ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 4, Winter 1998

MONOTONICITY IN TIME OF LARGE SOLUTIONS TO A NONLINEAR HEAT EQUATION

V.A. GALAKTIONOV AND A.A. LACEY

ABSTRACT. We consider the Cauchy problem for a two-dimensional semi-linear heat equation with radial symmetry

$$u_t = u_{rr} + u_r/r + e^u \quad \text{in} \quad \mathbf{R}^+ \times (0,T);$$

with smooth, bounded initial data $u_0(r)$. We prove that the solution $u(r,t)$ becomes strictly monotone in time, $u_t > 0$, at any point where u is large enough.

The proof is based on intersection comparison of $u(r,t)$ with the set $\{w(\cdot)\}$ of stationary solutions satisfying $w'' + w'/r + e^w = 0$ for $r > 0$. The above monotonicity result is shown to depend essentially on the global structure of the set $\{w\}$.

The same result is found to hold for positive solutions u to the equation with power nonlinearity

$$u_t = \Delta u + u^p, \quad 1 < p < (N+2)/(N-2)_+.$$

Several generalizations to boundary value problems and quasi-linear equations are given.

1. Introduction. Nonlinear parabolic equations of the general form

$$u_t = \Delta\phi(u) + f(u)$$

with $f > 0$, $\phi' > 0$, are known, under certain restrictions on f (and ϕ if the problem is based on a bounded domain with, say, Dirichlet conditions) to exhibit blow-up. By this it is meant that the solution ceases to exist at some finite time T with a classical solution u valid for $0 < t < T$ and $\sup_x u(x,t) \to \infty$ as $t \to T$. See, for example, the books [**2, 17**] and references therein.

Received by the editors on February 20, 1996, and in revised form on December 6, 1996.
AMS (MOS) *Mathematics Subject Classifications.* 35K55, 35K65.
Key words and phrases. Semilinear heat equation, stationary solutions, intersection comparison, maximum principle.

A good deal of attention has been focused on the manner of blow-up, in particular, the local behavior with respect to time t and position x, near any blow-up points, i.e., t close to T, x close to X for which there is some sequence $(x_n, t_n) \to (X, T)$ such that $u(x_n, t_n) \to \infty$. For some results it is necessary to make the assumption that the solution be increasing in time, $u_t \geq 0$, at least for x in a neighborhood of X and t close to T, see, for example, [**6, 7**] and references in [**2, 17**]. By standard use of the maximum principle this condition is certainly guaranteed if $u_t \geq 0$ initially and on the lateral boundary of the domain but is certainly not true in general.

In the context of combustion theory, blow-up of the variable u, representing temperature, is equivalent to ignition, provided that $u_t \geq 0$.

In the present paper we prove that the condition $u_t > 0$ must occur in a domain where u is large enough for a wide class of radial solutions. The result is true in the subcritical parameter range with respect to the critical Sobolev exponent for the elliptic operator of the nonlinear heat equation. The proof is based on the analysis of the structure of the family of stationary solutions.

Galaktionov and Posashkov [**9, 10, 12**] considered the one-dimensional problem for different uniformly parabolic and also degenerate equations and were able to show, with some assumptions on ϕ and f, and quite weak restrictions on the initial data $u(x, 0) = u_0(x)$, that there is some value M, depending upon the initial data, such that $u_t(x, t) \geq 0$ wherever $u(x, t) \geq M$.

In the present paper we first consider the two-dimensional, radially symmetric problems

$$u_t = \Delta u + e^u \quad \text{in} \quad \Omega \times (0, T),$$

where Ω is the ball $B(0, R) = \{x : |x| = r < R\}$ or the whole space $\mathbf{R}^2$. For simplicity, the case of $\Omega = \mathbf{R}^2$ is generally considered and we shall only remark on how the results extend to $\Omega = B(0, R)$ with Dirichlet boundary conditions imposed on $\partial\Omega = \{x : |x| = R\}$. We shall also note extensions to more general equations but still retaining radial symmetry.

We begin by studying the semi-linear heat equation

$$u_t = u_{rr} + u_r/r + e^u \quad \text{in} \quad Q_T = \mathbf{R}^+ \times (0,T); \tag{1.1}$$

$$u_r = 0 \quad \text{on} \quad r = 0, \quad 0 \le t < T; \tag{1.2}$$

$$u(r,0) = u_0(r) \quad \text{for} \quad r > 0. \tag{1.3}$$

Here $T \in [0,\infty]$ is assumed to be the maximal existence time of the classical solution, [**5**].

We shall suppose that u_0 is in C^1, more particularly, that there is some $K > 0$ such that

$$\sup |u_0| < K, \qquad \sup |u_0'| < K, \tag{1.4}$$

and also that $u_0'(0) = 0$ for regularity. We do not require that u_0 be radially decreasing. It may be noted that we can use the smoothing properties of heat equations to enlarge the class of initial data, replacing $t = 0$ by $t = \varepsilon$ for some small $\varepsilon > 0$:

$$\sup |u(\cdot,\varepsilon)| < K_\varepsilon, \qquad \sup |u_r(\cdot,\varepsilon)| < K_\varepsilon, \qquad u_r(0,\varepsilon) = 0.$$

Our main result is to obtain monotonicity in time where the solution is large, like Galaktionov and Posashkov for $N = 1$. In particular, we prove

Theorem 1. *There exists a constant M, depending upon the constant K in* (1.4)*, such that if $u(r_0,t_0) \ge M$ for some $(r_0,t_0) \in Q_T$, then $u_t(r_0,t) > 0$ for all $t \in [t_0,T)$.*

In the case of $r_0 = 0$ the above result can be proved using the approach given in [**8**].

Since our monotonicity result depends upon the global structure of the family of stationary solutions of the equation under consideration, we start with the particular semi-linear equation (1.1) in $\mathbf{R}^2$ which has explicit stationary solutions, see, for example, [**13**]. This simplifies the analysis and makes it possible to explain the basic comparison idea, which applies equally well to many nonlinear heat equations.

Specifically, we also discuss the following well-known semi-linear diffusion equation from physical chemistry

$$u_t = \Delta u + u^p = u_{rr} + (N-1)u_r/r + u^p \quad \text{in} \quad Q_T, \tag{1.5}$$

where $p > 1$ is a constant, $\mathbf{R}^2$ is replaced by $\mathbf{R}^N$ and symmetry condition (1.2) again applies. The initial data (1.3) satisfies (1.4) and is now nonnegative. We prove

Theorem 2. *The assertion of Theorem* 1 *is true for the equation* (1.5) *provided that*

$$1 < p < p_s \equiv \frac{N+2}{(N-2)_+}. \tag{1.6}$$

Finally we state the most general result on eventual monotonicity for equations with power-type nonlinearities of the form

$$u_t = \Delta u^m + u^p, \quad m > 1, \quad p > 1, \tag{1.7}$$

or

$$u_t = \nabla \cdot (|\nabla u|^m \nabla u) + u^p, \quad m > 0, \quad p > 1, \tag{1.8}$$

with bounded smooth enough data $u_0 \geq 0$,

$$\sup |u_0| < K; \qquad \sup |(u_0^{m-1})'| < K \quad \text{or} \quad \sup |u_0'| < K. \tag{1.9}$$

