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THE ERGODIC THEOREM FOR TOPOLOGICAL COCYCLES

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Cocycle occur in purely algebraic, measure theoretic, topological and differentiable settings. The focus of this paper is continuous vector-valued cocycles for lattice actions on compact metric spaces. Since these cocycles satisfy the same inequality as a bounded linear operator, it is natural to examine them for some kind of asymptotic linear behavior. Specifically, for most points in the space is a "time" average asymptotic to linear map? Any theorem of this type can be thought of as an ergodic theorem for cocycles.

In 1991 Boivin and Derriennic [**1**] published a definitive cocycle ergodic theorem which generalizes the Birkhoff individual ergodic theorem. The context of their results is a measurable action of the integer lattice $\mathbf{Z}^m$ or the vector space $\mathbf{R}^m$ and an invariant probability measure μ. When the acting group is $\mathbf{Z}^m$, restrictions of the cocycle to the generators of $\mathbf{Z}^m$ is required to be in the Lorentz space $L_1(m,\mu)$, and the proof depends on several delicate weighted maximal inequalities.

An ergodic theorem of the same type for $\mathbf{Z}^m$ was first stated by Katok [**2**] without proof for an ergodic probability measure and cocycles whose restrictions to generators were in $L_1(\mu)$. A full discussion of the relationship between these two theorems can be found in [**1**].

The purpose of this note is to show for continuous cocycles that an ergodic theorem of the same type can be proved by quite different and more elementary methods than those used by Boivin and Derriennic. Our proof uses the familiar Birkhoff individual ergodic theorem and the Poincaré recurrence theorem from the theory of a single measure preserving transformation and passes to the $\mathbf{Z}^m$ action with an application of the Ascoli lemma which is the distinctive feature of our approach.

The restriction to continuous cocycles for commuting homeomorphisms is not artificial. These cocycles provide a way of modeling

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and studying continuous actions of $\mathbf{R}^m$ on metric spaces [**9, 4**]. Furthermore, the growth and asymptotic properties of these cocycles are important ingredients in their study [**3, 8**], and asymptotic averages of the kind considered here play an important role in earlier works. (See [**5, 6, 7**].) This paper closes a gap in the theory of continuous cocycles by providing a simple proof of a cocycle ergodic theorem in that context and thereby a better understanding of the asymptotic behavior of continuous cocycles.

Throughout this paper, X will be a compact metric space on which the integer lattice group $\mathbf{Z}^m$ is acting as a group of homeomorphisms. The action of $a \in \mathbf{Z}^m$ on $x \in X$ will be denoted by ax. It will be convenient to use the norm

$$|v| = \sum_{i=1}^{m} |v_i|$$

on $\mathbf{Z}^m$ and $\mathbf{R}^m$.

A *topological cocycle* or simply a *cocycle* in the present context is a continuous function $h : X \times \mathbf{Z}^m \to \mathbf{R}$ satisfying

$$h(x, a + b) = h(x, a) + h(ax, b)$$

for all $x \in X$ and all $a, b \in \mathbf{Z}^m$. Setting

$$\|h\| = \sup\left\{\frac{|h(x,a)|}{|a|} : x \in X \text{ and } a \in \mathbf{Z}^m, a \neq 0\right\}$$

defines a norm on the vector space $\mathcal{C}$ of all topological cocycles. With this norm $\mathcal{C}$ is a separable Banach space. Several essential properties of continuous cocycles readily follow from these definitions. First $h(bx, -b) = -h(x, b)$ because $0 = h(x, 0) = h(x, b - b)$. It then follows that $h(bx, a - b) = h(x, a) - h(x, b)$. From the definition of the norm on $\mathcal{C}$ it is obvious that $|h(x, a)| \leq \|h\||a|$ and then, using the preceding observations, it follows that

$$|h(x, a) - h(x, b)| \leq \|h\||a - b|.$$

The linear maps from $\mathbf{R}^m$ to $\mathbf{R}$ will play a special role, and the vector space of these linear maps will be denoted by $\mathcal{L}$. Given $L \in \mathcal{L}$, setting

$h(x,a) = L(a)$ defines a cocycle, usually called a constant cocycle, and so $\mathcal{L} \subset \mathcal{C}$.

