}\| \leq\ & c \cdot \omega_s(f^{(R')}; 1/n) \\
& \cdot \Big\{ n^{-R'+k} + \sum_{0 \leq j \leq R'} n^{-R'+j} \cdot \Big\| \sum_{e_{i,j}=1} |A_{i,j}^{(k)}| \Big\| \Big\} \\
& + c \cdot \sum_{R'+1 \leq j \leq R} n^{-R'+j} \cdot \omega_{j-R'}(f^{(R')}; 1/n) \\
& \cdot \Big\| \sum_{e_{i,j}=1} |A_{i,j}^{(k)}| \Big\|.
\end{aligned}
$$

Remark 2.5. (i) For j_0 fixed,the quantities $\| \sum_{e_{i,j_0}=1} |A_{i,j_0}^{(k)}| \|$ figuring in Theorem 2.3 are bounded from above by $\sum_{e_{i,j_0}=1} \|A_{i,j_0}^{(k)}\|$. The latter terms were used in our previous papers [**1**] and [**2**]. It is, however, of advantage to use the former instead, as will become clear from the applications below.

(ii) A straightforward way to obtain upper bounds for the quantities just mentioned is to use Markov's inequality, giving

$$\Big\| \sum_{e_{i,j_0}=1} |A_{i,j_0}^{(k)}| \Big\| \le c \cdot n^{2k} \cdot \Big\| \sum_{e_{i,j_0}=1} |A_{i,j_0}| \Big\|.$$

However, this approach should only be used if no better inequalities are available for the derivatives of the fundamental functions.

3. Applications. In this section we will demonstrate what the above approach implies in several concrete cases. As will become clear from the statements below, our results generalize and either improve or at least reproduce all earlier quantitative assertions obtained in special cases.

In the following, the point sequence Δ_n will always consist of the zeros of the polynomial

$$\pi_n(x) = (1-x^2) \cdot P'_{n-1}(x) = -n \cdot (n-1) \cdot \int_{-1}^{x} P_{n-1}(t)\, dt,$$

where $P_n(x)$ is the Legendre polynomial of degree n, normed such that $P_n(1) = 1$.

3.1. $(0, \dots, R-2, R)$ *interpolation.* For this case Theorem 2.3 implies Theorem 3.2 in [**2**]. Thus, it also covers all of the special cases considered in Section 4 of this paper.

3.2. $(0, \dots, R-3, R)$ *interpolation.* The most recent paper in which a nonmodified case was considered is one by Szabados and Varma [**5**], in which convergent $(0,3)$ interpolation processes were investigated. They showed that the "*pure*" $(0,3)$ *interpolation* operators (case $R' = 0$ in our notation) based on the roots of $\pi_n(x)$ converge for all continuous functions.

The incidence matrix has the following form:

$$E = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

The interpolation polynomials then take the form

$$P(x) = \sum_{i=1}^{n} a_i^{(0)} \cdot A_{i,0}(x) + \sum_{i=1}^{n} a_i^{(3)} \cdot A_{i,3}(x), \quad x \in [-1,1].$$

For the fundamental functions, the authors proved the following:

Lemma 3.1. (i)

$$\left\| \sum_{i=1}^{n} |A_{i,0}| \right\| \le c,$$

(ii)

$$\left\| \sum_{i=1}^{n} |A_{i,3}| \right\| \le c \cdot n^{-3} \cdot \log n.$$

Applying Theorem 2.3 for the case $R' = 0$ yields

Proposition 3.2 (see [**5**, Theorem 3]). *For $f \in C[-1,1]$ and $n > 3$, we have*

$$\|L_{0,2n-1} f - f\| \le c \cdot \omega_3\left(f; \frac{\log^{1/3} n}{n} \right),$$

where the constant c does not depend on n or f.

