ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 1, Spring 1999

SURFACES IN $\mathbf{P}^5$ WHICH DO NOT ADMIT TRISECANTS

E. BALLICO AND A. COSSIDENTE

ABSTRACT. We study the trisecant lines of surfaces embedded in $\mathbf{P}^5$. We are mainly interested in surfaces defined over the algebraic closure of a finite field, embedded in the Grassmannian $G(1,3)$ of lines of $\mathbf{P}^3$ and having no trisecant line.

1. Introduction. The main actors in this paper are surfaces embedded in $G(1,3)$, the Grassmannian of lines of $\mathbf{P}^3$ over any algebraically closed field $\mathbf{K}$ and in particular we are interested in the case in which $\mathbf{K}$ is the algebraic closure of a finite field $GF(q)$, $q = p^h$, $h \geq 1$, p prime. We will study surfaces X contained in $G(1,3)$ which have the particularity to contain no trisecant lines, in the sense of Definition 2.1, and as we will see these are very few. Actually, since our tools are algebraic geometric, most of our results hold for a surface in $\mathbf{P}^5$, not just for a surface in $G(1,3)$ seen as a smooth quadric hypersurface of $\mathbf{P}^5$, see Theorems 4.1 and 4.2. Denote by $PG(n,q)$ the projective space of dimension n over $GF(q)$. There is a close relation between such surfaces and objects coming from Galois geometries, namely, K-caps in $PG(n,q)$.

A *K-cap* in $PG(n,q)$ is a set of K points, no three of which are collinear, cf. [**11**, p. 285]. A K-cap of $PG(2,q)$ is also called a K-*arc*. The maximum value of K for which there exists a K-cap in $PG(n,q)$ is denoted by $m_2(n,q)$, cf. [**11**, p. 285]. This number $m_2(n,q)$ is only known, for arbitrary q, when $n \in \{2,3\}$. With respect to the other values of $m_2(n,q)$, only upper bounds are known. Constructing a K-cap of size $m_2(n,q)$, $n \geq 4$ seems to be an extremely hard problem.

Some authors looked at caps contained in algebraic varieties such as quadrics or Hermitian varieties. Here we are substantially interested in

Received by the editors on November 26, 1996, and in revised form on February 14, 1997.
AMS *Mathematics Subject Classification.* 14J25, 14N05, 51E21, 51E22.
Research supported by GNSAGA of CNR and the Italian Ministry for Research and Technology.

caps which have an algebraic structure, namely, algebraic surfaces in the Klein quadric. For instance, conics in the Galois plane are examples of arcs, elliptic quadrics of $PG(3,q)$ are (q^2+1)-caps, the Veronese surface of $PG(5,q)$ is a (q^2+q+1)-cap.

However, when we claim to have proved the existence of trisecant lines to our surface, which are always defined over $GF(q)$, it could happen that such trisecants are not defined over $GF(q)$ but only over some (a priori unknown) extension of $GF(q)$.

2. Preliminary results. We start with the following

Definition 2.1. Let $X \subset \mathbf{P}^N$ be a reduced projective scheme and $L \subset \mathbf{P}^N$ be a line. We will say that L is *trisecant* to X if the scheme $X \cap L$ contains a length 3 subscheme of L, i.e., if and only if either L is contained in X or $X \cap L$ is finite but contains at least 3 points or $\mathrm{card}\,(X \cap L) = 1$ or 2, but the sum of the multiplicities of the divisor $X \cap L$ of the smooth curve L at the points of X, i.e., its degree, is at least 3.

We need the following well-known lemma.

Lemma 2.2. *Let $X \subset \mathbf{P}^4$ be an integral, i.e., reduced and irreducible, surface. Then X has at least one, and indeed infinitely many, trisecant lines.*

Proof. If X is a plane or a quadric (even singular), then it contains infinitely many lines. Hence we may assume $d := \deg\,(X) \geq 3$. The result is obvious if X is contained in a hyperplane. Hence, we may assume that X spans $\mathbf{P}^4$. Let $Y \subset \mathbf{P}^3$ be a general integral hyperplane section of X. Projecting from a smooth point of Y and using the genus formulas for plane curves and Castelnuovo bound for $p_q(Y)$ (for the positive characteristic case, see, e.g., [**18**, Section 2]), we see that Y has infinitely many trisecant lines unless $\deg\,(Y) \leq 4$. Hence, we may assume $d = 3$ or $d = 4$. By the characteristic free classification of low degree surfaces, see, e.g., [**21**], we see that if $d = 3$, then X is a scroll, possibly singular, i.e., a cone, and that if $d = 4$, then X is the complete intersection of two quadric hypersurfaces, say U and

V, of $\mathbf{P}^4$. Hence, if $d = 3$, then X contains infinitely many lines. Assume $d = 4$. We reduce easily to the case in which U and V are smooth. Let $F(Z)$ be the smooth dimension 3 projective variety of lines contained in a smooth quadric hypersurface Z of $\mathbf{P}^4$. $F(Z)$ is seen as a subvariety of the Grassmannian $G(1,4)$ of lines in $\mathbf{P}^4$. Since the class $F(Z)^2$ in the Chow ring of $G(1,4)$ is a nonzero integer, we have that $F(U) \cap F(V) \neq \varnothing$. Every line $L \in F(U) \cap F(V)$ is a trisecant line to X. □

Now we will consider the case of a reducible surface $X \subset \mathbf{P}^5$.