We can show that eventual monotonicity crucially depends on the general structure of the family of stationary solutions and holds in the subcritical range:

Theorem 3. *Under the hypotheses* (1.9) *the assertion of Theorem* 1 *is true for* (1.7) *or* (1.8) *provided that* $p \in (1, p_s)$ *where* p_s *is the critical Sobolev exponent corresponding to the elliptic operators in* (1.7) *or* (1.8), *i.e.,*

$$\begin{aligned} p_s &= m\frac{N+2}{(N-2)_+} \qquad \textit{for } (1.7), \\ p_s &= \frac{N(m+1)+m+2}{[N-(m+2)]_+} \qquad \textit{for } (1.8). \end{aligned} \tag{1.10}$$

For (1.8) *the value* p_s *in* (1.10) *is critical for the compact embedding* $W_1^{m+2}(B_1) \subset L^{p+1}(B_1)$, B_1 *is a ball in* $\mathbf{R}^N$, *which holds for* $p < p_s$ *but not for* $p \geq p_s$.

We show that such monotonicity results near $t = T$ for $1 < p < p_s$ imply directly that blow-up is complete: no continuation of solutions into $t > T$ is possible.

We conclude this introduction by noting that the result for (1.1) appears to fail for three-dimensional problems. Eberly and Troy [**4**], see also Lacey and Tzanetis [**15**], found a similarity solution for $u_t = u_{rr} + 2u_r/r + e^u$ valid in $r \geq 0$, $t < 0$, which blows up at $r = 0$ as $t \to 0^-$. Close examination of this solution shows that $u_t(r,t) < 0$ for r and t arbitrarily close to 0, where u is arbitrarily large.

2. Preliminaries.

2.1. *Intersection properties.* We start by outlining the key method used in the papers of Galaktionov and Posashkov and which is the basis of the proofs of the present results.

Suppose that we choose a steady state $w(r)$, which then satisfies

$$w'' + w'/r + e^w = 0 \quad \text{for} \quad r > 0, \tag{2.1}$$

which intersects $u_0(r)$ at just two points $0 < r_1 < r_2$. This means that the difference $z(r,0) = u_0(r) - w(r)$ changes sign in any small neighborhoods of $r = r_1$ and $r = r_2$. For simplicity we consider the case $(r - r_1)(r - r_2)(u_0(r) - w(r)) \geq 0$ for all $r \geq 0$. From the strong maximum principle, [**5**], there are some continuous $r_1(t), r_2(t)$ defined for $0 \leq t < t_0 \in (0,T]$ with $r_1(t) < r_2(t)$, $u(r,t) > w(r)$ for $r > r_2(t)$ or $r < r_1(t)$ and $u(r,t) < w(r)$ for $r_1(t) < r < r_2(t)$; i.e., for times t less than t_0, the function u intersects w at precisely two points in r. Results of this kind for linear parabolic equations may be found in references in [**17, 8, 10**]. The basic ideas for such analysis go back to [**18**]. Moreover, if the maximal t_0 for which this holds is less than T, then there is some $s \geq 0$ such that $u(r,t_0) > w(r)$ for $r \neq s$ while $u(s,t_0) = w(s)$ and $u_r(s,t_0) = w'(s)$.

By the maximum principle we know that if the solution to the parabolic problem $u(r,t)$ and a steady state $w(r)$ intersect just once in some interval $0 < R_1 < r < R_2$ for $t = t_1$, and $u(R_j, t)$ does not cross $w(R_j)$ for $t \in [t_1, t_2]$, $j = 1, 2$, then again the comparison principle for parabolic equations gives, for all t in $[t_1, t_2]$, a unique intersection of $u(r, t)$ with $w(r)$ in (R_1, R_2). More precisely, the number of intersections for $t = t_2$ does not exceed the number of sign changes of $z(r, t) = u(r, t) - w(r)$ on the parabolic boundary of the domain $(R_1, R_2) \times (t_1, t_2)$. Moreover, suppose that $u(r, t)$ and $w(r)$ are tangent at some point $R_0 \in (R_1, R_2)$ at some time, say $t_0 \in (t_1, t_2)$, i.e., $r = R_0$ is a zero-tangency point of $z(r, t_0)$ (which satisfies $z(R_0, t_0) = z_r(R_0, t_0) = 0$). Then, by taking a slightly different steady state w_ε which crosses w at $r = R_0$, we can arrange things so that $u(r, t_0)$ crosses $w_\varepsilon(r)$ at least twice while $z_\varepsilon = u - w_\varepsilon$ again has a single sign-change on the parabolic boundary. This gives a contradiction. See [**9, 10**] for details. Thus, we have under the above hypothesis $|u_r - w'| > 0$ at the point of intersection for $t \in (t_1, t_2]$.

Applying the result to two isolated points $r_1(t), r_2(t)$, as above, we see that $u_r(r_1(t), t) - w'(r_1(t)) < 0 < u_r(r_2(t), t) - w'(r_2(t))$ for $0 < t < t_0$. Tangency only occurs when two or more points of intersection come together.

Now, choosing some $r_0 > 0$, let us assume that there is some value $M_0 \geq u_0(r_0)$ so that any steady state w which satisfies $w(r_0) \geq M_0$ intersects u_0 at most twice. Then, if at some time $t = t_0 > 0$, $u = M \geq M_0$ at $r = r_0$, then the "tangent steady state" w, with $w(r_0) = M$ and $w'(r_0) = u_r(r_0, t_0)$, crosses u_0 at most twice and is tangent to $u(\cdot, t_0)$ at $r = r_0$. From the above there are precisely two points of intersection between w and $u(\cdot, t)$ for $t < t_0$, with $u(r, t_0) > w(r)$ for $r \neq r_0$. It follows that $u_{rr}(r_0, t_0) \geq w''(r_0)$ and, from (1.1) and (2.1), $u_t(r_0, t_0) \geq 0$. Thus, for a given $r_0 \geq 0$, we have

$$u_t(r_0, t) \geq 0 \text{ at any time at which } u(r_0, t) \geq M_0. \tag{2.2}$$