Proof. In Theorem 2.3 put $s = 3$ and $h = (\log^{1/3} n)/n$. □

The second case we consider is $R' = 3$. Here we have the following

Proposition 3.3. *For $f \in C^3[-1,1]$, $s \geq 1$ and all $n \geq s+3$, we have for $0 \leq k \leq 3$,*

$$\|(L_{3,2n-1}f - f)^{(k)}\| \leq c \cdot n^{2k-3} \cdot \log n \cdot \omega_2(f^{(3)}; 1/n),$$

with the constant c depending neither on n nor on f.

Proof. We apply Corollary 2.4. Note first that the terms involving $\omega_{j-R'}(f^{(3)}; 1/n)$ will not be present, which is due to the choice $R' = R = 3$. What remains as the upper bound is

$$c \cdot \omega_s(f^{(3)}; 1/n) \cdot \left\{ n^{-3+k} + n^{-3} \cdot \left\| \sum_{e_{i,0}=1} |A_{i,0}^{(k)}| \right\| + \left\| \sum_{e_{i,3}=1} |A_{i,3}^{(k)}| \right\| \right\}.$$

An application of Markov's inequality gives

$$\left\| \sum_{e_{i,j}=1} |A_{i,j}^{(k)}| \right\| \leq c \cdot n^{2k} \cdot \left\| \sum_{e_{i,j}=1} |A_{i,j}| \right\| \quad \text{for} \quad j \in \{0,3\}.$$

This implies

$$\begin{aligned} \|(L_{3,2n-1}f - f)^{(k)}\| &\leq c \cdot \omega_s(f^{(3)}; 1/n) \\ &\quad \cdot \left\{ n^{-3+k} + n^{-3} \cdot n^{2k} \cdot \left\| \sum_{e_{i,0}=1} |A_{i,0}^{(k)}| \right\| \right. \\ &\quad \left. + n^{2k} \cdot \left\| \sum_{e_{i,3}=1} |A_{i,3}^{(k)}| \right\| \right\} \\ &\leq c \cdot \omega_s(f^{(3)}; 1/n) \{ n^{-3+k} + n^{-3+2k} + n^{-3+2k} \cdot \log n \} \\ &\leq c \cdot n^{2k-3} \cdot \log n \cdot \omega_s(f^{(3)}; 1/n). \qquad \square \end{aligned}$$

We next consider a *modified* case of $(0,3)$ *interpolation*, which was also investigated in [**5**]. Here "modified" refers to the fact that the interpolation conditions can now be visualized by the incidence matrix

$$E = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

The interpolation polynomial is then given by

$$P(x) = \sum_{i=1}^{n} a_i^{(0)} \cdot A_{i,0}(x) + a_1^{(1)} \cdot A_{1,1}(x) + a_n^{(1)} \cdot A_{n,1}(x) + \sum_{i=2}^{n-1} a_i^{(3)} \cdot A_{i,3}(x).$$

For the fundamental functions, one has

Lemma 3.4 (see [**5**]).

(i)
$$\left\| \sum_{i=1}^{n} |A_{i,0}| \right\| = \mathcal{O}(1),$$

(ii)
$$\|A_{1,1}\| = \|A_{n,1}\| = \mathcal{O}(n^{-2}),$$

(iii)
$$\left\| \sum_{i=2}^{n-1} |A_{i,3}| \right\| = \mathcal{O}\left(\frac{\log n}{n^3} \right).$$

For the case $R' = 0$, we get the following result.

Proposition 3.5. *For the modified* $(0,3)$ *interpolation operators* $L_{0,2n-1}$*, we have*

$$\|L_{0,2n-1}f - f\| \le c \cdot \left\{ \omega_3\left(f; \frac{\log^{1/3} n}{n} \right) + \frac{1}{n \cdot \log^{1/3} n} \cdot \omega_1\left(f; \frac{\log^{1/3} n}{n} \right) \right\},$$

for $f \in C[-1,1]$ *and* $n > 3$.