Lemma 2.3. *Let $X \subset \mathbf{P}^5$ be a reducible surface. Assume that one of the irreducible components of X is not a Veronese surface. Then X has infinitely many trisecant lines.*

Proof. Let A be an irreducible component of X which is not a Veronese surface. If A is degenerated, then A, and hence X, has infinitely many trisecant lines by Lemma 2.2. Hence we may assume A nondegenerate. We may also assume that X is not a cone. Hence, by a theorem of Severi proved in positive characteristic by Dale [**8**, Theorem 6], the secant variety $\mathbf{Sec}\,(A)$ of A is $\mathbf{P}^5$. Hence, for a general point P of a component B of X, with $B \neq A$, hence with $P \notin A$, there is a line L with $P \in L$ and L secant to A. L is a trisecant line to X. □

By Lemma 2.3 X has at least one trisecant line (and indeed infinitely many ones) unless each irreducible component of X is a Veronese surface. From now on, we assume that X has no trisecant lines and that each irreducible component $X(i)$, $1 \leq i \leq s$, with $s \geq 2$, of X is a Veronese surface.

Remark 2.4. Let X and X' be different integral surfaces in $\mathbf{P}^5$. We want to check that $\mathbf{P}^5$ is the join $J(X, X')$ of X and X', i.e., by definition of join, $\mathbf{P}^5$ is the closure of the union of the lines spanned by a point $P \in X$ and a point $P' \in X'$ with $P \neq P'$. Fix a general point $Q \in \mathbf{P}^5$ and let Y, respectively Y', be the projection of X, respectively X', into $\mathbf{P}^4$. Since Q is general, we have $Y \neq Y'$. Since Y and Y' are surfaces, we have $Y \cap Y' \neq \varnothing$, i.e., $Q \in J(X, X')$.

Remark 2.5. By Remark 2.4 we have $s = 2$. Indeed, if $s \geq 3$, since $J(X(1), X(2)) = P^5$, for a general point $P \in X(3)$, there is a line L with $P \in L$, such that $X(1) \cap L \neq \varnothing$, $X(2) \cap L \neq \varnothing$ and $\mathrm{card}\,(L \cap (X(1) \cup X(2) \cup X(3)) \geq 3$.

3. Examples. (i) In the projective space $PG(5, q)$, q odd, where $(Z_{11}, Z_{12}, Z_{22}, Z_{13}, Z_{23}, Z_{33})$ are homogeneous projective coordinates, define $F_{ij} = Z_{ij}^2 - Z_{ii}Z_{jj}$, $i, j \in \{1, 2, 3\}$. Set $F := \cap X_{ij}$. Then F is a degree eight surface which is the union of two Veronese surfaces of $PG(5, q)$, say $X(1)$ and $X(2)$. The intersection of these two Veronese surfaces is the union of three conics pairwise intersecting in one point. Furthermore, F has $(2q^2 - q + 2)$ points over $GF(q)$ and admits no trisecant, namely, F is a $(2q^2 - q + 2)$-cap of $PG(5, q)$. For more details, see [**5**].

(ii) In the projective space $PG(3, q)$, $q \neq 3$, consider a twisted cubic in its canonical form:

$$\mathcal{C} = \{P(t) = P(t^3, t^2, t, 1) : t \in \gamma^+\},$$

where $t = \infty$ gives the point $(1, 0, 0, 0)$.

We recall that a *chord* of $\mathcal{C}$ is a line of $PG(3, q)$ joining either a pair of real points, defined over $GF(q)$ of $\mathcal{C}$, possibly coincident, or a pair of complex conjugate points of $\mathcal{C}$, namely points of $\mathcal{C}$ defined over a quadratic extension of $GF(q)$, cf. [**10**, Chapter 21]. Let $l(t_1, t_2) = P(t_1)P(t_2)$. Then

$$\begin{aligned} l(t_1, t_2) &= I(t_1^2 t_2^2, t_1 t_2 (t_1 + t_2), t_1^2 + t_1 t_2 + t_2^2, t_1 t_2, -(t_1 + t_2), 1) \\ &= I(\alpha_2^2, \alpha_1 \alpha_2, \alpha_1^2 - \alpha_2, \alpha_2, -\alpha_1, 1), \end{aligned}$$

where $\alpha_1 = t_1 + t_2$ and $\alpha_2 = t_1 t_2$. For $p \neq 3$, dual to the chords of $\mathcal{C}$ are the *axes* of the osculating developable Γ.

