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AN EXPLICIT UPPER BOUND FOR THE RIEMANN ZETA-FUNCTION NEAR THE LINE $\sigma = 1$

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ABSTRACT. In this paper we give the following explicit estimate for the Riemann zeta-function. Let $t \geq 2$. For $1/2 \leq \sigma \leq 1$,

$$|\zeta(\sigma + it)| \leq 175 t^{46(1-\sigma)^{3/2}} \log^{2/3} t;$$

for $\sigma \geq 1$,

$$|\zeta(\sigma + it)| \leq 175 \log^{2/3} t.$$

1. Introduction. In regard to the prime number theorem, the zero-free region of the Riemann zeta-function plays an important role. The best known zero-free region of the Riemann zeta-function asserts that there are no zeros of $\zeta(\sigma + it)$ for $\sigma > 1 - c(\log t)^{-2/3}(\log\log t)^{-1/3}$ and $t \geq t_0$ where c and t_0 are absolute positive constants. This zero-free region was established by using the methods from Vinogradov and Korobov, and the principal tool was an upper bound for the Riemann zeta-function near the line $\sigma = 1$.

In 1963, Richert proved the following result, see [**11**]. For $1/2 \leq \sigma \leq 1$ and $t \geq 2$, there exists an absolute constant A such that

$$(*) \qquad |\zeta(\sigma + it)| \leq A t^{B(1-\sigma)^{3/2}} \log^{2/3} t,$$

with $B = 100$. In 1975, Ellison proved the same result as $(*)$ with $B = 86$ and $A = 2100$, see [**7**]. There are also other results with sharpened numbers B, see [**10, 1**].

In some applications one needs to have the result completely explicit. That is, to have the number A calculated out. To prove the above result with a reasonable size of the number A, it is convenient to use

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a slightly sub-optimal method and be satisfied with a slightly bigger B. In this paper, based on an explicit version of Vinogradov's mean value theorem in [**9**], we give the same result as $(*)$ with $B = 46$ and $A = 175$, the inequality holding for all $t \geq 2$ and $1/2 \leq \sigma \leq 1$. We also prove for $t \geq 2$ and $\sigma \geq 1$ that

$$|\zeta(\sigma + it)| \leq 175 \log^{2/3} t.$$

The main tools we use include the Euler summation formula, the double sum method of Korobov, an explicit version of the Vinogradov mean value theorem from Korobov, and an idea from Ellison.

In a subsequent paper, we hope to use the results of this paper to establish an explicit zero-free region for $\zeta(s)$ of the shape found by Vinogradov and Korobov. This in turn will allow us to establish the prime number theorem with an explicit error term of the sharpest known form, i.e., to find specific constants, say C, D and x_0 such that, for $x \geq x_0$,

$$\pi(x) = \int_2^x \frac{1}{\log t}\, dt + E(x),$$

with $|E(x)| \leq Cx \exp\{-D(\log x)^{3/5}(\log\log x)^{-1/5}\}$. Such an explicit form of the prime number theorem has useful applications. For example, in [**5**], a tool involved is an explicit form of the prime number theorem established by Rosser and Schoenfeld, see [**12**] and [**14**]. For related references, one may see [**3**].

After we have an explicit zero-free region for $\zeta(s)$, we may also use it in giving an explicit form of Ingham's theorem [**8**], which states that there is a prime between two sufficiently large consecutive cubes. To make this theorem explicit means to find a numerical value for the number n_0 such that, for every $n \geq n_0$, there is a prime between n^3 and $(n+1)^3$.

2. Approximation formula for $\zeta(\sigma + it)$. We first give the following approximate functional equation.

Proposition 1. *Let $\sigma \geq 1/2$ and $t \geq 2$. Then*

$$\zeta(s) = \sum_{n=1}^{[t]} \frac{1}{n^s} + C(s), \tag{1}$$

where $|C(s)| \leq 5.505$.

Remark. A result without an explicit number is standard. See [**15**, Theorem 4.11]. We have adapted the idea of using the Euler summation formula and the Fourier expansion of $x - [x] - 1/2$ from Titchmarsh's proof. We give a slightly weaker result with explicit constant, which is sufficient for our application later. Our proof is also slightly different, using the approximate formula in (2).

For $\sigma > 0$ and $s = \sigma + it \neq 1$, the Riemann zeta-function can be represented by

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^2} - s \int_N^\infty \frac{u - [u]}{u^{s+1}}\, du + \frac{1}{(s-1)N^{s-1}},$$

where N is a positive integer. For reference, see [**2**, p. 69]. Starting from this point, under the condition $\sigma > 0$ and $s \neq 1$, we note that

$$\left| \frac{1}{(s-1)N^{s-1}} \right| \leq \frac{N^{1-\sigma}}{t}.$$

For $\sigma \geq 1/2$, $t \geq 2$, we have

$$\left| s \int_N^\infty \frac{u - [u]}{u^{s+1}}\, du \right| \leq |s| \int_N^\infty \frac{1}{u^{\sigma+1}}\, du = \frac{\sqrt{(t/\sigma)^2 + 1}}{N^\sigma}.$$

Let $N = [t^2]$. Then $t^2 - 1 < N \leq t^2$. It follows for $\sigma \geq 1/2$, $t \geq 2$, that

$$\begin{aligned} \left| \zeta(s) - \sum_{n \leq t^2} \frac{1}{n^s} \right| &\leq \frac{N^{1-\sigma}}{t} + \frac{\sqrt{(t/\sigma)^2 + 1}}{N^\sigma} \\ &\leq 1 + \sqrt{\frac{4t^2 + 1}{t^2 - 1}} \leq 3.381. \end{aligned} \tag{2}$$

To get a suitable approximate formula for our purpose, we need the following lemmas.

Lemma 1. *If f and g are both differentiable real-valued functions and f is either positive decreasing or negative increasing, then*

$$\left| \int_a^b f(x) g'(x)\, dx \right| \leq 2|f(a)| \max_{a \leq x \leq b} |g(x)|.$$

Proof. We may assume that f is positive decreasing. Integrating by parts, we have

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)\,dx.$$

Using the fact that the modulus of an integral does not exceed the integral of the modulus, $\int_a^b F(x)\,dx \le \int_a^b G(x)\,dx$ whenever $F(x) \le G(x)$, and $|f'(x)| = -f'(x)$, we obtain that

$$\begin{aligned}\int_a^b f(x)g'(x)\,dx &\le f(b)g(b) - f(a)g(a) - \max_{a\le x\le b}|g(x)|\int_a^b f'(x)\,dx\\ &= f(b)\Big\{g(b) - \max_{a\le x\le b}|g(x)|\Big\}\\ &\quad + f(a)\Big\{\max_{a\le x\le b}|g(x)| - g(a)\Big\}\\ &\le 2f(a)\max_{a\le x\le b}|g(x)|.\end{aligned}$$