Again, assuming that w and u_0 cross precisely twice and that w and $u(\cdot, t_0)$ touch at $r = r_0$, (so w and $u(\cdot, t_0)$ are tangent, without crossing, at that point) we now want a rather stronger result, namely,

$$u_t(r_0, t_0) > 0. \tag{2.3}$$

Fix $\delta > 0$ small enough and take an arbitrary $(r_1, t_1) \in Q_{\delta, t_0} = (r_0 - \delta, r_0 + \delta) \times (t_0 - \delta, t_0)$. Denote by $w_1(r)$ the stationary solution which is tangent to the profile $u(r, t_1)$ at $r = r_1$. By continuous dependence of solutions of the stationary ordinary differential equation (2.1) and by continuity of $u(r, t)$, we conclude that, for a given small $\varepsilon > 0$ (chosen so that the intersections of u with w lie in $(\varepsilon, 1/\varepsilon)$),

$$w_1(r) - w(r) \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow 0, \tag{2.4}$$

uniformly on $(\varepsilon, 1/\varepsilon)$. Note that $w_1(r)$ can be singular at the origin, see Section 2.2. Assuming, without loss of generality, that both intersections of u_0 and w are transversal (if not, then this becomes true after a small change of time origin), we can take δ small enough so that w_1 intersects u_0 twice in $[\varepsilon, 1/\varepsilon]$ and $|w_1(\varepsilon^{\pm 1}) - u(\varepsilon^{\pm 1}, t)| > 0$ for $0 \le t \le t_1$, we conclude from (2.4) that the number of intersections between u_0 and w_1 is less than or equal to three. However, in view of (2.4) and a possible singularity of w_1, we also deduce from the known regularity of the solution $u(r, t)$ that the third intersection (if it exists for all $t \in (0, t_0)$) stays near $r = 0$. Hence, whether or not the third intersection exists, we have only two points of intersection between $u(\cdot, t)$ and w_1 in $(\varepsilon, 1/\varepsilon)$ for all $t \in (0, t_1)$. As above, we conclude that

$$u_t(r_1, t_1) \ge 0 \quad \text{for all} \quad (r_1, t_1) \in Q_{\delta, t_0}. \tag{2.5}$$

Now, using the fact that the derivative $p = u_t$ solves a linear parabolic equation with smooth coefficients in the neighborhood of (r_0, t_0),

$$p_t = p_{rr} + p_r/r + e^u p,$$

either $p \equiv 0$ or, by the strong maximum principle, $p > 0$ in Q_{δ, t_0} [**5**]. The same argument applies for some (possibly smaller) $\delta > 0$, as u and its derivatives are bounded in $[\varepsilon, 1/\varepsilon] \times [0, t_0]$ for any $t_2 \le t_0$ such that $u(r_0, t_2) = u(r_0, t_0)$. Taking the smallest such t_2 we see that $p \not\equiv 0$. We arrive at (2.3).

We conclude that, if $u(\cdot, t)$ develops a point of tangency with the steady state w at $r = r_0$ through the coalescence of precisely two points of intersection, say $r_1(t) < r_2(t)$ with $(r - r_1)(r - r_2)(u - w) > 0$ for $r_1 \ne r \ne r_2$, then $u_t > 0$ at that point. (If $(r - r_1)(r - r_2)(u - w) < 0$ for $r_1 \ne r \ne r_2$, then $u_t < 0$ at the point of tangency.) Conversely, if $u_t = 0$ at $r = r_0$, $t = t_0 > 0$ with $u(r_0, t_0) = w(r_0)$, $u_r(r_0, t_0) = w'(r_0)$, then for $t < t_0$ there must be at least three distinct points of intersection between $u(\cdot, t)$ and w, all of which tend to r_0 as $t \to t_0$. Here $r = r_0$ is a point of inflection of the difference $z(r, t_0)$.

2.2. *Explicit steady states.* For the radially symmetric problem, a steady state $w = w(r)$ solves (2.1). If it is a classical steady solution, then w is bounded for r bounded and $w' \to 0$ as $r \to 0$. The ordinary differential equation may be solved explicitly (by putting $w = \psi - 2\ln r$ and $r = e^s$ to get an autonomous equation; see, for example, [**13**]) to find the general solution

$$w(r; a, \gamma) = -2\ln[ar^\gamma + r^{2-\gamma}/8a(1-\gamma)^2] \tag{2.6}$$

with a, γ arbitrary constants satisfying $a > 0$, $\gamma < 1$. (At times it is more convenient to write w in other forms.) All steady states are of this type so that, given any differentiable function $f(r)$ and any point $r = s$, it is possible to choose a unique pair of values (a, γ), $a > 0$ and $\gamma < 1$, such that $w = w(\cdot; a, \gamma)$ is tangent to f at $r = s : w(s) = f(s)$, $w'(s) = f'(s)$.

The properties of these steady solutions are discussed in detail in the Appendix. There are two key results of that analysis. The first is that there are three types of solution:

(i) for $\gamma = 0$, w is classical. It is bounded above with $w'(0) = 0$ and $w(0) = -2\ln a$ finite;

(ii) for $\gamma < 0$, w is singular but bounded above, $w \to -\infty$ as $r \to 0$. There is then a point $r = s$, where w achieves its maximum;

(iii) for $0 < \gamma < 1$, w is singular and unbounded both above and below but such w lie below the envelope, $-\ln(r^2/2)$ of the regular ($\gamma = 0$) solutions.

The second result is that there is some function $k(K)$ such that if w has a maximum $\geq M$ (so that $\gamma \leq 0$) and $M \geq k(K)$, then $|w'| > K$ wherever $|w| \leq K$.

These results could be obtained from the analysis of [**13**], as is done for the power case in Section 6, but they are contained in the Appendix as an alternative approach and for completeness of the present paper.

3. Use of intersection comparison with steady states.

Partial proof of Theorem 1. We take initial data $u(r,0) = u_0(r)$ for $r \geq 0$ satisfying $|u_0|, |u_0'| \leq K$ for some K. We now use the techniques of intersection comparison by proving the following result which is a weakened version of the theorem.

Proposition 1. *Assume that, for some* $r_0 \geq 0$, $t_0 \in (0,T)$,

$$u(r_0,t_0) > k(K), \tag{3.1}$$

where either

$$\begin{gathered} r_0 = 0 \quad \textit{or} \\ r_0 > 0 \quad \textit{and} \quad u(r_0,t_0) \geq -\ln(r_0^2/2). \end{gathered} \tag{3.2}$$

Then

$$u_t(r_0,t) > 0 \quad \textit{for all} \quad t \in [t_0,T). \tag{3.3}$$

Proof. Let us choose $M > k(K)$, which automatically means that the inequalities $M > K + 2\ln 2$, $M > 3K + 4\ln K$ are satisfied. Now suppose that, for some position $r = r_0 \geq 0$ and $t \geq t_0$, $u(r_0,t) \geq M$, and either $r_0 = 0$ or $M \geq -\ln(r_0^2/2)$.

Take the steady state $w = w(\cdot; a, \gamma)$ such that w is tangent to $u(\cdot, t_0)$ at $r = r_0$. Then, since $r_0 = 0$ or $w(r_0) \geq -\ln(r_0^2/2)$ for $r_0 > 0$, we must have $\gamma \leq 0$. With $\gamma \leq 0$ and w reaching or exceeding M we know that

(a) if $\gamma = 0$, then $w' < -K$ where $-K \leq w \leq K$ and so w crosses u_0 precisely once;

(b) if $\gamma < 0$, then $w' > K$ where $-K \leq w \leq K$ and $r < s$ while $w' < -K$ where $-K \leq w \leq K$ and $r > s$ so w crosses u_0 precisely twice.