Let $l'(v_1, v_2) = \pi(v_1) \cap \pi(v_2)$. Then

$$\begin{aligned} l'(v_1, v_2) = I(&v_1^2 v_2^2, v_1 v_2 (v_1 + v_2), 3 v_1 v_2, \\ &(v_1^2 + v_1 v_2 + v_2^2)/3, -(v_1 + v_2), 1) \\ = I(&\beta_2^2, \beta_1 \beta_2, 3\beta_2(\beta_1^2 - \beta_2)/3, -\beta_1, 1) \end{aligned}$$

where $\beta_1 = v_1 + v_2$ and $\beta_2 = v_1 v_2$. From [**10**, Lemma 21.1.4] we have that the total number of chords of $\mathcal{C}$ is $q^2 + q + 1$. Dually, the number of axes of Γ is q^2+q+1. Now, it is easy to see that $l(t_1,t_2)$ represents the generic point of a Veronese surface $X(1)$ of $PG(5,q)$ embedded in $G(1,3)$. Similarly, $l'(v_1,v_2)$ represents the generic point of another Veronese surface $X(2)$ of $PG(5,q)$ embedded again in $G(1,3)$. So we obtain a degree eight surface F of $PG(5,q)$ embedded in $G(1,3)$ as the union of such two Veronese surfaces. Clearly, card $(X(1)\cap X(2)) = q+1$, since the tangent lines to $\mathcal{C}$ are self-dual. These tangents represented in $G(1,3)$ form a conic if q is even and a normal rational curve in $PG(4,q)$ if q is odd. In particular, F has $2q^2+q+1$ rational points and admits no trisecant, namely, F is a $(2q^2+q+1)$-cap of $PG(5,q)$. For more details, see [**7**].

Remark. Let $M(i)$, $i=1,2$, be the cubic hypersurface which is the secant variety of a Veronese surface $X(i)$.

If $M(i)\cap X(j)\neq X(i)\cap X(j)$, $i,j=1,2$, $i\neq j$, then, for infinitely many points of $M(i)\cap X(j)$, there is a line L meeting $X(j)$ and intersecting $X(i)$ at two other points, i.e., with card $(L\cap X)\geq 3$. So in order for $X = X(1)\cup X(2)$ to have no trisecants, $M(i)\cap X(j) = X(1)\cap X(2)$. This is the case in example (i). Here $X(1)$ and $X(2)$ are given in parametric equations by (u^2,uv,v^2,uw,vw,w^2) and $(u^2,-uv,v^2,uw,vw,w^2)$, respectively, where u,v,w are projective homogeneous coordinates in $PG(2,q)$.

The equations of $M(1)$ and $M(2)$ are

$$Z_0Z_2Z_5 - Z_0Z_4^2 - Z_1^2Z_5 + 2Z_1Z_3Z_4 - Z_2Z_3^2 = 0$$

and

$$Z_0Z_2Z_5 - Z_0Z_4^2 - Z_1^2Z_5 - 2Z_1Z_3Z_4 - Z_2Z_3^2 = 0,$$

respectively, where (Z_i), $0\leq i\leq 5$, are projective homogeneous coordinates in $PG(5,q)$. It is easy to see that $M(1)\cap X(2) = M(2)\cap X(1) = X(1)\cap X(2)$ is the locus $-4u^2v^2w^2=0$. In particular we may assume that $(X(1)\cap X(2))_{\mathrm{red}}$ is the support of a plane curve (possibly reducible or not reduced). In example (i), $X(1)\cap X(2)$ is a reducible degree 6 plane curve.

4. The main results.

The trisecant formula. Let $C\subset\mathbf{P}^4$ be a smooth genus g curve of degree d. In 1895, Berzolari "proved" that the number of trisecant lines to C is $((d-2)(d-3)(d-4))/6-g(d-4)$ in the following sense: either C has only finitely many (or none) trisecant lines and then their number, "counting multiplicities" is exactly $((d-2)(d-3)(d-4))/6-g(d-4)$ or C has infinitely many trisecant lines. In particular, if we assume that C has no trisecant lines, then $((d-2)(d-3)(d-4))/6 = g(d-4)$. This was made rigorous by several mathematicians and the formula verified in characteristic 0, see, for instance, [**6, 12, 13, 14, 15, 18**]. Here we want to explain why these foundational works allow us to get in positive characteristic the same assertion: if C has no trisecant lines, then $((d-2)(d-3)(d-4))/6 - g(d-4) = 0$. The foundational works in [**6, 12**] and [**18**] are characteristic free. However, the approach of [**12**] does not allow us to conclude that if there is no trisecant line, then a suitable number, say $t(C)$, is zero. The other approaches allow this crucial assertion, see the introduction of [**18**], and hence, [**6**] or [**18**], we may use the triple-point formula of [**12, 6**] or [**18**] to compute the number $t(C)$. In characteristic zero always holds true that $t(C) = ((d-2)(d-3)(d-4))/6 - g(d-4)$. As remarked in [**15**, first part of page 2], we may use in the following way the triple-point formula to compute $t(C)$. Since the triple-point formula has no term depending on the characteristic and we do not care about the multiplicities of the solutions, we will obtain in this way that $t(C) = 0$ is equivalent to $((d-2)(d-3)(d-4))/6 - g(d-4) = 0$, as wanted.

Let Z be a subvariety of $\mathbf{P}^N$, and let $G(1,N)$ be the Grassmannian of lines in $\mathbf{P}^N$.