This proves Lemma 1. $\square$

Lemma 2. *Let $b > a > t/(2\pi)$. Then, for any positive integer ν and any $\sigma \ge 0$, we have*

$$\left|\int_a^b \frac{\sin(t\log x \pm 2\pi\nu x)}{x^{1+\sigma}}\,dx\right| \le \frac{2}{a^\sigma(2\pi\nu a \pm t)}, \tag{3}$$

where the three $\pm$ signs are all $+$ or all $-$. Also,

$$\left|\int_a^b \frac{\sin(t\log x + 2\pi\nu x) \pm \sin(t\log x - 2\pi\nu x)}{x^{1+\sigma}}\,dx\right| \le \frac{8\pi\nu a}{a^\sigma(4\pi^2\nu^2a^2 - t^2)}. \tag{4}$$

The above statements are also valid if we change the sine functions to cosine functions. We also have

$$\left|\int_a^b \frac{\sin(t\log x)\sin(2\pi\nu x)}{x^{1+\sigma}}\,dx\right| \le \frac{8\pi\nu a}{a^\sigma(2\pi^2\nu^2a^2 - t^2)} \tag{5}$$

and

$$\left| \int_a^b \frac{\cos(t \log x) \sin(2\pi\nu x)}{x^{1+\sigma}}\, dx \right| \le \frac{8\pi\nu a}{a^\sigma (4\pi^2\nu^2 a^2 - t^2)}. \tag{6}$$

Proof. Trivially, the integral on the left of (3) is

$$-\int_a^b \frac{d\{\cos(t \log x \pm 2\pi\nu x)\}}{x^{1+\sigma}(t/x \pm 2\pi\nu)} = -\int_a^b \frac{d\{\cos(t \log x \pm 2\pi\nu x)\}}{x^\sigma (t \pm 2\pi\nu x)}.$$

Note that

$$f(x) = \frac{1}{x^\sigma(2\pi\nu x \mp t)}$$

is a positive decreasing function for $x \ge a > t/(2\pi)$. It is easy to get (3) by applying Lemma 1 for $f(x)$ and $-f(x)$.

The inequality (4) is a direct consequence of (3), noting that

$$\frac{1}{2\pi\nu a - t} + \frac{1}{2\pi\nu a + t} = \frac{4\pi\nu a}{4\pi^2\nu^2 a^2 - t^2}.$$

As for (5) and (6), note that

$$\begin{aligned}
\sin(t \log x) \sin(2\pi\nu x) &= \frac{1}{2}(-\cos(t \log x + 2\pi\nu x) + \cos(t \log x - 2\pi\nu x)), \\
\cos(t \log x) \sin(2\pi\nu x) &= \frac{1}{2}(\sin(t \log x + 2\pi\nu x) - \sin(t \log x - 2\pi\nu x)).
\end{aligned}$$

Applying (4) and the analog with cosines to the integrals, it is straightforward to prove (5) and (6). $\square$

For brevity, especially for later utilization, we denote $\eta(x) = x - [x] - 1/2$ and define $((x)) = \eta(x)$ if x is not an integer and $((x)) = 0$ if x is an integer. Note that, by our definition,

$$\lim_{\delta \to 0^+} ((x - \delta)) + \lim_{\delta \to 0^+} ((x + \delta)) = 2((x)).$$

By Jordan's test, see [**6**], we can easily see that the function $((x))$ converges to its Fourier series. That is,

$$((x)) = -\frac{1}{\pi}\sum_{\nu=1}^{\infty}\frac{\sin(2\pi\nu x)}{\nu}.$$

This result can be found in [**15**, p. 74]. We then have

Lemma 3. *Suppose $\sigma \geq 1/2$ and $t \geq 2$. Then we have*

$$\left|\sqrt{t^2+1/4}\int_t^{t^2}((x))\frac{\sin(t\log x)}{x^{1+\sigma}}\,dx\right| \leq 0.176; \tag{7}$$

and

$$\left|\sqrt{t^2+1/4}\int_t^{t^2}((x))\frac{\cos(t\log x)}{x^{1+\sigma}}\,dx\right| \leq 0.176. \tag{8}$$

Proof. Replacing the Fourier expansion of the function $((x))$ inside the integral, the left side of (7) is

$$\left|\sqrt{t^2+1/4}\int_t^{t^2}\sum_{\nu=1}^{\infty}\frac{\sin(t\log x)\sin(2\pi\nu x)}{\pi\nu x^{1+\sigma}}\,dx\right|.$$

We may exchange the order of the integral and the summation in the expression by applying Lebesgue's bounded convergence theorem, see [**13**]. This leads to the estimation

$$\begin{aligned}
&\left|\sqrt{t^2+1/4}\int_t^{t^2}\sum_{\nu=1}^{\infty}\frac{\sin(t\log x)\sin(2\pi\nu x)}{\pi\nu x^{1+\sigma}}\,dx\right| \\
&\qquad\le \sqrt{t^2+1/4}\sum_{\nu=1}^{\infty}\frac{1}{\pi\nu}\left|\int_t^{t^2}\frac{\sin(t\log x)\sin(2\pi\nu x)}{x^{1+\sigma}}\,dx\right| \\
&\qquad\le 8\sqrt{t^2+1/4}\sum_{\nu=1}^{\infty}\left(\frac{t^{1-\sigma}}{4\pi^2\nu^2t^2-t^2}\right) \\
&\qquad= 8\sqrt{1+1/(4t^2)}\sum_{\nu=1}^{\infty}\left(\frac{1}{t^{\sigma}(4\pi^2\nu^2-1)}\right) \\
&\qquad\le 4\sqrt{1+1/64}\sum_{\nu=1}^{\infty}\left(\frac{1}{(4\pi^2\nu^2-1)}\right)\le 0.106\sum_{\nu=1}^{\infty}\frac{1}{\nu^2} \\
&\qquad= 0.106\frac{\pi^2}{6}\le 0.176,
\end{aligned}$$

using (5). This concludes (7). The inequality (8) may be proved similarly by using (6). □

Remark. If we replace $((x))$ by $\eta(x)$, at most finitely many values in the interval are changed. Hence, we also have the same result with $\eta(x)$ in the place of $((x))$.