Using the results from Section 2.1 concerning intersections of u with w, we have that $u(r,t_0) > w(r)$ for $r \neq r_0$ and $u_t(r_0,t_0) > 0$. For any time t while $u(r_0,t) > k(K)$, we have, by the above, $u_t(r_0,t) > 0$ so that $u(r_0,t) \geq u(r_0,t_0)$. This completes the proof. $\square$

Note. The special case $r_0 = 0$ requires that the argument be suitably modified, for example, as follows (or see either [**8**] or [**17**, p. 420]). A new comparison function $v(r,t;\varepsilon,\delta)$ can be defined as a solution to the same nonlinear heat equation as u but with initial data v_0 close to the regular steady state w which is tangent to $u(\cdot,t_0)$ as t_0. To be more precise, we can take $v_0(r)$ given by

$$v_0'' + v_0'/r + e^{v_0} = \begin{cases} \varepsilon & r < 1, \\ 0 & r > 1 \end{cases}$$

with $v_0'(0) = 0$, $v_0(0) = u(0,t_0) - \delta$. With δ small and positive $\varepsilon(\delta)$ is chosen so that $v(0,t_0;\varepsilon,\delta) = u(0,t_0)$. Such an ε is also small and positive, which guarantees that v_0 and u_0 cross precisely once, on again taking the intersection of w and u_0 to be transversal. Now $v(\cdot,t)$ and $u(\cdot,t)$ have at most one point of intersection, and, in particular, $v(0,t_0) = u(0,t_0)$, $v(r,t_0) < u(r,t_0)$ for $r > 0$, so $v(\cdot,t) < u(\cdot,t)$ for $t > t_0$. Also $v_t > 0$ for $t > 0$ since v_0 is a lower solution: $u_t(0,t_0) \geq v_t(0,t_0) > 0$.

4. Consequences.

4.1. *Blow-up at* $r = 0$. We say that blow-up occurs at $r = 0$ if there are some sequences $r_n \to 0$, $t_n \to T$ with $u(r_n,t_n) \to \infty$.

Let us suppose that there are points arbitrarily close to zero such that $u(r,t) \leq -\ln(r^2/2)$ for $0 \leq t < T$. Now choose $r = R$ to be such a position with $-\ln(R^2/2) \geq K$. We can pick the value $a = R/2\sqrt{2}$ so that $w = w(\cdot;a,0)$ touches the envelope $-\ln(r^2/2)$ at $r = R$. Then $u(R,t) \leq w(R)$ for $0 \leq t < T$, $u(r,0) \leq w(r)$ for $0 \leq r \leq R$ and consequently w is an upper solution for u in $0 \leq r \leq R$ so $u \leq w \leq -2\ln a = -\ln(R^2/8)$ for $0 \leq r \leq R$, $0 \leq t \leq T$. This would contradict the supposition that u blows up at $r = 0$.

Thus, for all r in $(0,R)$, for some $R > 0$, there is a time $t(r)$, $0 < t(r) < T$, such that $u(r,t)$ exceeds both $k(K)$ and $-\ln(r^2/2)$. Then it follows from Proposition 1 that $u_t > 0$ for $t > t(r)$, $0 < r < R$ and, in particular, if the solution still exists (in some weak sense) at $t = T$, so that for $0 < r < R, u(r,T) < \infty$, $u_t(r,T) > 0$ for $r < R$.

Should u continue to exist beyond the blow-up time T then the earlier arguments would continue to hold so that $u_t > 0$ and $u > -\ln(r^2/2)$

for $0 < r < R$, $T \leq t < T + \varepsilon$ for some $\varepsilon > 0$. Then for $t > T$, $u(0,t) = \infty$, the arguments of [**14**] may be applied, and u is actually infinite everywhere, see also [**1**]. We conclude that u blows up completely (so there is no continuation) at $t = T$.

4.2. *Blow-up at* $r = R > 0$. We denote the one-dimensional family of stationary solutions satisfying $w(R) = K$ by S. Let $W(r)$ be the envelope of S, $W(r) = \sup_{w \in S} w(r)$. Similar arguments to those of Section 4.1 then give that $u(r,T) > W(r)$ and $u_t(r,T) > 0$ in either a left or right neighborhood of $r = R$ with complete blow-up occurring at $t = T$.

5. Monotonicity for $u > M$.

Proof of Theorem 1. The results of Section 3 are incomplete in that, in view of Hypothesis 3.2, there appears to be a possibility of having points arbitrarily close to the origin with u arbitrarily large (but less than $-\ln(r^2/2)$) and $u_t \leq 0$.

Let us now assume that this is true so there are some sequences $r_n \to 0$, $t_n \to T$, $U_n \equiv u(r_n, t_n) \to \infty$, with $u_t(r_n, t_n) \leq 0$. We can suppose, omitting the first few values of n if necessary, that $U_n > K$ for all n, in which case $u(r_n, t_n) > u(r_n, 0)$. It follows that, for some values of t in $(0, t_n]$, $u_t(r_n, t) > 0$ and there must exist some time such that $u_t(r_n, t) = 0$ and $u(r_n, t) \geq U_n$. With an appropriate redefinition of t_n and U_n, we may then take $u_t(r_n, t_n) = 0$.

Assuming further that $U_n > k(K)$ for all n (which again can be ensured by omitting some n) and recalling by Proposition 1 that $u_t > 0$ if $u > k(K)$ and $u \geq -\ln(r^2/2)$, we must have $r_n < \sqrt{2}e^{-U_n/2}$.

Now let $w_n = w(\cdot; a_n, \gamma_n)$ be the steady state tangent to $u(\cdot, t_n)$ at $r = r_n$. Since $u(r_n, t_n) = U_n > k(K)$ any steady state with $\gamma_n \leq 0$ must intersect u_0 at most twice (Sections 2, 3) which would give $u_t(r_n, t_n) > 0$ and contradict the assumed property $u_t(r_n, t_n) = 0$. Thus $0 < \gamma_n < 1$ (as well as $a_n > 0$). Indeed, from Section 2, w_n must intersect $u(\cdot, t)$ at least three times near $u = U_n$ and $r = r_n$.

Now choose n large enough and a_0 small enough to make $w_0 = w(\cdot; a_0, 0)$ intersect u_0 just once and the first intersection point, $r = R$, between w_0 and w_n be such that $u_0(r) < w_0(r) \leq w_n(r)$ for $r \leq R$ (recall that w_n lies below the envelope $-\ln(r^2/2)$ which w_0 touches at $r = 2\sqrt{2}a_0$, $w = -2\ln(2a_0)$). See Figure 1(a).