Let $T := \{(P,L) \in Z \times G(1,N) : P \in L\}$ be the incidence variety. Note that T is smooth if Z is smooth (as in our case in which Z is the smooth curve C). The trisecant lines to Z correspond to the locus of triple points, in the sense of [**6**] or [**12**], of the projection $T \to G(1,N)$.

Theorem 4.1. *Let $X \subset \mathbf{P}^5$ be an integral surface without trisecant lines. Assume that X has at most isolated singularities. Then either X is a Veronese surface or it is a degree 8 surface, which is the complete intersection of three quadric hypersurfaces or it has the following description*:

$\deg(X) = 6$*, the general hyperplane section of X has genus 2, X is linearly normal and contained in four linearly independent quadrics,* $\mathrm{card}\,(\mathbf{Sing}\,(X)) \leq 3$*, a desingularization of X is a rational surface.*

Vice versa, a complete intersection of three quadric hypersurfaces admits a trisecant line if and only if it contains a line, and this is not the case for the general complete intersection of three quadric hypersurfaces.

Proof. The last assertion was proved in the last part of [**3**, Theorem 1] without using the characteristic zero assumption. We divide the proof of the main assertion into eight steps. Set $d := \deg(X)$.

Step 1. By Lemma 2.2 we may assume that X spans $\mathbf{P}^5$.

Step 2. Since $\dim(\mathbf{Sing}(X)) \le 0$ by assumption, for a general hyperplane H, the integral curve $C := X \cap H$ is smooth. Set $g := p_a(C)$. Since C has no trisecant lines, by the trisecant formula, we have $((d-2)(d-3)(d-4))/6 - g(d-4) = 0$. If $d = 4$, by the characteristic free description of surfaces of minimal degree, see [**21**] for the case $d = 4$, either X is the Veronese surface or X is a scroll (possibly singular, i.e., a cone over a rational normal curve of $\mathbf{P}^4$). In the latter case, X contains infinitely many lines, a contradiction. Hence, we may assume that $d \ge 5$ and $g = ((d-2)(d-3))/6$. Set $m_1 := [(d-1)/4]$, $\varepsilon_1 := d - 4m_1 - 1$ and $\mu_1 := 1$ if $\varepsilon_1 = 3$ and $\mu_1 := 0$ otherwise. Set $\pi_1(d,4) := 4m_1(m_1-1)/2 + m_1(\varepsilon_1+1) + \mu_1$. First, assume that $g > \pi_1(d,4)$. If $d > 10$, by [**1**, p. 123], (in positive characteristic, see for instance, the proof of [**2**, Proposition 2.8]), C is contained in a minimal degree surface T of H, $\deg(T) = 3$. Hence, by the characteristic free classification of such surfaces, see, e.g., [**21**], T is either a smooth rational scroll or the cone over a rational normal curve of $\mathbf{P}^3$. First assume that T is a smooth rational scroll. Hence $T \equiv F_1$ (the Segre-Hirzebruch surface) and $\mathrm{Pic}(T) \simeq \mathbf{Z}^2$, with base h, f, where h is a hyperplane section of T and f is a line of the ruling, such that $h^2 = -1$, $h \cdot f = 1$ and $f^2 = 0$. Furthermore, $\mathcal{O}_T(1) \equiv h + 2f$ and $\omega_T \equiv -2h - 3f$. We have $C \in |ah + bf|$ with $a > 0$ and $b \ge a$. We have $d := (ah+bf)\cdot(h+2f) = a+b$ and $2g-2 = (ah+bf)\cdot((a-2)\cdot h+(b-3)\cdot f)$ (adjunction formula), i.e., $2g-2 = -a^2 - a + 2ab - 2b$. We have that h is embedded as a line, say J. Since $C \cdot h = (ah+bf)\cdot h = b - a$, if $b \ge a + 3$, J is a trisecant line to C. Hence we may assume that $b \le a + 2$. Note that $C \cdot f = (ah + bf) \cdot f = a$. Since $d \ge 5$, we have $a \ge 3$ and so C and X have infinitely many trisecant lines given by the lines of the ruling of T. Now assume that T is the cone over a rational normal curve. Let $\pi : U \to T$ be the minimal desingularization of T and C' the strict transform of C in U. Since C is smooth, $C' \equiv C$. We have $U \equiv F_3$ and $\mathrm{Pic}\,(U) \simeq \mathbf{Z}^2$ with basis h, f such that $h^2 = -3$, $h \cdot f = 1$ and $f^2 = 0$. Furthermore, $\pi^*(\mathcal{O}_T(1)) \equiv h + 3f$ and $\pi(h)$ is the vertex of T. Set $C' \in |ah + bf|$ with $a > 0$ and $b \geq 3a$. Since C is smooth, we have $b - 3a \leq 1$ and $b = 3a$ if and only if C does not contain the vertex of T. We have $d = (ah + bf) \cdot (h + 3f) = b$. If either $b = 3a + 1$ and $a \geq 2$ or $b = 3a$ and $a \geq 3$, C and X have infinitely many trisecant lines. Since $b = d > 10$, we conclude

Step 3. Note that $d \neq 10$ because $(d-2)(d-3)/6 = g$ is an integer. Here we assume $d = 9$ and so $g = 7$. Set $m := [(d-1)/3]$, $\varepsilon := d-1-3m$, $\pi(d, 4) := 3m(m-1)/2 + m\varepsilon$, [**1**, p. 116]. Note that $\pi(9, 4) = 7 = g$. By [**1**, Theorem 2.5], see [**18**, Section 2] or [**2**, proof of Proposition 2.8], in the case of positive characteristic, C has infinitely many trisecant lines, a contradiction.