Proof of Proposition 1. It follows from (2) that

$$\zeta(s)=\sum_{n=1}^{[t]}\frac{1}{n^s}+\sum_{t<n\le t^2}\frac{1}{n^s}+E(s), \tag{9}$$

with $|E(s)|\le 3.381$. We use Euler's summation formula, see [**15**, Theorem 2] to estimate the second sum on the right side of (9). Note that

$$\sum_{t<n\le t^2}\frac{1}{n^s}=\int_t^{t^2}\frac{dx}{x^s}+\frac{((t))}{t^s}-\frac{((t^2))}{t^{2s}}-s\int_t^{t^2}\frac{((x))}{x^{1+s}}\,dx.$$

For the first item, we have

$$\left| \int_t^{t^2} \frac{ds}{x^s} \right| = \left| \frac{x^{1-s}}{1-s} \right|_t^{t^2} \le \frac{1}{\sqrt{t^2+(1-\sigma)^2}} (t^{2-2\sigma} + t^{1-\sigma})$$
$$\le \frac{1}{t}(t + t^{1/2}) \le \frac{3}{2},$$

since $\sigma \ge 1/2$ and $t \ge 2$. For the second and third items, we have the trivial estimate

$$\left| ((t))\frac{1}{t^s} - ((t^2))\frac{1}{t^{2s}} \right| \le \frac{1}{2}\left(\frac{1}{t^\sigma} + \frac{1}{t^{2\sigma}} \right) \le \frac{1}{2t^{1/2}} + \frac{1}{2t} \le \frac{3}{8},$$

by using the fact that $|((x))| \le 1/2$. For the fourth item, we have that

$$\int_t^{t^2} \frac{((x))}{x^{1+s}}\, dx = \int_t^{t^2} \frac{((x))\cos(t\log x)}{x^{1+\sigma}}\, dx$$
$$- i \int_t^{t^2} \frac{((x))\sin(t\log x)}{x^{1+\sigma}}\, dx.$$

Recalling Lemma 3, we get that

$$\left| \sum_{t<n\le t^2} \frac{1}{n^s} \right| \le \frac{3}{2} + \frac{3}{8} + 0.176 \times \sqrt{2} \le 2.124..$$

Noting that $2.124 + 3.381 = 5.505$, we have finished the proof of Proposition 1. □

3. Zeta-sums. In this section we are concerned with an estimate of the so-called zeta-sum

$$S(G) = \sum_{n=G+1}^{G+H} n^{-it},$$

where G and H are positive integers, $H \le G$, and H is chosen as to maximize $|S(G)|$.

The following proposition is almost the same as [**9**, Lemma 27], except for the constants involved. The proof is also similar.

Proposition 2. *Let $k \geq 2$ and $S(G)$ be defined as above, and*

$$S_F(M,M) = \sum_{m_1,m_2=1}^{M} e^{-itF_k(m_1 m_2)}, \tag{10}$$

where $F_k(x) = (1/n)x - (1/(2n^2))x^2 + \cdots + (-1)^{k-1}(1/(kn^k))x^k$. Then, for any positive integer M with $M < G^{1/2}$, we have

$$|S(G)| \leq \frac{G}{M^2} \max_{G<n\leq G+H} |S_F(M,M)| + \frac{tM^{2k+2}}{(k+1)G^k} + \frac{(M+1)^2}{2}. \tag{11}$$

Vinogradov's mean value theorem establishes an upper bound for $N(M,l,k)$, the number of integral solutions of the system of equations

$$\begin{cases} x_1 + \cdots + x_l = y_1 + \cdots + y_l \\ \cdots\cdots\cdots\cdots \\ x_1^k + \cdots + x_l^k = y_1^k + \cdots + y_l^k \end{cases}, \quad 1 \leq x_j, y_j \leq M.$$

Our next lemma contributes to the estimate for the sum $S_F(M,M)$.

Lemma 4. *Let k, l and M be positive integers and $S_F(M,M)$ be defined as in* (10)*. Then*

$$|S_F(M,M)|^{4l^2} \leq M^{8l^2-4l} N(M,l,k)^2 V_F,$$

where

$$V_F = \sum_{\vec{\nu}} \left| \sum_{\vec{\mu}} \exp(-it\{\alpha_1\mu_1\nu_1 + \cdots + \alpha_k\mu_k\nu_k\}) \right|,$$

$\alpha_j = (-1)^{j-1}(1/(jn^j))$ for $j = 1, \ldots, k$, and the sums run over the set of integer vectors $\vec{\mu} = \langle \mu_1, \cdots, \mu_k \rangle$ and $\vec{\nu} = \langle \nu_1, \ldots, \nu_k \rangle$ such that $|\mu_j| < lM^j$ and $|\nu_j| < lM^j$ for $j = 1, \ldots, k$.

Proof. See [**9**, Lemma 25 and the first six rows on p. 120] and use the result [**9**, (162), p. 80]. □

Our estimate of zeta sums shall come out from the estimate of the following sum:

$$V_F = \sum_{\mu_1,\cdots,\mu_k} \Big| \sum_{\nu_1,\cdots,\nu_k} \exp\bigg(-it\Big\{ \frac{1}{n_0}\mu_1\nu_1 - \frac{1}{2n_0^2}\mu_2\nu_2 + \cdots + (-1)^{k-1}\frac{1}{kn_0^k}\mu_k\nu_k \Big\}\bigg)\Big|,$$

where n_0 is an integer with $G < n_0 \le G+H \le 2G$. We have

$$\begin{aligned}\Big| \sum_{\nu_1,\cdots,\nu_k} &\exp\bigg(-it\Big\{ \frac{1}{n_0}\mu_1\nu_1 - \frac{1}{2n_0^2}\mu_2\nu_2 + \cdots + (-1)^{k-1}\frac{1}{kn_0^k}\mu_k\nu_k \Big\}\bigg)\Big| \\ &= \Big| \sum_{|\nu_1|<lM} \exp\Big(it\frac{1}{n_0}\mu_1\nu_1 \Big)\Big| \cdots \Big| \sum_{|\nu_k|<lM^k} \exp\Big(it\frac{1}{kn_0^k}\mu_k\nu_k \Big)\Big|.\end{aligned}$$

For each $j = 1, 2, \ldots, k$, using Weyl's lemma, see [**9**, Lemma 1], we have

$$\Big| \sum_{|\nu_j|<lM^j} \exp\Big(it\frac{1}{jn_0^j}\mu_j\nu_j \Big)\Big| \le \min\Big\{ 2lM^j, \frac{1}{2\|t\mu_j/(2\pi j n_0^j)\|} \Big\}.$$

We thus have

$$\begin{aligned}V_F &= \sum_{\mu_1,\ldots,\mu_k} \prod_{j=1}^{k} \min\Big\{ 2lM^j, \frac{1}{2\|t\mu_j/(2\pi j n_0^j)\|} \Big\} \\ &= \prod_{j=1}^{k} \sum_{|\mu_j|<lM^j} \min\Big\{ 2lM^j, \frac{1}{2\|t\mu_j/(2\pi j n_0^j)\|} \Big\}.\end{aligned} \tag{12}$$

Let λ be such that $t = G^\lambda$. For $j \le \lambda$, we shall use the trivial estimate that

$$\sum_{|\mu_j|<lM^j} \min\Big\{ 2lM^j, \frac{1}{2\|t\mu_j/(2\pi j n_0^j)\|} \Big\} \le 4l^2M^{2j}. \tag{13}$$

For $j > \lambda$, let

$$\alpha_j = \frac{t}{2\pi j n_0^j} = \frac{1}{2\pi j} G^\lambda n_0^{-j}.$$

Using $G < n_0 \le 2G$, we have $\alpha_j < 1$. Let

$$d_j = \left[\frac{1}{\alpha_j}\right] \quad \text{and} \quad \theta_j = d_j^2\left(\alpha_j - \frac{1}{d_j}\right).$$

For each j with $\lambda < j \le k$, we have $|\theta_j| \le 1$ and

$$\alpha_j = \frac{1}{d_j} + \frac{\theta_j}{d_j^2}.$$

It follows that

$$\pi G^{j-\lambda} < \pi j G^{j-\lambda} \le d_j \le 2^{j+1}\pi j G^{j-\lambda} \le 2^{k+1}\pi k G^{j-\lambda}. \tag{14}$$

$\square$

We need the following lemma, which is similar to [**9**, Lemma 14].