The nonempty set of invariant Borel probability measures for the $\mathbf{Z}^m$ action will be denoted by $\mathcal{M}$. A Borel set E of X is said to have *invariant measure one* provided $\mu(E) = 1$ for all $\mu \in \mathcal{M}$.

We can now state the main theorem:

Theorem 1. *There exists a Borel set Q of invariant measure one and for each $x \in Q$ a linear map $L_x : \mathcal{C} \to \mathcal{L}$ such that*

$$\lim_{|a|\to\infty} \frac{|h(x,a) - L_x(h)(a)|}{|a|} = 0$$

for all $h \in \mathcal{C}$ and $x \in Q$.

The proof has two main components, one measure theoretic and the other topological. The measure theoretic part is concerned with an analysis of limits of the form

$$\lim_{N\to\infty} \frac{h(x,Na)}{N}$$

for fixed $a \in \mathbf{Z}^m$. In the topological component, the cocycle is extended to $X \times \mathbf{R}^m$ and limits in the compact open topology are linked to those of the form described above.

Let $\mu \in \mathcal{M}$, and let $h \in \mathcal{C}$. For fixed $a \in \mathbf{Z}$, the transformation $x \to ax$ preserves the measure μ and the individual ergodic theorem for $\mathbf{Z}$ actions can be applied to it. Since

$$\frac{h(x,Na)}{N} = \frac{1}{N} \sum_{k=0}^{N-1} h((ka)x, a)$$

when N is a positive integer, by the individual ergodic theorem

$$\lim_{N\to\infty} \frac{h(x,Na)}{N} = h^*(x,a)$$

μ-almost everywhere and $h^*(,a) \in L_1(\mu)$.

Because $\mathbf{Z}^m$ is countable, there exists a set $E_0(h)$ of μ measure one such that, for all $a \in \mathbf{Z}^m$,

$$\lim_{N\to\infty} \frac{h(x,Na)}{N} = h^*(x,a)$$

when $x \in E_0(h)$. Notice that, from the equation,

$$h(bx,Na) = h(x,Na) - h(x,b) + h(Nax,a)$$

and the continuity of h, it follows that

$$h^*(bx,a) = h^*(x,a).$$

Thus, we may assume without loss of generality that the set $E_0(h)$ is invariant under $\mathbf{Z}^m$.

Proposition 1. *Given $\mu \in \mathcal{M}$ and $h \in \mathcal{C}$, there exists a set $E(h)$ of μ measure one such that, for all $x \in E(h)$ and $a, b \in \mathbf{Z}^m$,*

$$h^*(x,a+b) = h^*(x,a) + h^*(x,b).$$

Proof. For any pair of positive integers K and M, set

$$A(K,M) = \left\{ x \in E_0(h) : \left| \frac{h(x,\pm Ne_i)}{N} - h^*(x,\pm e_i) \right| < \frac{1}{M} \text{ for all } N \geq K \right\},$$

where $\{e_1, e_2, \ldots, e_m\}$ is the standard set of generators for $\mathbf{Z}^m$. Clearly

$$A(K,M) \subset A(K+1,M)$$

and

$$E_0(h) = \bigcap_{M=1}^{\infty} \left(\bigcup_{K=1}^{\infty} A(K,M) \right)$$

and, consequently,

$$\lim_{K\to\infty} \mu(A(K,M)) = 1.$$

Now set

$$\begin{aligned}&B(K,M)\\&=\{x\in A(K,M):\forall\, a\in \mathbf{Z}^m \&\forall\, L>0\ \exists k\ge L\ni (ka)x\in A(K,M)\}.\end{aligned}$$

It follows that $\mu(B(K,M))=\mu(A(K,M))$ by applying the Poincaré recurrence theorem to each transformation $x\to ax$. Also note that

$$B(K,M)\subset B(K+1,M).$$

Define the set $E(h)$ by

$$E(h)=\bigcap_{M=1}^{\infty}\Big(\bigcup_{K=1}^{\infty}B(K,M)\Big).$$

Clearly, $\mu(E(h))=1$.