Step 4. Now assume $g \leq \pi_1(d, 4)$. Checking the four possible congruence classes of $d \bmod (4)$, we obtain that $d \leq 8$ and so $g \leq 5$. Furthermore, $d = 7$ is excluded since g is an integer. If $(d, g) = (5, 1)$ or $(6, 2)$, we have $h^1(C, \mathcal{O}_C(1)) = 0$ and so C and X are linearly normal. If $d = 8$, C must be linearly normal by the bound of the genus for nondegenerate curves in $\mathbf{P}^6$ [**18**, Section 2] in the case of positive characteristic. Hence, X is always linearly normal, i.e., $h^1(\mathbf{P}^5, \mathcal{I}_X(1)) = 0$. From the exact sequence

$$0 \longrightarrow \mathcal{I}_X(1) \longrightarrow \mathcal{I}_X(2) \longrightarrow \mathcal{I}_{C,H}(2) \to 0$$

and the linear normality of X, we obtain that every quadric hypersurface of H containing C lifts to a quadric containing X. First assume $(d, g) = (8, 5)$. Since $h^0(C, \mathcal{O}_C(1)) \geq g = 5$ and $d \leq 2g - 2$, we see that C is embedded as a canonical curve. By a theorem of Petri and Saint-Donat, see [**19**] for the positive characteristic case, C is contained in three linearly independent quadrics and either C is trigonal or C is the complete intersection of these three quadrics. Furthermore, if C is trigonal, then it is contained in a degree 3 scroll T and hence C admits infinitely many trisecant lines, as checked in Step 2. Since these quadrics lift to quadrics containing X, if C is a complete intersection we easily see that X is the complete intersection of three quadrics, as wanted.

Step 5. Here we assume that X is smooth and $(d, g) = (5, 1)$. Set $M := \mathcal{O}_X(1)$. By the adjunction formula we have that $(K_X + M) \cdot M = 0$ and so X has Kodaira dimension < 0. Furthermore, taking $X \cap H'$ with H' tangent hyperplane to X, we see that through a general point $P \in X$, there is a singular degree 5 rational curve. Hence, by the classification of surfaces, X is rational. Since $(K_X + M) \cdot M = 0$, we have that $h^2(X, K_X + M) = 0$. Hence, by Riemann-Roch's theorem, $h^0(X, K_X + M) \geq 1$. Hence, since $(K_X + M) \cdot M = 0$, it follows that $K_X \equiv M^*$, namely, X is embedded as a Del Pezzo surface. By the classification of such surfaces [**16**], we know that X is isomorphic to the blowing-up Y of $\mathbf{P}^2$ at five points P_i, $1 \leq i \leq 5$, such that these five points are contained into a unique conic, say D, which is irreducible. The strict transform of D in Y is embedded as a degree 1 curve in $\mathbf{P}^5$, i.e., X contains a line.

Step 6. Here we assume that X is smooth and $(d, g) = (6, 2)$. Set $M := \mathcal{O}_X(1)$ and so $M^2 = 6$. By the adjunction formula, we have that $K_X \cdot M = -4$. Hence, X has Kodaira dimension < 0. Furthermore, taking the intersection of X with a general tangent hyperplane, we see that X contains many singular elliptic or rational curves. Hence, we easily see that X is rational. Thus, $K_X + M$ satisfies Kodaira vanishing $h^i(X, K_X + M) = 0$ for $i > 0$, see, e.g., [**20**, Corollary 8], but the result was previously known and very elementary for X rational and M very ample. Hence, by Riemann-Roch, we have $h^0(X, K_X + M) = 1 + (K_X + M)M/2 = 2$. Thus $|K_X + M|$ is a pencil of conics. Since we have assumed that X contains no lines, we may also assume that each of these conics is irreducible. Hence, the base locus of $|K_X + M|$ is finite and nonempty. Let $Z \to X$ be the resolution of this base locus. This pencil of conics induces a morphism $\pi : Z \to \mathbf{P}^1$ such that every fiber of π is a smooth rational curve. Hence π is a $\mathbf{P}^1$-bundle, $Z \equiv F_e$, the Segre-Hirzebruch surface, with $e \geq 0$, Z contains at most one curve with self-intersection $-e < 0$ and the contraction of this curve is smooth, as was assumed to be X, if and only if $e = 1$. If $e = 1$ we have $X \equiv \mathbf{P}^2$. Since the plane has no degree 6 embedding, we obtain a contradiction.