Lemma 5. *Let $\alpha = (a/d) + (\theta/d^2)$, where a and d are integers with $d > 0$, $(a, d) = 1$ and $|\theta| \le 1$. Then for any integers H, Q and an arbitrary real number β we have*

$$\sum_{|x|<Q} \min\left\{H, \frac{1}{2\|\alpha n + \beta\|}\right\} \le 3\left(\frac{2Q}{d} + 1\right)(H + d)\log H.$$

For the proof, refer to Korobov's proof with some slight change. Applying the lemma and using (14), we get

$$\begin{aligned}
\sum_{|\mu_j|<lM^j} &\min\left\{2lM^j, \frac{1}{2\|t\mu_j/(2\pi j n_0^j)\|}\right\} \\
&\le 3\left(\frac{2lM^j}{d_j} + 1\right)(2lM^j + d_j)\log(2lM^j) \\
&= 3\left(\frac{4l^2M^{2j}}{d_j} + 2lM^j + 2lM^j + d_j\right)\log(2lM^j) \\
&\le 3((4/\pi)l^2M^{2j}G^{\lambda-j} + 4lM^j + 2^{k+1}\pi k G^{j-\lambda})\log(2lM^j).
\end{aligned}$$

Let us assume that $k \ge 3$ and $l \ge 8k^2/3$ as we shall have. Note that $3((4/\pi)l^2 + 4l + 2^{k+1}\pi k) \le 2^k l^2$. Thus,

$$\begin{aligned}
\sum_{|\mu_j|<lM^j} &\min\left\{2lM^j, \frac{1}{2\|t\mu_j/2\pi j n_0^j}\|\right\} \\
&\le 2^k l^2(M^{2j}G^{\lambda-j} + M^j + G^{j-\lambda})\log(2lM^j).
\end{aligned}$$

Let μ be a number such that

$$\frac{\lambda}{1-2\mu} \le k. \tag{15}$$

We shall choose the integer k so that $1/\log G < \mu < 1/2$. (So, in addition, in what follows, we shall have $G \ge 8$.) We shall let

$$M = [G^{\mu}], \quad \text{so that} \quad 1 < G^{\mu} - 1 < M \le G^{\mu}. \tag{16}$$

Thus, for $\lambda < j \le k$, we have

$$\begin{aligned}\sum_{|\mu_j|<lM^j} \min\left\{2lM^j, \frac{1}{2\|t\mu_j/(2\pi j n_0^j\|}\right\} \\ \le 2^k l^2 (G^{2\mu j+\lambda-j} + G^{\mu j} + G^{j-\lambda}) \log(2lM^j).\end{aligned}$$

Noting that $2\mu j + \lambda - j = \mu j = j - \lambda$ at $j = \lambda/(1-\mu)$, we see that

$$\max\{2\mu j + \lambda - j, \mu j, j - \lambda\} = \begin{cases} 2\mu j + \lambda - j & \text{if } \lambda < j \le \lambda/(1-\mu), \\ j - \lambda & \text{if } \lambda/(1-\mu) \le j \le k. \end{cases}$$

Then note that $j - \lambda = 2\mu j$ for $j = \lambda/(1-2\mu)$. Also, we have $\log(2l) \le l$ and

$$\begin{aligned}\log(2lM^j) &\le k\log(M) + \log(2l) \le (k+l)\log(M) \\ &\le \frac{1}{2}(k+l)\log G \le l \log G.\end{aligned}$$

Recalling the trivial estimate in (13) for $j \le \lambda$, which we shall also use when $\lambda/(1-2\mu) < j \le k$, we obtain that

$$\sum_{|\mu_j|<lM^j} \min\left\{2lM^j, \frac{1}{2\|t\mu_j/(2\pi j n_0^j)\|}\right\} \le 2^k l^3 G^{g_{\lambda,\mu}(j)} \log(G),$$

where

$$g_{\lambda,\mu}(j) = \begin{cases} 2\mu j & \text{if } j \le \lambda, \\ 2\mu j + \lambda - j & \text{if } \lambda < j \le \lambda/(1-\mu), \\ j - \lambda & \text{if } \lambda/(1-\mu) < j \le \lambda/(1-2\mu), \\ 2\mu j & \text{if } \lambda/(1-2\mu) < j \le k. \end{cases}$$

We obtain that

$$V_F \le \prod_{j=1}^{k} 2^k l^3 G^{g_{\lambda,\mu}(j)} \log G \le 2^{k^2} l^{3k} G^{h(\lambda,\mu)} \log^k(G), \tag{17}$$

where

$$\begin{aligned} h(\lambda,\mu) &= \sum_{1\le j\le \lambda} 2\mu j + \sum_{\lambda<j\le \lambda/(1-\mu)} (\lambda - j + 2\mu j) \\ &\quad + \sum_{\lambda/(1-\mu)<j\le\lambda/(1-2\mu)} (j-\lambda) + \sum_{\lambda/(1-2\mu)<j\le k} 2\mu j \\ &= \sum_{1\le j\le k} 2\mu j + \sum_{\lambda<j\le\lambda/(1-\mu)} (\lambda - j) \\ &\quad + \sum_{\lambda/(1-\mu)<j\le\lambda/(1-2\mu)} (j-\lambda-2\mu j). \end{aligned}$$

We get

$$h(\lambda,\mu) = \mu k^2 + \mu k - \frac{\mu^2}{(1-2\mu)(1-\mu)}\lambda^2 + E(\lambda,\mu), \tag{18}$$

with

$$E(\lambda,\mu) = \frac{1}{2}\eta(\lambda)^2 - (1-\mu)\eta\left(\frac{\lambda}{1-\mu}\right)^2 + \left(\frac{1}{2}-\mu\right)\eta\left(\frac{\lambda}{1-2\mu}\right)^2. \tag{19}$$

Using (18) and (19) in (17) finishes the estimate of V_F.