To complete the proof it suffices to show that

$$h^*(x,a\pm e_i)=h^*(x,a)\pm h^*(x,e_i)$$

for $x\in E(h)$. First

$$\begin{aligned}h^*(x,a+e_i)&=\lim_{N\to\infty}\frac{h(x,N(a+e_i))}{N}\\&=\lim_{N\to\infty}\frac{h(x,Na)}{N}+\frac{h((NA)x,Ne_i)}{N}\\&=h^*(x,a)+\lim_{N\to\infty}\frac{h((Na)x,Ne_i)}{N}.\end{aligned}$$

In particular, the limit as N goes to infinity of $h((Na)x,Ne_i)/N$ exists and the problem is to compute its value.

Because $x\in E(h)$, given a positive integer M, there exists another positive integer $k(M)$ such that $x\in B(k(M),M)$. It follows from the construction of $B(K,M)$ that there exists $n(M)>k(M)$ satisfying $n(M)ax\in A(k(M),M)$. Therefore,

$$\left|\frac{h(n(M)ax,n(M)e_i)}{n(M)}-h^*(n(M)ax,e_i)\right|<\frac{1}{M}$$

and, by the invariance of $h^*(\cdot, e_i)$,

$$\left| \frac{h(n(M)ax, n(M)e_i)}{n(M)} - h^*(x, e_i) \right| < \frac{1}{M}.$$

A similar argument shows that

$$h^*(x, a - e_i) = h^*(x, a) + h^*(x, -e_i).$$

Finally, $0 = h^*(x, 0) = h^*(x, -e_i + e_i) = h^*(x, -e_i) + h^*(x, e_i)$ implies

$$h^*(x, -e_i) = -h^*(x, e_i)$$

to finish the proof. $\square$

Proposition 2. *For every $\mu \in \mathcal{M}$ there exists a set $F(\mu)$ of μ measure one such that, for all $h \in \mathcal{C}$, $x \in F(\mu)$ and $a, b \in \mathbf{Z}^m$,*

$$\lim_{N \to \infty} \frac{h(x, Na)}{N} = h^*(x, a)$$

and

$$h^*(x, a + b) = h^*(x, a) + h^*(x, b).$$

Proof. Let h_n, $n = 1, 2, \ldots$, be a countable dense subset of $\mathcal{C}$, and set

$$F(\mu) = \bigcap_{n=1}^{\infty} E(h_n).$$

For $h \in \mathcal{C}$ and $x \in F(\mu)$,

$$\begin{aligned} \left| \frac{h(x, Na)}{N} - \frac{h(x, Ma)}{M} \right| &\le \left| \frac{h(x, Na)}{N} - \frac{h_k(x, Na)}{N} \right| \\ &\quad + \left| \frac{h_k(x, Na)}{N} - \frac{h_k(x, Ma)}{M} \right| \\ &\quad + \left| \frac{h_k(x, Ma)}{M} - \frac{h(x, Ma)}{M} \right| \\ &\le 2\|h - h_k\||a| + \left| \frac{h_k(x, Na)}{N} - \frac{h_k(x, Ma)}{M} \right| \end{aligned}$$

from which it follows that $h(x, Na)/N$ is a Cauchy sequence. Further use of the triangle inequality shows that the limit satisfies

$$h^*(x, a+b) = h^*(x, a) + h^*(x, b)$$

and completes the proof. $\square$

Corollary 1. *For every ergodic $\mu \in \mathcal{M}$ there exists a set $F(\mu)$ of μ measure one such that, for all $h \in \mathcal{C}$, $x \in F(\mu)$, and $a, b \in \mathbf{Z}^m$*

$$\lim_{N\to\infty} \frac{h(x, Na)}{N} = \int_X h(x, a)\, d\mu$$

and

$$\int_X h(x, a+b)\, d\mu = \int_X h(x, a)\, d\mu + \int_X h(x, b)\, d\mu.$$

Proof. If μ is ergodic, then, because of the invariance of $h^*(\cdot, a)$,

$$h^*(x, a) = \int_X h(x, a)\, d\mu$$

on a set of μ measure one. Thus, without loss of generality, $h_n^*(x, a) = \int_X h_n(x, a)\, d\mu$ on the sets $E(h_n)$, as described in the proof of Proposition 1. Because

$$\left| \int h(x, a)\, d\mu - \frac{h(x, Na)}{N} \right| \le 2\|h - h_n\||a| \\ + \left| \int h_n(x, a)\, d\mu - \frac{h_n(x, Na)}{N} \right|$$

it follows that $h^*(x, a) = \int_X h(x, a)\, d\mu$ on $F(\mu)$ for all $h \in \mathcal{C}$. $\square$

Proposition 3. *There exists a Borel set Q of invariant measure one and for each $x \in Q$ a linear map $L_x : \mathcal{C} \to \mathcal{L}$ such that when $h \in \mathcal{C}$ and $x \in Q$*

$$\lim_{N\to\infty} \frac{h(x, Na)}{N} = L_x(h)(a)$$

for all $a \in \mathbf{Z}^m$.