Proof. Applying Theorem 2.3 with $s = 3$ and $h = (\log^{1/3} n)/n$ immediately yields our statement. $\square$

Corollary 3.6 (see [**5**]).

$$\|L_{0,2n-1}f - f\| = \begin{cases} \mathcal{O}(\omega_1(f; (\log^{1/3} n)/n)) & f \in C[-1,1], \\ \mathcal{O}(\omega_3(f; (\log^{1/3} n)/n)) + \mathcal{O}(1/n^2) & f' \in C[-1,1]. \end{cases}$$

Next we consider the case $R' = 3$. Here we obtain the following result.

Proposition 3.7. *For $f \in C^3[-1,1]$, $s \geq 1$ and $n \geq s+3$, we have for $0 \leq k \leq 3$,*

$$\|(L_{3,2n-1}f - f)^{(k)}\| \leq c \cdot n^{2k-3} \cdot \log n \cdot \omega_s(f^{(3)}; 1/n),$$

with constant c depending neither on n nor on f.

Proof. The proof is completely analogous to that of Proposition 3.3. $\square$

Now we consider *modified* $(0,1,4)$ *interpolation*, as investigated in the recent paper of Varma and the two Saxenas [**7**]. The incidence matrix in this case is

$$E = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

The explicit form of the interpolation polynomial is

$$\begin{aligned} P(x) = \sum_{j=0}^{1} \sum_{i=1}^{n} a_i^{(j)} \cdot A_{i,j}(x) + a_1^{(2)} \cdot A_{1,2}(x) \\ + a_n^{(2)} \cdot A_{n,2}(x) + \sum_{i=2}^{n-1} a_i^{(4)} \cdot A_{i,4}(x). \end{aligned}$$

Estimates for the fundamental functions read as follows.

Lemma 3.8 (see [**7**]).

(i)

$$|A_{1,0}(x)| \leq c, \qquad |A_{n,0}(x)| \leq c, \qquad \sum_{i=2}^{n-1} |A_{i,0}(x)| \leq c \cdot \log n,$$

(ii)

$$|A_{1,1}(x)| \le \frac{c}{n^2}, \qquad |A_{n,1}(x)| \le \frac{c}{n^2}, \qquad \sum_{i=2}^{n-1} |A_{i,1}(x)| \le c \cdot \frac{\log n}{n},$$

(iii)

$$|A_{1,2}(x)| \le \frac{c}{n^4}, \qquad |A_{n,2}(x)| \le \frac{c}{n^4},$$

(iv)

$$\sum_{i=2}^{n-1} |A_{i,4}(x)| \le c \cdot \frac{\log n}{n^4}.$$

Instead of considering separately all of the meaningful choices for R', $0 \le R' \le 4$, we give the following general result covering all these possibilities.

Theorem 3.9. *Let $0 \le R' \le 4$. Then for the modified $(0,1,4)$ lacunary interpolation operators $L_{R',3n-1}$, the following inequalities hold*

$$\|(L_{R',3n-1}f - f)^{(k)}\| \le c \cdot n^{2k-R'} \cdot \log n \cdot \omega_{s_{R'}}(f^{(R')}; 1/n)$$

for all $f \in C^{R'}[-1,1]$, $k = 0, \dots, R'$ and $R' \in \{0,1,2,4\}$ with $s_0 = s_1 = 1$, $s_2 = 2$ and $s_4 \in \mathbf{N}$ arbitrary, $n > \max\{4, R' + s_{R'}\}$.

Proof. From Corollary 2.4 and (many) applications of Markov's inequality, we arrive at the upper bound

$$\begin{aligned} c \cdot \omega_s(f^{(R')}; 1/n) \cdot \bigg\{ n^{-R'+k} + \sum_{0 \le j \le 4} n^{-R'+j+2k} \cdot \bigg\| \sum_{e_{i,j}=1} |A_{i,j}| \bigg\| \bigg\} \\ + c \cdot \sum_{R'+1 \le j \le 4} n^{-R'+j+2k} \cdot \omega_{j-R'}(f^{(R')}; 1/n) \\ \cdot \bigg\| \sum_{e_{i,j}=1} |A_{i,j}| \bigg\|. \end{aligned}$$