4. Upper bounds for the zeta sums. Using Vinogradov's mean value theorem, see [**9**, Theorem 16], we have

$$N(M,l,k)^2 \le (2l)^{4l}(2k)^{2k^3} M^{4l-k^2-k+(k^2-k)(1-1/k)^r}, \tag{20}$$

with r being a nonnegative integer and

$$l = \frac{k(k+1)}{2} + rk. \tag{21}$$

Note that $x^{1/x}$ is a decreasing function of x for $x \geq 4$, as are x^{1/x^2} and x^{1/x^3} for $x \geq 2$. For $k \geq 3$ and $l \geq 8k^2/3 \geq 24$, we have

$$((2l)^{4l}(2k)^{2k^3}2^{k^2}l^{3k})^{1/(4l^2)} = 2^{1/l+k^3/(2l^2)+k^2/(4l^2)}l^{l/1+3k/(4l^2)}k^{k^3/(2l^2)} < 1.245.$$

Recalling (10), Lemma 4, (17), (18), (19), (20), and using our assumption (16), we have

$$\begin{aligned}
&|S_F(M,M)|^{4l^2} \\
&\quad \leq M^{8l^2-4l}(2l)^{4l}(2k)^{2k^3}M^{4l-k^2-k+(k^2-k)(1-1/k)^r} \\
&\qquad \times 2^{k^2}l^{3k}G^{h(\lambda,\mu)}\log^k(G) \\
&\quad \leq 1.245^{4l^2}G^{8\mu l^2-\mu k^2-\mu k+\mu(k^2-k)(1-1/k)^r+h(\lambda,\mu)}\log^k(G) \\
&\quad \leq 1.245^{4l^2}G^{8\mu l^2-\mu^2/((1-2\mu)(1-\mu))\lambda^2+\mu(k^2-k)1-1/k)^r+E^{(\lambda,\mu)}}\log^k(G).
\end{aligned}$$

If $x > 1$, then it is easy to see that $\log x \leq yx^{1/y}$ for any positive number y. Taking $y = 4 \times 10^6$, we get $\log G \leq 4 \times 10^6 G^{1/(4\times 10^6)}$, so $(\log G)^{k/(4l^2)} \leq (4 \times 10^6)^{k/(4l^2)}G^{k/(16\times 10^6 l^2)}$. We shall be taking $l \geq 8k^2/3$ and $k \geq 2\lambda$. Thus,

$$(\log G)^{k/(4l^2)} \leq (4 \times 10^6)^{1/768}G^{9/(1024\times 10^6 k^3)} < 1.02G^{1/(10^9\lambda^2)}.$$

Thus,

$$|S_F(M,M)| \leq 1.2699G^{2\mu+g(\lambda,\mu)/(4l^2)+1/(10^9\lambda^2)}, \tag{22}$$

where

$$g(\lambda,\mu) = -\frac{\mu^2}{(1-2\mu)(1-\mu)}\lambda^2 + \mu(k^2-k)\left(k-\frac{1}{k}\right)^r + E(\lambda,\mu), \tag{23}$$

with $E(\lambda,\mu)$ being defined in (19).

We claim the following lemma.

Lemma 6. *For each $\lambda \geq 1$, there are choices of μ, k, r where k and r are positive integers, $5/18 \leq \mu \leq 1/3$, $k \geq 3$, (15) holds, and such that if l is given by (21), then $l \geq 8k^2/3$ and we have*

$$\frac{g(\lambda,\mu)}{4l^2} \leq -\frac{1}{14101.3\lambda^2}. \tag{24}$$

We shall prove this lemma in the next two sections. We now use Lemma 6 to prove the following proposition.

Proposition 3. *Let G and H be any positive integers with $H \leq G$ and $t = G^{\lambda}$. Then, for $\lambda \geq 1$, we have*

$$|S(G)| \leq 1.26992 G^{1-1/(14102\lambda^2)}, \quad \text{for any } G \geq 1. \tag{25}$$

Proof. Proposition 3 follows from the trivial estimate $|S(G)| \leq G$ when $G \leq 10^{1400}$.

We then assume that $G > 10^{1400}$. Using $5/18 \leq \mu \leq 1/3$ and $M = [G^{\mu}]$, we have

$$\frac{G^{2\mu}}{M^2} \leq \left(\frac{G^{\mu}}{G^{\mu}-1}\right)^2 \leq 1 + 10^{-10}.$$

Now, applying the result (22) to $|S_F(M, M)|$ defined in (10), we get

$$\begin{aligned}
&\frac{G}{M^2} \max_{G<n\leq G+H} |S_F(M, M)| \\
&\qquad\qquad \leq 1.2699 G \frac{G^{2\mu}}{M^2} G^{-1/(14101.3\lambda^2)+1/(10^9\lambda^2)} \\
&\qquad\qquad \leq 1.2699(1+10^{-10}) G^{1-1/(14101.3\lambda^2)+1/(10^9\lambda^2)} \\
&\qquad\qquad < 1.26991 G^{1-1/(14101.4\lambda^2)}.
\end{aligned}$$

Recalling that λ is defined by $t = G^{\lambda}$ and that $\lambda \leq (1-2\mu)k$, we have $tM^{2k+2}/G^k \leq G^{\lambda+(2k+2)\mu-k} \leq G^{2\mu}$. From our assumption $k \geq 3$ and $\mu \leq 1/3$, we obtain that

$$\frac{tM^{2k+1}}{(k+1)G^k} + \frac{(M+1)^2}{2} \leq \frac{1}{4}G^{2/3} + \frac{3}{4}G^{2/3} < 10^{-10} G^{1-1/(14101.4\lambda^2)}.$$

Finally, putting these estimates into (11) we complete the proof of Proposition 3. $\square$

We then have the following corollary.

Corollary. *Let $t \geq 2$, let $X \leq t$ be any real numbers, and let $1 \leq Y \leq X$. Then we have*

$$\left| \sum_{X<n\leq X+Y} n^{-it} \right| \leq 1.27X^{1-\log^2 X/(14102\log^2 t)}. \tag{26}$$

Proof. Similar to what we mentioned in the proof of Proposition 3, the result is trivial when $X < e^{18}$. We may assume $X \geq e^{18}$. Using (25) by letting $G = [X]$, $H = [Y]$ and $\lambda = \log t/\log G$, we get

$$\begin{aligned}\left| \sum_{X<n\leq X+Y} n^{-it} \right| &\leq \left| \sum_{n=G+1}^{G+H} n^{-it} \right| + \left| \sum_{[X]+[Y]<n\leq X+Y} n^{-it} \right| \\ &\leq S([X]) + 1 \leq 1.26992[X]^{1-1/(14102\lambda^2)} + 1 \\ &\leq 1.27X^{1-\log^2 X/(14102\log^2 t)}.\end{aligned}$$

This completes the proof of the corollary. □

5. The cases for $\lambda \geq 6$. In this section we prove Lemma 6 when $\lambda \geq 6$. Let $z = [\lambda]$ and assume that $z + (x-1)/3 \leq \lambda < z + x/3$ where $x \in \{1, 2, 3\}$. We let $\mu = 1/3$. Thus $\lambda/(1-2\mu) = 3\lambda$. Noting that $3z + x - 1 \leq 3\lambda = \lambda/(1-2\mu) < 3z + x$, we may let $k = 3z + x$, which is compatible with (15). We also let $2k < r = 3k - 4 = 9z + 3x - 4$. (Note that $2k < 3k - 4$ holds since $z \geq 6$, so that $k \geq 19$.) From (23), $l \geq 8k^2/3$.

Now we evaluate $E(\lambda, \mu)$. For brevity, we only exhibit the detail for the case that z is odd and $x = 1$, which is our assumption in the following evaluation.