Proof. First note that, for each $h \in \mathcal{C}$, the set

$$Q(h) = \left\{ x : \exists L \in \mathcal{L} \ni \lim_{N \to \infty} \frac{h(x, Na)}{N} = L(a) \ \forall \, a \in \mathbf{Z}^m \right\}$$

is a Borel set. Set

$$Q = \bigcap_{n=1}^{\infty} Q(h_n)$$

where h_n, $n = 1, 2, \dots$, is a countable dense subset of $\mathcal{C}$.

Arguing as in the proof of the previous proposition, it follows that $Q \subset Q(h)$ for all $h \in \mathcal{C}$ and $Q = \cap \{Q(h) : h \in \mathcal{C}\}$. Since $F(\mu) \subset Q(h)$ for all h, $F(\mu) \subset Q$ and $\mu(Q) = 1$ for every invariant measure which finishes the proof. □

We conclude with the proof of the main theorem. The proof involves the topological component of our argument, and for this we will need to extend h to $X \times \mathbf{R}^m$ as follows. Using the standard triangularization of $\mathbf{R}^m$ with $\mathbf{Z}^m$ as vertices, see Spanier [**10**, page 109], let $H(x, v)$ be the piecewise linear extension of $h(x, a)$ to $X \times \mathbf{R}^m$. It is easy to check that, for $a \in \mathbf{Z}^m$ and $v \in \mathbf{R}^m$

$$H(x, a + v) = h(x, a) + H(ax, v).$$

Moreover, there exists $B > 0$ such that

$$|H(x, v) - H(x, w)| \leq B|v - w|.$$

Proof of main theorem. Let Q be given by Proposition 3, let $h \in \mathcal{C}$ and construct H, the piecewise linear extension, as just described. For r real

$$\frac{H(x, ra)}{r} = \frac{h(x, [r]a)}{[r]} \frac{[r]}{r} + \frac{H([r]ax, (r - [r])a)}{r}$$

where $[r]$ denotes the integer part of r, and thus

$$\lim_{r \to \infty} \frac{H(x, ra)}{r} = L_x(h)(a)$$

because $H(y, ta)$ is bounded for $0 \le t < 1$.

Now consider $v \in \mathbf{Q}^m$, the subset of $\mathbf{R}^m$ with rational coordinates. There exists $q \in \mathbf{Z}$ such that $qv \in \mathbf{Z}^m$. Then

$$\frac{H(x, rv)}{r} = \frac{1}{q} \frac{H(x, (r/q)qv)}{r/q}$$

which implies that

$$\lim_{r \to \infty} \frac{H(x, rv)}{r} = \frac{1}{q} L_x(h)(qv) = L_x(h)(v).$$

The family of functions

$$\left\{ \frac{H(x, r\cdot)}{r} : r > 0 \right\}$$

is equicontinuous because

$$\left| \frac{H(x, rv)}{r} - \frac{H(x, rw)}{r} \right| \le \frac{1}{r} B|rv - rw| = B|v - w|.$$

Similarly, for fixed v,

$$\left\{ \frac{H(x, rv)}{r} : r > 0 \right\}$$

is bounded. Applying Ascoli's lemma, it follows that $H(x, rv)/r$ converges uniformly on compact subsets of $\mathbf{R}^m$ to $L_x(h)(v)$ when $x \in Q$.

In particular, $H(x, rv)/r$ converges uniformly to $L_x(h)(v)$ on $\{v : |v| = 1\}$. Therefore,

$$\lim_{|a| \to \infty} \left| \frac{H(x, |a|(a/|a|))}{|a|} - L_x(h)(a/|a|) \right| = 0.$$

The proof is completed by noting that

$$\frac{h(x, a) - L_x(h)(a)}{|a|} = \frac{H(x, |a|(a/|a|))}{|a|} - L_x(h)(a/|a|). \qquad \square$$

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