As long as $u(R,t)$ remains less than $w_0(R) = w_n(R) \equiv W$ ($> -2\ln a_0$ by the above) w_0 acts as an upper solution for u in $r < R$ so $u(r,t) < w_0(r) \leq w_n(r)$ for $r \leq R$. If $u(R,t)$ reaches W at time $t = T_1$, i.e., $u(R,T_1) = W$, then by the strong maximum principle, $u_r(R,T_1) > w_0'(R) > w_n'(R)$, $u(r,T_1) < w_0(r) < w_n(r)$ for $r < R$, $u(R,T_1) = w_0(R) = w_n(R) = W$, and $u(r,T_1) > w_0(r) > w_n(r)$ in some right neighborhood of $r = R$, see Figure 1(b). Thus, $u(\cdot,T_1)$ crosses w_n precisely once in $0 \leq r < R + \varepsilon$ for some $\varepsilon > 0$. If $u(R,t)$ then becomes greater than W for $t > T_1$, there is again precisely one intersection point between $u(\cdot,t)$ and w_n for $0 \leq r < R + \varepsilon$; indeed, as long as $u(R,t) > W$, $u(\cdot,t)$ and w_n cross at most once for $r \in [0,R]$, see Figure 1(c). Continuing this type of argument, suppose now that $u(R,t)$ falls back to W at time $t = T_2 > T_1$, then the arguments of Section 2, with the uniqueness of the intersection between u and w_0, show that the situation is essentially that of $t = T_1$. Since w_0 and $u(\cdot,t)$ continue to cross at most once, if $u(R,t) > W$ there is at most one point of intersection between $u(\cdot,t)$ and w_n in $(0,R)$, while if $u(R,t) \leq W$ there is no such crossing point in that interval, see Figure 1.

We see that, for all times $t < T$, there can be at most one point of intersection between w_n and $u(\cdot,t)$ for $r \in (0,R)$, i.e., above $u = W < -2\ln a_0$. That there can be at most one point of intersection between $u(\cdot,t)$ and w_n, with u greater than $w_0(0) = -2\ln a_0$, contradicts the need for at least three points of intersection to coalesce where $u = U_n$ when n is sufficiently large so that $U_n > -2\ln a_0$.

This completes the proof of Theorem 1. $\square$

6. Monotonicity for the equation with power-nonlinearity.

In this section we show that the method we have described above can be directly applied to equation (1.5), as long as (1.6) holds, to prove Theorem 2. Our analysis consists of several steps. We consider the nonnegative solution $u = u(r,t) \geq 0$ in Q_T of the Cauchy problem (1.5), (1.2), (1.3).

6.1. *Intersection properties.* We first note that all the properties of

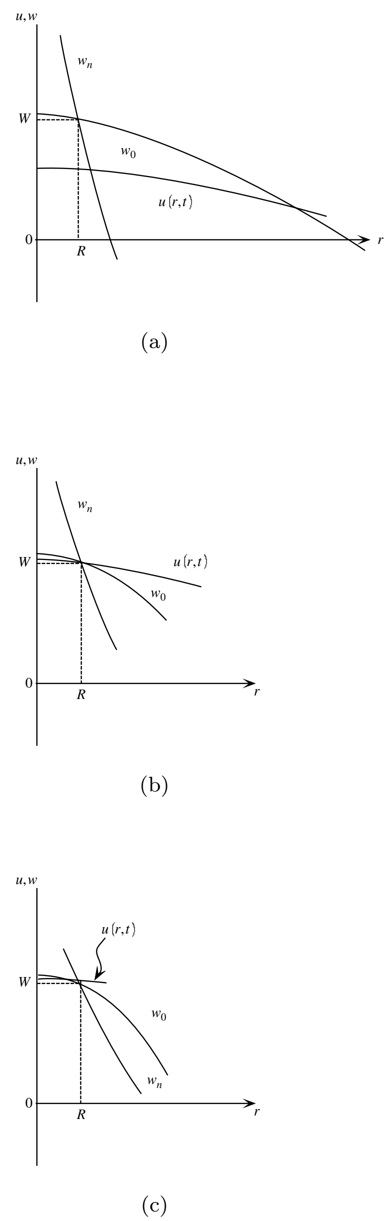

FIGURE 1. Crossings of $u(\cdot,t)$ with w_0, w_n: (a) $t < T_1$; (b) $t = T_1$ or T_2; (c) $T_1 < t < T_2$.

intersection with stationary solutions, that is, with solutions $w(r)$ of

$$Lw \equiv w'' + (N-1)w'/r + w^p = 0, \tag{6.1}$$

as given in Section 2.1, still hold.

6.2. *Steady states.* Properties of the set of stationary solutions $w = w(r; a, \mu, \lambda)$ satisfying equation (6.1) with the conditions

$$w = \lambda \quad \text{and} \quad w' = \mu \quad \text{at } r = a, \tag{6.2}$$

where $a \geq 0$, $\lambda > 0$, $\mu \in \mathbf{R}$, are well known [**13**]. (Note that this parameterization is not unique; we can have $w(\cdot; a_1, \mu_1, \lambda_1) = w(\cdot; a_2, \mu_2, \lambda_2)$ with $a_1 \neq a_2$, $\mu_1 \neq \mu_2$, $\lambda_1 \neq \lambda_2$.) We state two basic results which are necessary for our comparison argument:

If (1.6) holds, then any regular steady state $w_0(r; \lambda) \equiv w(r; 0, 0, \lambda)$, with $\lambda > 0$, vanishes at a finite value of r, r_0, and

$$r_0 = r_0(\lambda) = r_0(1)\lambda^{(1-p)/2} \longrightarrow 0 \quad \text{as} \quad \lambda \longrightarrow \infty. \tag{6.3}$$

Moreover, for a constant $c > 0$,

$$\begin{gathered} |w_0'(r; \lambda)| \longrightarrow \infty \quad \text{as} \quad \lambda \longrightarrow \infty \\ \text{on the level set } \{w_0 = c\}; \end{gathered} \tag{6.4}$$

indeed, this holds uniformly in $c \in (0, C]$ for any fixed $C > 0$.

Observe that, in view of the similarity invariance of equation (6.1), we have

$$w_0(r; \lambda) \equiv \lambda w_0(r\lambda^{(p-1)/2}; 1). \tag{6.5}$$

We now check that the properties of the derivatives for steady solutions which are bounded above, as found in Section 2.2, again hold.

Proposition 2. *Let* (1.6) *hold. There exists* $M = M(K) > 0$ *such that, for any* $\lambda \geq M$ *and any* $a \geq 0$, *the steady state* $w = w(r; a, 0, \lambda)$ *satisfies*

$$|w'| \geq K \quad \textit{on any level set} \quad \{w = c \in (0, K]\}. \tag{6.6}$$

Proof. It follows from (6.4) that (6.6) is true for $a = 0$. Since, if $a \gg 1$, the solution $w(r; a, 0, \lambda)$ is almost one-dimensional in the sense that, after the change of the independent variable, $r = a + y$, it solves an asymptotically small perturbed, one-dimensional equation

$$w_{yy} + w^p = -(N-1)w_y/(a+y) = O(1/a), \tag{6.7}$$

the result is true for all $a \gg 1$. Evidently (6.6) holds for solutions of the unperturbed equation

$$v_{yy} + v^p = 0, \tag{6.8}$$

(this property of the exact one-dimensional problem is fundamental to the result of [**9**]) and then, by continuity with respect to a small perturbation to the ordinary differential equation (6.8), it is true for (6.7).