First we have $\eta(\lambda) = \lambda - z - 1/2$, and

$$\frac{1}{2}\eta(\lambda)^2 = \frac{1}{2}\left(\lambda - z - \frac{1}{2}\right)^2 = \frac{1}{2}\lambda^2 - \lambda z - \frac{1}{2}\lambda + \frac{1}{2}z^2 + \frac{1}{2}z + \frac{1}{8}. \tag{27}$$

Second, noting that $\lambda/(1-\mu) = 3\lambda/2$, and

$$\left[\frac{\lambda}{1-\mu}\right] = \frac{3}{2}z - \frac{1}{2}, \tag{28}$$

we have
$$\eta\left(\frac{\lambda}{1-\mu}\right)=\frac{3}{2}\lambda-\frac{3}{2}z.$$
Noting also $-(1-\mu)=-2/3$, we obtain that
$$-(1-\mu)\eta\left(\frac{\lambda}{1-\mu}\right)^2=-\frac{2}{3}\eta\left(\frac{3}{2}\lambda\right)^2=-\frac{3}{2}\lambda^2+3z\lambda-\frac{3}{2}z^2.$$

Third, we see that $\eta(3\lambda)=3\lambda-3z-x+1/2$, so we have
$$\begin{aligned}\left(\frac{1}{2}-\mu\right)\eta\left(\frac{\lambda}{1-2\mu}\right)^2&=\left(\frac{1}{2}-\mu\right)\eta(3\lambda)^2\\&=\frac{3}{2}\lambda^2-3z\lambda-\frac{1}{2}\lambda+\frac{3}{2}z^2+\frac{1}{2}z+\frac{1}{24}.\end{aligned}$$
We acquire that
$$E(\lambda,1/3)=\frac{1}{2}\lambda^2-z\lambda-\lambda+\frac{1}{2}z^2+z+\frac{1}{6}.$$
Recalling (23), we have
$$g(\lambda,1/3)=-\frac{1}{2}\lambda^2+E(\lambda,1/3)+\frac{1}{3}k(k-1)\left(1-\frac{1}{k}\right)^r.$$
Substituting $k=3z+x$ and $r=9z+3x-4$ in the above equation for each case respectively, we have
$$g(\lambda,1/3)=-z\lambda-\lambda+\frac{1}{2}z^2+z+\frac{1}{6}+z(3z+1)\left(1-\frac{1}{3z+1}\right)^{9z-1}.$$
It is now obvious that $g(\lambda,1/3)$ is decreasing on the interval $z+(x-1)/3\le\lambda<z+x/3$. Hence, we get
$$g(\lambda,1/3)\le g\left(z+\frac{x-1}{3},\frac{1}{3}\right)=-\frac{1}{2}z^2+\frac{1}{6}+z(3z+1)\left(1-\frac{1}{3z+1}\right)^{9z-1}.$$
From this, we can easily see that
$$\begin{aligned}g\left(\lambda,\frac{1}{3}\right)&\le g\left(z+\frac{x-1}{3},\frac{1}{3}\right)\\&\le-\frac{1}{2}z^2+\frac{1}{6}+z(3z+1)\left(1-\frac{1}{3z+1}\right)^{9z-1}.\end{aligned}$$

Recalling (21) and our choices of k and r, we have $4l^2 = (63z^2+21z)^2$. As we shall see, the function $g(\lambda, 1/3)$ is less than zero and so is $g(\lambda, 1/3)/(4l^2)$. Thus, it will be true for each x that

$$\frac{g(\lambda, 1/3)\lambda^2}{4l^2} \leq \frac{g(z+(x-1)/3, 1/3)(z+(x-1)/3)^2}{4l^2},$$

and $g(\lambda, 1/3)\lambda^2/4l^2 \leq F_x(z)$ where

$$F_x(z) = \frac{(-(1/2)z^2 + 1/6 + z(3z+1)(1-1/(3z+1))^{9z-1})z^2}{(63z^2+21z)^2}.$$

We find out that $F_x(z)$ is a decreasing function for $z \geq 6$, so that $F_x(z) \leq F_x(6+(x-1)/3)$. We have $F_x(6+(x-1)/3) < -0.0000712$. The evaluation for other cases can be done similarly, only with $-.0000727$ and -0.0000755 in place of -0.0000712. We have proved (24) for $\lambda \geq 6$.

6. The cases for $\lambda < 6$. We now prove Lemma 6 when $1 \leq \lambda < 6$. Recall that $z+(x-1)/3 \leq \lambda < z+x/3$. To prove (24) for $1 \leq \lambda < 6$, we let $k = 3z+x-1$, $\mu = (k-\lambda)/(2k)$, which makes (15) actually an equality. We have $k \geq 3$. Note also

$$\begin{aligned}\frac{5}{18} \leq \frac{1}{3} - \frac{1}{18z+6x-6} &= \frac{1}{2} - \frac{z+x/3}{2(3z+x-1)} \\ &\leq \mu \leq \frac{1}{2} - \frac{z+(x-1)/3}{2(3z+x-1)} = \frac{1}{3}.\end{aligned}$$

Also, we let $2k \leq r = 3k-x = 9z+2x-3$ so that, from (21), $l \geq 8k^2/3$.

We again evaluate $E(\lambda, \mu)$. The evaluation for the first term in $E(\lambda, \mu)$ stays the same as in (27). Corresponding to the second term, we note that

$$\frac{\lambda}{1-\mu} = \frac{2k\lambda}{k+\lambda} = 2k - \frac{2k^2}{k+\lambda},$$

so that, for each fixed pair of integers z and x, the function $\lambda/(1-\mu)$ is increasing for $z+(x-1)/3 \leq \lambda < z+x/3$. Thus,

$$\begin{aligned}\frac{3}{2}z + \frac{x-1}{2} < \frac{9}{5}z + \frac{3}{5}(x-1) &= \frac{2k(z+(x-1)/3}{k+z+(x-1)/3} \\ &\leq \frac{\lambda}{1-\mu} \leq \frac{\lambda}{1-1/3} \\ &= \frac{3}{2}z + \frac{x}{2},\end{aligned}$$

so we still have (28). Also, note that

$$-(1-\mu) = -\left(\frac{1}{2}+\frac{\lambda}{2k}\right).$$

The following evaluation is for z being odd and $x=1$. We obtain that

$$\begin{aligned}-(1-\mu)\eta\left(\frac{\lambda}{1-\mu}\right)^2 &= -\frac{k+\lambda}{2k}\eta\left(\frac{2k\lambda}{k+\lambda}\right)^2\\ &= -\left(\frac{1}{2}+\frac{\lambda}{2k}\right)\left(\frac{2k\lambda}{k+\lambda}-\frac{3z}{2}\right)^2.\end{aligned}$$

As for the third term, we have that $\eta(\lambda/(1-2\mu)) = -1/2$, since $\lambda/(1-2\mu)$ is actually equal to the integer k. We get

$$\frac{1-2\mu}{2}\eta\left(\frac{\lambda}{1-2\mu}\right)^2 = \left(\frac{1}{2}-\frac{k-\lambda}{2k}\right)\frac{1}{4} = \frac{\lambda}{8k}.$$