Step 7. Here we assume that $(d, g) = (5, 1)$ and $\mathbf{Sing}(X)$ is finite and not empty. Fix $P \in \mathbf{Sing}(X)$. We may assume that X is not a cone with vertex P. Let X' be the projection of X from P into $\mathbf{P}^4$. Let Y be the cone with vertex P and base X', hence $X \subset Y$. X' is a surface of degree d-$\mathrm{mult}_P(X) \leq d - 2$. Hence, if $d = 5$, then X' is a rational scroll, possibly singular, i.e., a cone. Thus, X has a rational pencil of plane curves and hence infinitely many trisecant lines unless these plane curves are conics. We may also assume that all these conics are smooth. We easily obtain that X is rational. Hence, applying Riemann-Roch and the adjunction formula to a minimal desingularization of X, we obtain that X is a weak Del Pezzo surface, i.e., the anticanonical image of the blowing-up of $\mathbf{P}^2$ at 5 (possibly infinitely near) points. As in Step 6 we see that X contains at least one line.

Step 8. Here we assume $(d, g) = (6, 2)$ and $\mathbf{Sing}(X)$ finite and not empty. We copy the proof of Step 6. We obtain that X is rational and we conclude if X has one point of multiplicity ≥ 3. We assume that this is not the case. If $\mathrm{card}(\mathbf{Sing}(X)) \geq 2$, projecting X into $\mathbf{P}^3$ from the line spanned by two singular points (assuming that this line is not contained in X) we obtain an integral quadric surface $Y \subset \mathbf{P}^3$. Since Y has at most one singular point and is of multiplicity ≤ 2, we have that $\mathrm{card}(\mathbf{Sing}(X)) \leq 3$. Hence we are in the exceptional case. We do not know if the exceptional case (Theorem 4.1) occurs. □

Now we deal with surfaces embedded in $G(1, 3)$. We recall that an integral surface X contained in the Grassmannian $G(1, 3)$ in $\mathbf{P}^5$ has a bidegree (a, b) with a and b nonnegative integers. We have the following

Theorem 4.2. *Let X be an integral projective surface of bidegree (a, b) contained in the quadric $G(1, 3)$ of $\mathbf{P}^5$. Assume that, except at finitely many points, X has only planar singularities, i.e., such that the Zariski tangent space has dimension two. Assume that* $\min(a, b) \neq 2$. *Assume that X is neither the Veronese surface nor the complete intersection of three quadrics not containing a line nor the exceptional case, Theorem* 4.1. *Then there exists a trisecant line to X.*

Proof. In order to obtain a contradiction, we assume that X has no trisecant lines. By Lemma 2.3, we may assume X to be nondegenerate. Let H be a general tangent hyperplane to $G(1,3)$ at a general point $P \in X$. Set $Q' := G(1,3) \cap H$. Q' is a quadric cone of H with vertex P and base a smooth quadric Q of $\mathbf{P}^3$. Set $C := X \cap H$. By the generality of H we may assume that C has only planar singularities and that C is smooth at P. Since C has no trisecant lines and only planar singularities, the projection of C from P is a curve $Z \subset Q$ with $\deg(Z) = \deg(C) - 1$ and $Z \equiv C$. Furthermore, the isomorphism $Z \equiv C$ induces an isomorphism $\mathcal{O}_C(1)(-P) \equiv \mathcal{O}_Z(1)$. Call P' the point of Z corresponding to P under this isomorphism, i.e., the intersection of the tangent line to C at P with the $\mathbf{P}^3$ spanned by Q. Z is a curve of type (a,b) on Q with, say, $a \leq b$. Here we assume $a \geq 3$. By the adjunction formula $\omega_Z = \mathcal{O}_Z(a-2, b-2)$. Hence, the assumption $a \geq 3$ means that $\omega_Z(-1) \equiv \omega_Z(a-3, b-3)$ is base point free. By Riemann-Roch and Serre duality this means that $h^0(Z, \mathcal{O}_Z(1)(P')) = h^0(Z, \mathcal{O}_Z(1)) = 4$. The isomorphism $C \simeq Z$ maps $\mathcal{O}_Z(1)(P')$ to $\mathcal{O}_C(1)$. Hence $h^0(Z, \mathcal{O}_Z(1)(P')) = h^0(C, \mathcal{O}_C(1)) = 5$, a contradiction. If $a = 1$, then Z is smooth and rational and so C is smooth and rational. In particular, X has only isolated singularities and by Theorem 4.1 X has a trisecant line if it is not in one of the excluded cases. $\square$

We were unable to check and exclude the case $\min(a,b) = 2$ from the exceptional cases of Theorem 4.2.

5. Other definitions of multisecant lines. Alternative approaches. There is a definition of multisecant line to singular projective varieties which arises from the theory of special divisors on smooth curves mapped birationally to some projective space.