Observing that the sums of fundamental functions which are of interest here all obey the rule

$$\Big\| \sum_{e_{i,j}=1} |A_{i,j}| \Big\| \le c \cdot n^{-j} \cdot \log n, \quad j \in \{0,1,2,4\},$$

we can now replace the upper bound by

$$\begin{aligned}
c \cdot \omega_s(f^{(R')};1/n) \cdot \Big\{ n^{-R'+k} + \sum_{0\le j\le 4} n^{-R'+j+2k-j} \cdot \log n \Big\} \\
&+ c \cdot \sum_{R'+1 \le j \le 4} n^{-R'+j+2k-j} \cdot \log n \\
&\cdot \omega_{j-R'}(f^{(R')};1/n) \cdot (1-\delta_{j,3}) \\
&\le c \cdot \omega_s(f^{(R')};1/n) \cdot n^{-R'+2k} \cdot \log n \\
&\quad + c \cdot n^{-R'+2k} \cdot \log n \\
&\quad \cdot \sum_{R'+1\le j\le 4} \omega_{j-R'}(f^{(R')};1/n) \cdot (1-\delta_{j,3}) \\
&= c \cdot n^{-R'+2k} \cdot \log n \cdot \Big\{ \omega_s(f^{(R')};1/n) \\
&\quad + \sum_{R'+1\le j\le 4} \omega_{j-R'}(f^{(R')};1/n) \cdot (1-\delta_{j,3}) \Big\}.
\end{aligned}$$

Here $\delta_{j,3}$ is the Kronecker symbol, the use of which is justified due to the fact that $e_{i,3} = 0$ for $0 \le i \le n$, i.e., $\| \sum_{e_{i,3}=1} |A_{i,3}| \| = 0$.

For $R' = 4$, the second term is not present at all, and $s \ge 1$ are the possible orders of the moduli of continuity which can be used in this case.

For $R' = 2$, the second term involves only $\omega_2(f'';1/n)$,and because $s \ge R - R' = 4 - 2 = 2$, the upper bound can now be given in terms of ω_2 only.

For $R' = 1$ and $R' = 0$, we have $s \ge 3$ and $s \ge 4$, respectively. The second term contains in the case $R' = 1$ the quantities ω_1 and ω_3, and for $R' = 0$ the moduli ω_1, ω_2 and ω_4. Hence, in both cases, the dominant modulus is ω_1.

The proof is complete. $\square$

Theorem 3.9 yields a lot of statements for special cases. We refrain from going through them, except for one.

Proposition 3.10. *Let $R' = 0$. Then, for all $f \in C[-1,1]$ and $n > 4$, we have*

$$\|L_{0,3n-1}f - f\| \leq c \cdot \log n \cdot \omega_1(f, 1/n).$$

In particular, for $f \in \mathrm{Lip}_\alpha$, $0 < \alpha \leq 1$, we have

$$\|L_{0,3n-1}f - f\| \leq c \cdot n^{-\alpha} \log n.$$

The latter inequality is one of the main results in [**7**].

3.3. $(0, \dots, R-3, R-1, R)$ *interpolation.* A very recent paper considering this case and, in particular, *modified* $(0,2,3)$ *interpolation* is one by Sharma, Szabados, Underhill and Varma [**4**]. The incidence matrix in the latter case is as follows:

$$E = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

The interpolation polynomial is now

$$\begin{aligned} P(x) = \sum_{i=1}^{n} a_i^{(0)} \cdot A_{i,0}(x) + a_1^{(1)} \cdot A_{1,1}(x) + a_n^{(1)} \cdot A_{n,1}(x) \\ + \sum_{i=1}^{n} a_i^{(2)} \cdot A_{i,2}(x) + \sum_{i=2}^{n-1} a_i^{(3)} \cdot A_{i,3}(x). \end{aligned}$$

As a consequence, from the above authors' estimates for the fundamental functions, we get the following

Lemma 3.11.