Now fix $a > 0$. Then, by monotonicity, we have that

$$w'' + w^p = -(N-1)w'/r < 0 \quad \text{for} \quad r < a,$$

and one can see immediately by comparison that $w(r)$ lies below the one-dimensional function $v(r-a)$ for $r < a$. Taking other translations of v, the comparison argument can be extended to show that w is steeper than v; more precisely, if $w(r) = v(y)$ with $r < a$ and $y < 0$, then $0 < v'(y) < w'(r)$. Thus (6.6) holds for all those r where $w' > 0$.

Now take $r > a$. Then, for any $b \in (0, a]$, we see that the function $w_0(r-b; \lambda)$ is a subsolution for equation (6.1), that is,

$$Lw_0(r-b;\lambda) = -b(N-1)w_0'/r(r-b) > 0 \quad \text{for} \quad r > b.$$

Therefore, by a standard comparison argument, we can deduce that $w(r) \le w_0(r-a;\lambda)$ for $r > a$, and, moreover, that w is steeper than w_0 for $r > a$. Using (6.4), we can then conclude that (6.6) holds at points on the level set where $w' < 0$.

This completes the proof. $\square$

6.3. *Partial proof of Theorem* 2. Let us now introduce the envelope of the set of steady states $\{w_0(r;\lambda)\}$:

$$V(r) = \sup_{\lambda > 0} w_0(r;\lambda) = a_0 r^{-2/(p-1)} \quad \text{for} \quad r > 0, \tag{6.9}$$

where $a_0 = a_0(p, N)$ is a constant. (The equation for the envelope V is easily calculated by using the invariance (6.5), see [**17**, p. 424].)

It then follows from Proposition 2 that the result of Proposition 1 is also true in the present case. The only difference is that we have to modify (3.2) by using the new envelope:

$$u(r_0, t_0) \geq a_0 r_0^{-2/(p-1)}. \tag{6.10}$$

We also replace k in (3.1) by $M(K)$ given in Proposition 2.

6.4. *Proof of Theorem* 2. Following the scheme of the analysis in Section 5 we need only check the following property of the set $\{w\}$ of steady states. We retain the notation of Section 5. In particular, we denote by w_n a singular steady state and, as above, w_0 is a regular solution.

Proposition 3. *Any w_0 intersects an arbitrary singular steady state w_n. In other words we always have, qualitatively, the relative position of profiles w_n, w_0 and $u(\cdot, t)$ shown in Figure* 1.

Proof. This can be obtained from the general properties of stationary solutions $\{w\}$ studied, by reducing (6.1) to a first order ordinary differential equation, in [**13**]. (There is also an evolution argument, see [**17**, Chapter 4].

Using Proposition 3 we can directly apply our geometric analysis described step by step in Figure 1. This concludes the proof of Theorem 2. $\square$

The proof of Theorem 3 is very similar. Necessary properties of stationary solutions to (1.8) can be found in [**11**].

7. Other problems.

7.1. *Constant boundary condition on $r = R > 0$.* For an initial-value problem posed in $r < R < \infty$ with a constant Dirichlet condition at $r = R$, we can assume, without loss of generality, that $u(R, t) = 0 = u_0(R)$. The intersection arguments carry over essentially unchanged since, taking $\gamma \leq 0$ and $\max\{w\}$ large, we still know that $|w'| > K \geq |u_0'|$ where $|w| \leq K$, i.e., at any point where u_0 and w may intersect. Then w and u_0, and hence w and $u(\cdot, t)$, can cross at most twice. These conclusions also apply to an annular domain, $0 < R_1 < r < R_2 < \infty$.

The method may be adapted to obtain the same results for Robin conditions, say $u_r + bu = 0$ on $r = R$, and for Neumann boundary conditions, $u_r = b$ on $r = R$.

7.2. *Varying condition at* $r = R > 0$. Taking the time-dependent Dirichlet condition $u(R, t) = g(t)$ for $0 < t < T$, again the method extends, with little modification, if g is ultimately increasing, i.e., $g' \geq 0$ for $t \in (T - \varepsilon, T)$ for some $\varepsilon > 0$. Starting at $t = T - \varepsilon$, one more intersection between $u(\cdot, t)$ and some w with $\gamma \leq 0$ can be created at $r = R$ as $g(t)$ increases through $w(R)$. However, for $\max\{w\}$ large, this can still only give a maximum of two crossings.

If g is instead decreasing at times arbitrarily close to T, the argument fails as more intersection points may appear at $r = R$. This failure should not be too surprising, and our results are in some sense best possible, as we can observe that any similarity solution, $u = -\ln(T - t) + v(r/(T-t)^{1/2})$, where $v(\eta)$ satisfies $v'' + v'/\eta - \eta v'/2 + e^v - 1 = 0$, $v'(0) = 0$, has $v(\eta) \to -\infty$ exponentially fast as $\eta \to \infty$. Then $u(0, t) \to \infty$, $u(r, t) \to -\infty$ for $r > 0$, and there are points arbitrarily close to 0 with $u(r, t) \to \infty$, $u_t(r, t) < 0$ as $t \to T$, [**3**].

7.3. *The nonlinear diffusion problem* $u_t = \Delta\phi(u) + e^{\phi(u)}$ *with* $\phi' > 0$. The key steps in the original argument use intersections with w satisfying $\Delta w + e^w = 0$. The steady states of this new problem satisfy $\Delta\phi(u) + e^{\phi(u)} = 0$ so the arguments carry over using intersections between u and $\phi^{-1}(w)$ (or $\phi(u)$ and w).

7.4. *Higher dimensions for* $u_t = \Delta u + e^u$. Again we remark that our results cannot extend to three dimensions for $f(u) = e^u$, where there is a weak singular steady state, and solutions u exist which have points $r \to 0$, $t \to T$, so that $u(r, t) \to \infty$ but $u_t(r, t) < 0$. We may regard our present results as optimal.

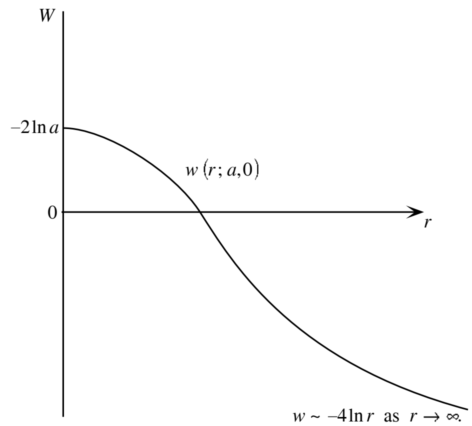

FIGURE 2. Regular steady state, $\gamma = 0$.

7.5. *Generalizations of equation* (1.7). There is, in principle, no difficulty in extending the result to a more general equation

$$u_t = \Delta\phi(u) + f(u),$$

where $\phi'(u) > 0$ and $f(u) > 0$ for $u > 0$ are assumed to be smooth and monotone increasing. The property given in Proposition 2 can be obtained by using Pohozaev's identity, see [**16**].

Appendix

We shall now briefly look at some properties of the three classes of radially symmetric steady state for the exponential in two dimensions.

(i) $\gamma = 0$. Here $w(r) = w(r; a, 0) = -2\ln(a + r^2/8a)$ is a regular (classical) steady state, see Figure 2.

The maximum of w is $w(0) = -2\ln a$.