Definition 5.1. Let C be a smooth projective curve. We do not assume that C is connected. Let $\pi : C \to \mathbf{P}^N$ be a rational morphism and $L \subset \mathbf{P}^N$ a line. L is the intersection of $N-1$ hyperplanes H_i, $1 \leq i \leq N-1$, and each hyperplane corresponds to an effective divisor D_i on C, with $\deg(D_i) = \deg(\pi(C))$. Let D be the intersection of these effective divisors (counting multiplicities). D does not depend on the choice of the hyperplanes H_i. Either $D = \varnothing$ or D is an effective divisor on C. In the latter case we will say that L is a *k-secant* to $\pi(C)$ if

$k \leq \deg(D)$.

Remark 5.2. Fix C, π, L as in the previous definition. Fix $P \in L \cap \pi(Y)$. Assume that $\pi(C)$ has at least t branches at P. Let D' be the union of the components of D with $\pi(D') = P$. Then $\deg(D') \geq t$.

Remark 5.3. If π is an embedding, i.e., $\pi(C)$ is smooth, then a line L is 3-secant to $\pi(C)$ in the sense of Definition 5.1 if and only if it is trisecant in the sense of Definition 1.1, and a similar observation is true for the corresponding definitions of k-secant lines, $k \geq 1$. Note that, by Remark 5.1 and its proof (plus a local computation) this is always false if $N > 2$ and L is a general line through a singular point P.

Remark 5.4. Let $C' \subset \mathbf{P}^4$ be an integral curve and $\pi : C \to C'$ the normalization. Set $d := \deg(C')$ and let $g := p_a(C')$ be the geometric genus of C'. We claim that if $\pi(C')$ has no 3-trisecant lines in the sense of Definition 5.1, then $(d-2)(d-3)(d-4) = g(d-4)$. The claim follows from the characteristic free computation of an enumerative formula due to Castelnuovo, see [**1**, Proposition 4.2] for a more general but less explicit formula due to Macdonald and [**9**, Theorem 2.1] for the proof of the Castelnuovo formula and the correction of some misprints in [**1**]. For the foundational work needed [**1**, Section 1], i.e., Grothendieck-Riemann-Roch formula, and [**1**, Chapter 8] in a positive characteristic, it is sufficient to use étale cohomology with value in $\mathbf{Z}/l\mathbf{Z}$, l a prime different from the characteristic, instead of singular cohomology with $\mathbf{Z}$ as coefficients. In particular, if C' is smooth and C' has no trisecant lines in the sense of Definition 1.1, by Remark 5.3 we have that $(d-2)(d-3)(d-4) = g(d-4)$.

Definition 5.5. Let Y be an integral, normal projective surface, $\pi : Y \to \mathbf{P}^N$ a birational morphism and $L \subset \mathbf{P}^N$ a line. We will say that L is 3-*secant* to $\pi(Y)$ if either L is trisecant to $\pi(Y)$ (in the sense of Definition 1.1) or there is a curve $C' \subset Y$, such that, calling $f : C \to C' \subset \mathbf{P}^N$ the normalization, L is 3-secant to $C' := f(C)$ in the sense of Definition 5.1.

Remark 5.6. Fix Y and π as in Definition 5.5. Fix $P \in \mathbf{Sing}(Y)$.

Assume that $\pi(Y)$ has no multiple linear space as a tangent cone at P. Then the intersection of $\pi(Y)$ with a general linear space of codimension $\dim(Y) - 1$ passing through P is a singular curve with at least two branches. Hence, every line L containing P is 2-secant to $\pi(Y)$ and so every line L with $P \in L$ and $L \cap \pi(Y) \neq \{P\}$ is trisecant to $\pi(Y)$.

6. Integral surfaces of $\mathbf{P}^5$ which contain every trisecant line. Our purpose in this last section is to give a few examples of integral surfaces $X \subset \mathbf{P}^5$ such that every trisecant line to X, in the sense of Definition 1.1, is contained in X. By virtue of Theorem 4.1, we will see that in many such examples X contains a line.

We fix a nondegenerate integral surface $X \subset \mathbf{P}^5$ and set $d := \deg(X)$. Let g be the arithmetic genus of a general hyperplane section of X.

Remark 6.1. Assume that X is linearly normal. Assume that there is a finite set W of X, possibly $W = \varnothing$, such that every hyperplane section of X disjoint from W is set-theoretically cut out by quadrics. We fix any such hyperplane section $C := X \cap H$ with $H \cap W = \varnothing$. Every trisecant line to C, in the sense of Definition 2.1, is contained in the intersection of the quadrics containing C. Hence, every trisecant line to C is an irreducible component of C. Now fix a trisecant line L to X. If $L \cap W \neq \varnothing$, we take a hyperplane H with $L \subset H$ and $H \cap W = \varnothing$. Then L is a component of $X \cap L$ and so $L \subset X$. In particular, if $W = \varnothing$, every trisecant line to X is contained in X. This is the case in the following