(i)
$$\Big\|\sum_{e_{i,0}=1} |A_{i,0}|\Big\| \le c\cdot \log n,$$

(ii)
$$\Big\|\sum_{e_{i,1}=1} |A_{i,1}|\Big\| \le c\cdot n^{-2},$$

(iii)
$$\Big\|\sum_{e_{i,2}=1} |A_{i,2}|\Big\| \le c\cdot n^{-2}\cdot \log n,$$

(iv)
$$\Big\|\sum_{e_{i,3}=1} |A_{i,3}|\Big\| \le c\cdot n^{-3}.$$

Proof. In [**4**] the authors showed

$$\sum_{e_{i,j}=1} \frac{|A_{i,j}(x)|}{\Delta_n(x_i)^j} \le c\cdot \log n, \quad \text{for } j=0 \text{ or } 2,$$

where $\Delta_n(x)=\sqrt{1-x^2}/n+1/n^2\le 2/n$. Hence,

$$\sum_{e_{i,j}=1} |A_{i,j}(x)| \le c\cdot n^j\cdot \log n \quad \text{for } j=0 \text{ or } 2;$$

this gives (i) and (iii). For $j=1$ they showed (ii) directly, and for $j=3$, an observation analogous to the one for $j=0$ or 2 immediately implies (iv). □

As a generalization of the above authors' convergence theorem for modified (0,2,3) interpolation, we now have

Proposition 3.12. *For any* $f\in C^{R'}[-1,1]$, $0\le R'\le 3$, *the following is true*:

$$\|(L_{R',3n-1}f-f)^{(k)}\| \le c\cdot n^{2k-R'}\cdot \log n\cdot \omega_{s_{R'}}(f^{(R')};1/n),$$

where $s_0 = s_1 = s_2 = 1$, $s_3 \in \mathbf{N}$ *is arbitrary,* $0 \leq k \leq R'$ *and* $n \geq \max\{2, R' + s_{R'}\}$.

Proof. Using Corollary 2.4 again and several applications of Markov's inequality, we get as the upper bound the following:

$$c \cdot \omega_s \cdot (f^{(R')}; 1/n)\Big\{n^{-R'+k} + \sum_{0\leq j\leq 3} n^{-R'+j+2k} \cdot \Big\|\sum_{e_{i,j}=1} |A_{i,j}|\Big\|\Big\}$$
$$+ c \cdot \sum_{R'+1\leq j\leq 3} n^{-R'+j+2k} \cdot \omega_{j-R'}(f^{(R')}; 1/n) \cdot \Big\|\sum_{e_{i,j}=1} |A_{i,j}|\Big\|.$$

In this case the sums of fundamental functions also obey the rule

$$\Big\|\sum_{e_{i,j}=1} |A_{i,j}|\Big\| \leq c \cdot n^{-j} \cdot \log n, \quad j \in \{0, 1, 2, 3\}.$$

Proceeding as in the proof of Theorem 3.9, the above upper bound can be replaced by

$$c \cdot n^{-R'+2k} \cdot \log n \cdot \Big\{\omega_s(f^{(R')}; 1/n) + \sum_{R'+1\leq j\leq 3} \omega_{j-R'}(f^{(R')}; 1/n)\Big\}.$$

Again, for $R' = 3$, the second term equals zero, so that any $s \geq 1$ can be used as the order of the modulus.

For $R' = 2$, we get an upper bound of the form

$$c \cdot n^{-2+2k} \cdot \log n \cdot \omega_1(f''; 1/n).$$

In the case $R' = 1$ we get the majorant

$$c \cdot n^{-1+2k} \cdot \log n \cdot \omega_1(f'; 1/n),$$

and for $R' = 0$, we obtain

$$c \cdot n^{2k} \cdot \log n \cdot \omega_1(f; 1/n).$$

This implies our claim. $\square$

Remark. A general treatment such as that given in Theorems 3.9 and 3.12 can also be carried out for the $(0,3)$ case. We chose to present our results separately in this latter case in order to give a more gradual development.

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