Differentiating the solution yields $-w' = r/2a(a + r^2/8a) > 0$. Indeed, $-w' \geq r/2ae^{K/2}$ at every point where $w \geq -K$ (so $a + r^2/8a \leq e^{K/2}$). Moreover, at points where additionally $w \leq K$, we have that $a + r^2/8a \geq e^{-K/2}$ so that $(r/a)^2 + 8 \geq 8e^{-K/2}/a$ and $(r/a)^2 > 4e^{-K/2}/a$ if $a < e^{-K/2}/2$. In this case $r/a > 2a^{-1/2}e^{-K/4}$ and $-w' > a^{-1/2}e^{-3K/4} > K$ for $a < K^{-2}e^{-3K/2}$ where $-K \leq w \leq K$.

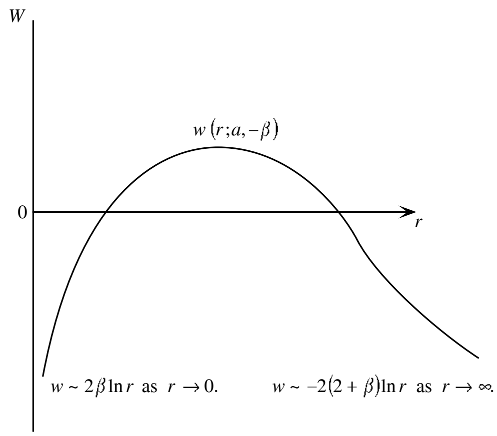

FIGURE 3. Singular steady state bounded above, $\gamma < 0$.

We see that, for one of these regular solutions to reach or exceed a value M ($\sup w \geq M$) $w(0) \geq M$ so $a \leq e^{-M/2}$. Thus, for $M > K + 2\ln 2$, $M > 3K + 4\ln K$, we may choose $a = e^{-M/2}$ and then $-w' > K$ wherever $-K \leq w \leq K$.

(ii) $\gamma < 0$. Now, putting $\gamma = -\beta$, $\beta > 0$, and

$$\begin{aligned} w(r) = w(r; a, -\beta) &= -2\ln[a/r^\beta + r^{2+\beta}/8a(1+\beta)^2] \\ &= \ln[8(1+\beta)^2] - 2\ln[A/r^\beta + r^{2+\beta}/A], \end{aligned}$$

where $A = 2\sqrt{2}(1+\beta)a > 0$. Here w is singular but bounded from above, see Figure 3.

Again, differentiating the solution, we obtain

$$w' = 2[\beta A/r^{\beta+1} - (2+\beta)r^{\beta+1}/A][A/r^\beta + r^{2+\beta}/A]^{-1} = 0$$

at the unique point $r = s$ with $s^{1+\beta}((2+\beta)/\beta)^{1/2} = A$.

At $r = s$, the solution takes its maximum value W given by

$$W = \ln[8(1+\beta)^2] - 2\ln\left[s\left(\left(\frac{2+\beta}{\beta}\right)^{1/2} + \left(\frac{2+\beta}{\beta}\right)^{-1/2}\right)\right].$$

Generally,

$$w - W = -2\ln\left\{\frac{R\left[((2+\beta)/\beta)^{1/2}/R^{1+\beta} + (\beta/(2+\beta))^{1/2}R^{1+\beta}\right]}{((2+\beta)/\beta)^{1/2} + (\beta/(2+\beta))^{1/2}}\right\},$$

where $R = r/s$.

For w to reach or exceed M somewhere $W \le M$, so where $w \le K$,

$$R\biggl[\biggl(\frac{2+\beta}{\beta}\biggr)^{1/2}\Big/R^{1+\beta}+\biggl(\frac{\beta}{2+\beta}\biggr)^{1/2}R^{1+\beta}\biggr]\Big/ \biggl[\biggl(\frac{2+\beta}{\beta}\biggr)^{1/2}+\biggl(\frac{\beta}{2+\beta}\biggr)^{1/2}\biggr] \ge 2B$$

with $2B = e^{(M-K)/2}$.

For $M > K + 2\ln 2$, we have $B > 1$.

There are now two possibilities:

(a) $R^{-\beta} > B$, in which case $R < 1$,

or

(b) $R^{2+\beta} > ((2+\beta)/\beta)B > B$, in which case $R > 1$.

Requiring that $-K \le w \le K$ so that we also have

$$\begin{aligned} w &= \ln[8(1+\beta)^2] \\ &\quad - 2\ln\biggl[s^{1+\beta}\biggl(\frac{2+\beta}{\beta}\biggr)^{1/2}\Big/r^{\beta}+r^{2+\beta}\biggl(\frac{\beta}{2+\beta}\biggr)^{1/2}\Big/s^{1+\beta}\biggr] \\ &\ge -K, s^{1+\beta}\biggl(\frac{2+\beta}{\beta}\biggr)^{1/2}\Big/r^{\beta}+r^{2+\beta}\biggl(\frac{\beta}{2+\beta}\biggr)^{1/2}\Big/s^{1+\beta} \\ &\le 2\sqrt{2}(1+\beta)e^{K/2}, \end{aligned}$$

and $w' = 2\sqrt{\beta(2+\beta)}[(s/r)^{\beta+1} - (r/s)^{\beta+1}]/[s^{1+\beta}(2+\beta)/\beta)^{1/2}/r^{\beta} + r^{2+\beta}(\beta/(2+\beta))^{1/2}/s^{1+\beta}]$ satisfies

$$w' \ge (1/\sqrt{2})e^{-K/2}\sqrt{(\beta(2+\beta)}/(1+\beta))(R^{-(1+\beta)} - R^{1+\beta})$$

where $R = r/s < 1$,

$$-w' \ge \frac{1}{\sqrt{2}}e^{-K/2}\frac{\sqrt{\beta(2+\beta)}}{(1+\beta)}(R^{1+\beta} - R^{-(1+\beta)}) \quad \text{where } R = r/s > 1.$$

We now examine the two separate cases in $-K \le w \le K$.

(a) $R^{-\beta} > B > 1$, $R < 1$. Then

$$\frac{\sqrt{\beta(2+\beta)}}{(1+\beta)}(R^{-(1+\beta)} - R^{1+\beta}) \geq \frac{\sqrt{\beta(2+\beta)}}{2(1+\beta)}R^{-(1+\beta)},$$

on taking M sufficiently large so that $B \geq \sqrt{2}$, i.e., $M \geq K + 3\ln 2$, which ensures that $R^{1+\beta}/R^{-(1+\beta)} < 1/R^{-2\beta} < 1/B^2 \leq 1/2$. Now

$$\frac{\sqrt{\beta(2+\beta)}}{2(1+\beta)}R^{-(1+\beta)} \geq \frac{\sqrt{\beta(2+\beta)}}{2(1+\beta)}B^{(1+\beta)/\beta} > \frac{B}{2}\sqrt{\frac{\beta}{1+\beta}}B^{1/\beta} \geq CB,$$

where $C = (1/2)\inf_{\beta\in\mathbf{R}^+}\{\sqrt{\beta/(1+\beta)}2^{1/2\beta}\} = (1/2)\sqrt{(e/2)\ln 2}$.