Example 6.2. Assume X is smooth and $d \leq 8$ and either $(d, g) = (8, 5)$ or $g + 4 = d \leq 6$. If $(d, g) = (8, 5)$ we checked that X is the complete intersection of three quadrics, and so from Bezout's theorem if a line is trisecant to X it will be completely contained in X. Hence we may assume $(d, g) = (6, 2)$ or $(5, 1)$ or $(4, 0)$. In the latter case, we conclude by the classification of varieties of degree 4 [**21**]. Hence we may assume $(d, g) = (6, 2)$ or (5,1). In all cases we checked in the proof of Theorem 4.1 that X is linearly normal and that every smooth hyperplane section of X has homogeneous ideal generated by quadrics. Let H be a hyperplane such that $H \cap X$ is integral. Since $d \geq 2g + 2$ and $g := p_a(X \cap H)$, the classical proof of the fact that a linearly normal curve of that degree and genus, has homogeneous ideal generated by quadrics, gives that the integral curve $X \cap H$ is set-theoretically cut out by quadrics. In order to obtain a contradiction, assume the existence of a trisecant line L to X with $\mathrm{card}\,(X \cap L)$ finite. Since $\mathrm{card}\,(X \cap L)$ is finite, and X is smooth, for a general hyperplane H, with $L \subset H$, the scheme $X \cap H$ is smooth at all points of $X \cap L$. Since X is smooth, $X \cap H$ is also locally Cohen-Macaulay, i.e., it has no embedded components. Since $X \cap H$ is smooth along $X \cap L$, $X \cap H$ cannot be a multiple of a curve. By the generality of H, $X \cap H$ contains no line and all the irreducible components of $(X \cap H)_{\mathrm{red}}$ have the same degree, and all of them appear with the same multiplicity in the scheme $X \cap H$. Since 5 is prime, only the possibility $d = 6$ remains and $X \cap H$ reduced but reducible, with two components of degree 3 or three components of degree 2. First assume that the rational projection of X from L into $\mathbf{P}^3$ has a surface image. Since X is irreducible, by Bertini's theorem we obtain that, for a general hyperplane H with $L \subset H$, the scheme $(X \cap H)_{\mathrm{red}}$ is irreducible (and in characteristic $0, X \cap H$ is also irreducible). Hence, $X \cap H$ is irreducible and so set-theoretically intersection of quadrics.

REFERENCES

1. E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, *Geometry of algebraic curves*, Grundlehren Math. Wiss. **267** (1985).

2. E. Ballico, *On singular curves in the case of positive characteristic*, Math. Nachr. **141** (1989), 267–273.

3. ———, *A characterization of the Veronese surface*, Proc. Amer. Math. Soc. **105** (1989), 531–534.

4. M. Beltrametti and A.J. Sommese, *On k-spannedness for projective surfaces*, Lecture Notes in Math. **1417** (1989), 24–51.

5. A. Bonisoli and A. Cossidente, *Inscribed bundles, Veronese surfaces and caps*, Cambridge University Press, Cambridge, to appear.

6. S. Colley, *Enumerating stationary multiple-points*, Adv. in Math. (China) **66** (1987), 149–170.

7. A. Cossidente, J.W.P. Hirschfeld and L. Storme, *Applications of line geometry over finite fields* III: *The quadric Veronesean and the chords of a twisted cubic*, submitted.

8. M. Dale, *Severi's theorem on the Veronese surface*, J. London Math. Soc. **32** (1985), 419–425.

9. D. Eisenbud, H. Lange, G. Martens and F.O. Schreyer, *The Clifford dimension of a projective curve*, Compositio. Math. **72** (1989), 173–204.

10. J.W.P. Hirschfeld, *Finite projective spaces of three dimensions*, Clarendon Press, Oxford, 1985.

11. J.W.P. Hirschfeld and J.A. Thas, *General Galois geometries*, Oxford University Press, Oxford, 1991.

12. S. Kleiman, *Multiple-point formulas* I: *Iteration*, Acta Math. **147** (1981), 13–49.

13. S. Kleiman (with the collaboration of A. Thorup), *Intersection theory and enumerative geometry*: *a decade in review*, in *Algebraic geometry*, Proc. Sympos. Pure Math. **46** (1985), 321–370.

14. P. Le Barz, *Quelques calculus dans les variétes d'alignement*, Adv. in Math. (China) **64** (1987), 87–117.

15. ———, *Formules pour les trisécant des surfaces algébriques*, Enseign. Math. **64** (1987), 1–66.

16. Y. Manin, *Cubic forms*, North Holland, Amsterdam, 1986.

17. Z. Ran, *Curvilinear enumerative geometry*, Acta Math. **155** (1985), 81–101.

18. J. Rathmann, *The uniform position principle for curves in characteristic* p, Math. Ann. **276** (1987), 565–579.

19. B. Saint-Donat, *On Petri's analysis of the linear system of quadrics through a canonical curve*, Math. Ann. **206** (1973), 157–175.

20. N.I. Sheperd-Barron, *Unstable vector bundles and linear systems on surfaces in characteristic* p, Invent. Math. **106** (1991), 243–262.

21. H.P.F. Swinnerton-Dyer, *An enumeration of all varieties of degree* 4, Amer. J. Math. **95** (1973), 403–418.

Department of Mathematics, University of Trento, via Sommarive 14, 38050 Povo (TN), Italy
E-mail address: Ballico@alpha.science.unitn.it

Department of Mathematics, University of Basilicata, via N. Sauro 85, 85100 Potenza
E-mail address: Cossidente@unibas.it