Recall the function $E(\lambda,\mu)$ from (19). With $\mu = (k-\lambda)/(2k)$, denoting the resulting function $E_0(\lambda)$, we have that

$$E_0(\lambda) = E(\lambda,\mu) = \frac{1}{2}\left(\lambda-z-\frac{1}{2}\right)^2 + \frac{\lambda}{8k} - \left(\frac{k+\lambda}{2k}\right)\left(\frac{2k\lambda}{k+\lambda}-\frac{3z}{2}\right)^2.$$

Substituting $\mu=(k-\lambda)/(2k)$ in (23), denoting the resulting function from $g(\lambda,\mu)$ by $g_0(\lambda)$, we obtain that

$$g_0(\lambda) = g(\lambda,\mu) = -\frac{(k-\lambda)^2\lambda}{2(k+\lambda)} + \frac{1}{2}(k-\lambda)(k-1)\left(1-\frac{1}{k}\right)^r + E_0(\lambda).$$

We then estimate $\lambda^2 g_0(\lambda)$. Using (19) and replacing $k=3z+x-1$ and $r=9z+2x-3$, we obtain

$$\begin{aligned}g_0(\lambda) = &-\frac{(3z-\lambda)^2\lambda}{2(3z+\lambda)} + \frac{1}{2}(3z-\lambda)(3z-1)\left(1-\frac{1}{3z}\right)^{9z-1}\\ &+\frac{1}{2}\left(\lambda-z-\frac{1}{2}\right)^2 + \frac{\lambda}{24z} - \frac{3z+\lambda}{6z}\left(\frac{6z\lambda}{3z+\lambda}-\frac{3z}{2}\right)^2.\end{aligned}$$

We find that $f_0(\lambda) := \lambda^2 g_0(\lambda)$ is a decreasing function of λ on the corresponding interval $[z+(x-1)/3, z+x/3)$. Thus,

$$f_0(\lambda) \le f_0\left(z+\frac{x-1}{3}\right) = -\frac{1}{2}z^4 + (3z-1)z^3\left(1-\frac{1}{3z}\right)^{9z-1} + \frac{1}{6}z^2.$$

Recalling (21) and our choices of k and r, we have $4l^2 = (63z^2-3z)^2$, and

$$\frac{f_0(z)}{4l^2} \le -\frac{(1/2)z^2 - (3z-1)z(1-1/(3z))^{9z-1} - 1/6}{(63z-3)^2}.$$

This value is less than $-1/14101.3$. For other cases this is also valid. This shows that $f_0(z)/4l^2 < -1/14101.3$ for $1 \le z < 6$ and completes the proof of (24) for $1 \le \lambda < 6$, which finishes the proof of Lemma 6. $\square$

7. The upper bound for $\zeta(s)$. Let us first note trivially that we have

$$|S_0| := \left|\sum_{0<n\le T} n^{-\sigma-it}\right| \le \sum_{0<n\le T} n^{-\sigma}.$$

If $0<\sigma<1$, we get an upper bound of the above sum as

$$\begin{aligned}|S_0| &\le 1+\int_1^T \frac{dx}{x^\sigma} = 1+\frac{1}{1-\sigma}T^{1-\sigma} - \frac{1}{1-\sigma}\\ &= \frac{1}{1-\sigma}T^{1-\sigma} - \frac{\sigma}{1-\sigma} \le \frac{1}{1-\sigma}T^{1-\sigma}.\end{aligned}$$

If $\sigma \le 1$, then $n^{1-\sigma} \le T^{1-\sigma}$. For $T \ge e$, we see that

$$|S_0| \le \sum_{0<n\le T} n^{-\sigma} \le T^{1-\sigma} \sum_{0<n\le T} \frac{1}{n} T^{1-\sigma}(\log T + 1).$$

We let $T = \exp(\log^{2/3} t)$, so that $\log T = \log^{2/3} t$. We then consider two cases. For $1/2 \le \sigma \le 1 - 1/\log^{2/3} t$, we have $(1-\sigma)\log^{2/3} t \ge 1$. It follows that

$$|S_0| \le \frac{1}{1-\sigma}T^{1-\sigma} = \frac{1}{1-\sigma}e^{(1-\sigma)\log^{2/3 t}} \le e^{(1-\sigma)^{3/2}\log t}\log^{2/3} t.$$

For $1 - 1/\log^{2/3} t \le \sigma \le 1$, we get $(1-\sigma)\log^{2/3} t \le 1$, and

$$|S_0| \le T^{1-\sigma}(\log T + 1) = e^{(1-\sigma)\log^{2/3} t}(\log^{2/3} t + 1) \le e\log^{2/3} t + e.$$

We conclude from the above that, for $t \ge e$, $T \ge e$ and $0 \le \sigma \le 1$,

$$\begin{aligned} |S_0| &\le \max\{e^{(1-\sigma)^{3/2}\log t}\log^{2/3} t, e\log^{2/3} t + e\} \\ &\le e^{1+(1-\sigma)^{3/2}\log t}\log^{2/3} t + e. \end{aligned} \tag{29}$$

For $\sigma \ge 1$, we also have a similar estimate, since in this case

$$|S_0| \le \sum_{0<n<T} \frac{1}{n} \le \log T + 1 = \log^{2/3} t + 1. \tag{30}$$

From Theorem 1 in Section 2, it remains to estimate the sum $S = \sum_{T<n\le t} n^{-\sigma-it}$ for $T = \exp(\log^{2/3} t)$.

The next step is to split the sum S into the zeta sums. Let r be the least integer such that $2^r T \ge t$. Then

$$2^r T \ge t \quad \text{or} \quad r \ge \frac{\log t - \log T}{\log 2}.$$

We have

$$r = \left\lceil \frac{\log t - \log T}{\log 2} \right\rceil \le \frac{\log t - \log T + \log 2}{\log 2} \le \frac{\log t}{\log 2},$$

if $T \ge 2$, which of course is the case. Thus, we have that

$$S = \sum_{j=0}^{r-1} \sum_{2^j T<n\le U_j} n^{-\sigma-it},$$

where

$$U_j = \begin{cases} 2^{j+1}T & \text{if } j = 0, 1, \dots, r-2, \\ t & \text{if } j = r-1. \end{cases}$$

It follows that

$$|S| \le \sum_{j=0}^{r-1} \left| \sum_{2^j T<n\le U_j} n^{-\sigma-it} \right|.$$

For $\sigma > 0$, note that $n^{-\sigma}$ is decreasing as n tends to infinity. We can use the partial summation formula to estimate the inner sum, getting that

$$\begin{aligned}\left|\sum_{2^j T<n\le U_j} n^{-\sigma-it}\right| &\le (2^j T)^{-\sigma}\max_{V\le U_j}\left|\sum_{2^j T<n\le V} n^{-it}\right| \\ &\le (2^j T)^{-\sigma}\max_{V\le 2^{j+1}T}\left|\sum_{2^j T<n\le V} n^{-it}\right|.\end{aligned}$$