(b) $R^{2+\beta} > ((2+\beta)/\beta)B > B > 1$, $R > 1$. Since $R > 1$, we have

$$R^{1+\beta} = (R^{2+\beta})^{(1+\beta)/(2+\beta)} > (R^{2+\beta})^{1/2} > \sqrt{\beta}$$
$$\text{as } \frac{1+\beta}{2+\beta} > \frac{1}{2} \quad \text{for } \beta > 0$$

and $R^{-(1+\beta)}/R^{1+\beta} < 1/B$. Then

$$\frac{\sqrt{\beta(2+\beta)}}{(1+\beta)}(R^{1+\beta} - 1/R^{1+\beta}) \geq \frac{\sqrt{\beta(2+\beta)}}{2(1+\beta)}R^{1+\beta}$$

on taking M sufficiently large so that $B \geq 2$, i.e., $M \geq K + 4\ln 2$. Now

$$\begin{aligned}\frac{\sqrt{\beta(2+\beta)}}{2(1+\beta)}R^{1+\beta} &> \frac{1}{2}\sqrt{\frac{\beta}{1+\beta}}\left[\frac{(2+\beta)}{\beta}B\right]^{(1+\beta)/(2+\beta)}\\ &> \frac{1}{2}\sqrt{\frac{\beta}{1+\beta}}\left[\frac{(2+\beta)}{\beta}B\right]^{1/2} > \frac{\sqrt{\beta}}{2},\end{aligned}$$

as $B > 1$ and $\beta > 0$.

Taking $M > k(K) = \max\{K + 4\ln 2, 2K + 2\ln K + 3\ln 2 - 2\ln C, 3K + 4\ln K + 8\ln 2\}$ then ensures that, for $\gamma < 0$, i.e., $\beta > 0$, $w' > K$ where $-K \leq w \leq K$ and $r < s$, and $w' < -K$ where $-K \leq w \leq K$ and $r > s$.

(iii) $0 < \gamma < 1$. Here $w(r) = w(r; a, \gamma) = -2\ln[ar^\gamma + r^{2-\gamma}/8a(1-\gamma)^2]$ is unbounded from above and $w' < 0$ for all r, see Figure 4.

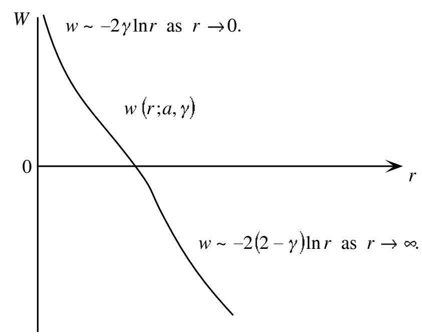

FIGURE 4. Singular steady state without upper bound ($0 < \gamma < 1$).

Fixing γ and r, $\partial w/\partial a = -2[r^{\gamma} - r^{2-\gamma}/8a^2(1-\gamma)^2]/[ar^{\gamma} + r^{2-\gamma}/8a(1-\gamma)^2]$ so we see that w achieves its maximum value (as a function of a) for $a = r^{1-\gamma}/2\sqrt{2}(1-\gamma)$. Then $w = -2\ln[r/\sqrt{2}(1-\gamma)]$. We see that these singular solutions all lie below $-2\ln(r/\sqrt{2}) = -\ln(r^2/2)$ which is the envelope of the regular ($\gamma = 0$) steady solutions.

Acknowledgments. This work was supported by the British Council and Spanish Ministry of Education and Science (Grant no. 92/7161) and the Science and Engineering Research Council (Grant no. GR/F86427).

REFERENCES

1. P. Baras and L. Cohen, *Complete blow-up after* $T_{\max}$ *for the solution of a semilinear heat equation*, J. Funct. Anal. **71** (1987), 142–174.

2. J. Bebernes and D. Eberly, *Mathematical problems from combustion theory*, Appl. Math. Sci. **83** (1989).

3. D. Eberly, *On the nonexistence of solutions to the Kassoy problem in dimensions* 1 *and* 2, J. Math. Anal. Appl. **129** (1988), 401–408.

4. D. Eberly and W. Troy, *On the existence of logarithmic-type solutions to the Kassoy-Kapila problem in dimensions* $3 \leq n \leq 9$, J. Differential Equations **70** (1987), 309–324.

5. A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

6. A. Friedman and J.B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J. **34** (1985), 425–447.

7. V.A. Galaktionov, *Asymptotics of unbounded solutions of nonlinear heat equation* $u_t = (u^\sigma u_x)_x + u^\beta$ *in a neighbourhood of a 'singular' point*, Dokl. AN SSSR **288** (1986), 1293–1297, in Russian; Soviet Math. Dokl. **33** (1986), 840–844, in English.

8. ———, *Monotonicity in time at the single point for the semilinear heat equation with source*, J. Differential and Integral Equations **4** (1991), 1089–1099.

9. V.A. Galaktionov and S.A. Posashknov, *Applications of new comparison theorems in the investigation of unbounded solutions of nonlinear parabolic equations*, Differential'nye Uraveniya **22** (1986), 1165–1173, in Russian; Differential Equations **22** (1986), 809–815, in English.

10. V.A. Galaktionov and S.A. Posashkov, *Any large solution of a non-linear heat conduction equation becomes monotonic in time*, Proc. Roy. Soc. Edinburgh. Sect. A **118** (1991), 13–20.

11. ———, *Single point blow-up for N-dimensional quasilinear equations with gradient diffusion and source*, Indiana Univ. Math. J. **40** (1991), 1041–1060.

12. ———, *On some monotonicity in time properties for a quasilinear parabolic equation with source*, in *Nonlinear diffusion equations and their equilibrium states*, Vol. 3 (N.G. Lloyd et al., eds.), Birkhauser, Boston, 1992, 77–93.

13. D.D. Joseph and T.S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rat. Mech. Anal. **49** (1973), 241–269.

14. A.A. Lacey and D.E. Tzanetis, *Complete blow-up for a semilinear diffusion equation with a sufficiently large initial condition*, IMA J. Appl. Math. **41** (1988), 207–215.

15. ———, *Global, unbounded solutions to a parabolic equation*, J. Differential Equations **101** (1993), 80–102.

16. S.I. Pohozaev, *Eigenfunction of the equation* $\Delta u + \lambda f(u) = 0$, Dokl. Acad. Nauk SSSR **165** (1965), 36–39, in Russian; Soviet Math. Dokl. **6** (1965), 1408–1411, in English.

17. A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov and A.P. Mikhailov, *Blow-up in quasilinear parabolic equations*, Nauka, Moscow, 1987, in Russian; Walter de Gruyter, Berlin, 1995, in English.

18. C. Sturm, *Mémoire sur une classe d'équations à différences partielles*, J. Math. Pure Appl. **1** (1836), 373–444.

Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Miusskaya Sq. 4, 125047 Moscow, Russia

Current address: Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK

Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, UK
E-mail address: A.A.Lacey@ma.hw.ac.uk