The last sums are zeta sums, and we have the estimate from the corollary in Section 4 that

$$\left|\sum_{2^j T<n\le V} n^{-it}\right| \le 1.27(2^j T)^{1-\log^2(2^j T)/(14102\log^2 t)}. \tag{31}$$

It follows that

$$|S| \le 1.27\sum_{j=0}^{r-1} e^{j(1-\sigma)\log 2+(1-\sigma)\log T-(j\log 2+\log T)^3/(14102\log^2 t)}. \tag{32}$$

For the next step we split the exponent into the sum of two functions as Ellison did in [**7**]. We let $g_1(j) = -a(j\log 2+\log T)^3/\log^2 t$, and $g_2(j) = j(1-\sigma)\log 2 + (1-\sigma)\log T - b(j\log 2+\log T)^3/\log^2 t$. If $a+b = c := 1/14102$, then

$$g_1(j)+g_2(j) = j(1-\sigma)\log 2+(1-\sigma)\log T - c\frac{(j\log 2+\log T)^3}{\log^2 t}.$$

We have

$$\left|\sum_{T<n\le t} n^{-\sigma-it}\right| \le 1.27\sum_{j=0}^{r-1} e^{g_1(j)+g_2(j)}. \tag{33}$$

It follows that

$$\frac{d}{dj}g_2(j) = (1-\sigma)\log 2 - 3b\log 2\frac{(j\log 2+\log T)^2}{\log^2 t}.$$

We find the critical point

$$j_0 = \frac{(1-\sigma)^{1/2}\log t}{\sqrt{3b}\log 2} - \frac{\log T}{\log 2} \quad \text{and} \quad \frac{d}{dj}g_2(j) \le 0, \quad \text{for } \sigma \ge 1. \tag{34}$$

Since

$$\frac{d^2}{dj^2}g_2(j) = -6b\log^2 2\frac{j\log 2 + \log T}{\log^2 t} < 0,$$

for $j \ge 0$ we know that the function $g_2(j)$ takes its maximal value at $j = j_0$ for $0 \le j < \infty$. Thus, for $1/2 \le \sigma \le 1$,

$$g_2(j) \le g_2(j_0) = \frac{2}{3\sqrt{3b}}(1-\sigma)^{3/2}\log t. \tag{35}$$

For our later use, we note here from (34) and the definition of $g_2(j)$ that

$$g_2(j) \le 0 \quad \text{for} \quad \sigma \ge 1. \tag{36}$$

Now

$$\begin{aligned}\sum_{j=0}^{r-1} e^{g_1(j)} &= \sum_{j=0}^{r-1} e^{-a(j\log 2+\log T)^3/\log^2 t} \\ &= \sum_{j=0}^{r-1} e^{-a(j\log 2/\log^{2/3} t+1)^3} \\ &\le \int_{-1}^{\infty} e^{-a(j\log 2/\log^{2/3} t+1)^3}\,dj.\end{aligned}$$

Changing the variable by letting $w = a^{1/3}(j\log 2/\log^{2/3} t + 1)$, we get

$$\begin{aligned}\int_{-1}^{\infty} e^{-a(j\log 2/\log^{2/3} t+1)^3}\,dj &= \frac{\log^{2/3} t}{a^{1/3}\log 2}\int_{a^{1/3}(1-\log 2/\log^{2/3} t)}^{\infty} e^{-w^3}\,dw \\ &\le \frac{\log^{2/3} t}{a^{1/3}\log 2}\int_0^{\infty} e^{-w^3}\,dw,\end{aligned}$$

since $1 - \log 2/\log^{2/3} t \ge 0$ for $t \ge 2$.

Here we use Mathematica finding the value of the last integral, which is $\Gamma(1/3)/3 < 0.893$. It implies that

$$\sum_{j=0}^{r-1} e^{g_1(j)} \leq \frac{.893}{a^{1/3}\log 2}\log^{2/3} t. \tag{37}$$

We conclude from (33), (35) and (37) that, for $t \geq 2$ and $1/2 \leq \sigma \leq 1$,

$$\left|\sum_{T<n\leq t} n^{-\sigma-it}\right| \leq \frac{1.637}{a^{1/3}} t^{B(1-\sigma)^{3/2}} \log^{2/3} t, \tag{38}$$

with $B = 2/(3\sqrt{3b})$ and, for $t \geq 2$ and $\sigma \geq 1$, recalling (36), we have

$$\left|\sum_{T<n\leq t} n^{-\sigma-it}\right| \leq \frac{1.637}{a^{1/3}} \log^{2/3} t, \tag{39}$$

where a and b are positive numbers such that $a + b = c = 1/14102$.

We are in a position to prove our main theorem.

Theorem 1. *Let $t \geq 2$. Then, for $1/2 \leq \sigma \leq 1$, we have*

$$|\zeta(s)| \leq 175 t^{46(1-\sigma)^{3/2}} \log^{2/3} t;$$

and, for $\sigma \geq 1$, we have

$$|\zeta(s)| \leq 175 \log^{2/3} t.$$

Proof. Recalling Proposition 1 in Section 2, we see that, for $1/2 \leq \sigma \leq 1$, $t \geq 2$,

$$|\zeta(s)| \leq \left|\sum_{0<n\leq t} n^{-\sigma-it}\right| + 5.505.$$

If $2 \leq t < e^6$, then we use the estimate $|1/n^s| = 1/n^\sigma \leq n^{-1/2}$ for each item in the above sum. We get $|\sum_{n=1}^{[t]} 1/n^s| \leq 1 + 2(t^{1/2} - 1)$ and

$$|\zeta(s)| \leq 2t^{1/2} + 1 + 5.505 \leq 175 \log^{2/3} t.$$

We then assume $t \geq e^6$. Taking $b = 1/14283$ in $a + b = 1/14102$, we have $a > 1/11128112$. By the formula $B = 2/(3\sqrt{3b})$, we get $B = 46$; from (29) and (38) we have

$$\begin{aligned}
|\zeta(s)| \leq & \left| \sum_{1<n\leq \exp(\log^{2/3} t)} n^{-\sigma-it} \right| \\
& + \left| \sum_{\exp(\log^{2/3} t)<n\leq t} n^{-\sigma-it} \right| \\
& + \left| \sum_{t<n<\infty} n^{-\sigma-it} \right| \\
& \leq e e^{(1-\sigma)^{3/2} \log t} \log^{2/3} t + e \\
& + \left(\frac{1.637}{a^{1/3}} \right) t^{B(1-\sigma)^{3/2}} \log^{2/3} t + 5.505 \\
& \leq (e + 169.7 + (e + 5.505) 6^{-2/3}) t^{46(1-\sigma)^{3/2}} \log^{2/3} t \\
& \leq 175 t^{46(1-\sigma)^{3/2}} \log^{2/3} t.
\end{aligned}$$

Recalling Proposition 1 in Section 2, (30) and (39) for $\sigma \geq 1$ and $t \geq 2$, similarly, we can prove that $|\zeta(s)| \leq 175 \log^{2/3} t$. This completes the proof of the theorem. □

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