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CYCLIC COMPOSITION OPERATORS ON SMOOTH WEIGHTED HARDY SPACES

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ABSTRACT. We determine hypercyclic and cyclic composition operators induced by a linear fractional self map of the unit disc, acting on a special class of weighted Hardy spaces. We establish the extreme possible cases and provide examples of spaces where they occur.

1. Introduction. Let H be a Hilbert space of functions analytic in the unit disc $\mathbf{D}$, and let ϕ be a nonconstant self map of $\mathbf{D}$. The *composition operator* C_ϕ on H is defined by $C_\phi f = f \circ \phi$ for all f in H.

When H is the classical Hardy space H^2, the operator C_ϕ is bounded. Some general properties of C_ϕ on H^2 are known, but still there are a lot of open basic questions. The situation becomes more complicated as we turn to some general classes of Hilbert spaces, for example, weighted Hardy spaces. Then it is still an open question precisely which analytic self maps of $\mathbf{D}$ will induce bounded composition operators. Nevertheless, composition operators provide a very interesting and important class of concrete examples of operators and, like multiplication operators, give a natural connection between operator theory and analytic function theory.

For an extensive reference on composition operators in general, see [**4**] and [**10**].

This paper deals with the problem of cyclicity of composition operators. Recall that the operator T on a Hilbert space H is *cyclic* if there is a vector f, called *cyclic vector*, whose orbit $\{T^N f | n \geq 0\}$ has a dense linear span in H, and T is *hypercyclic* if there is a vector, called *hypercyclic vector*, whose orbit is dense in H. Hypercyclicity is a much stronger property than cyclicity and, clearly, every hypercyclic operator is cyclic.

Bourdon and Shapiro have done an extensive study of cyclic and hypercyclic linear fractional composition operators on H^2, see [**1**] and [**2**].

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We shall determine cyclic and hypercyclic linear fractional composition operators on smooth weighted Hardy spaces. As in [**1**], the cyclic behavior will be determined by the position of the fixed points of the inducing function. However, the classification itself reveals a completely different situation.

Non-elliptic linear fractional transformations induce composition operators on disc-automorphism invariant smooth weighted Hardy spaces that are:

(a) never hypercyclic,

(b) cyclic only when ϕ has two fixed points in $\mathbf{C}\cup\{\infty\}$, one of which is exterior, i.e., outside $\overline{\mathbf{D}}$.

The main results containing the above classification and a few other results in a more general setting are presented in Section 2, after the preliminaries. Section 3 contains results on the cyclicity of composition operators on non disc-automorphism invariant smooth weighted Hardy spaces and some open problems.

2. Preliminaries. Let us define first the weighted Hardy spaces $H^2(\beta)$. For more details, see Shield's paper on weighted shifts and analytic function theory, [**11**].

Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be a positive sequence with $\beta_0 = 1$, and let

$$H^2(\beta) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n;\ a_i \in \mathbf{C},\ \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}.$$

The inner product inducing the norm in $H^2(\beta)$ is defined by

$$(f,g)_\beta = \sum_{n=0}^{\infty} a_n \bar{b}_n \beta_n^2,$$

where f and g are formal power series in $H^2(\beta)$ with coefficients a_n and b_n, respectively. If the sequence β has the property that $(\beta_{n+1}/\beta_n) \to 1$ as $n \to \infty$, the Hilbert space $H^2(\beta)$ is a space of functions analytic in the unit disc $\mathbf{D} = \{z : |z| < 1\}$.

Examples. 1) If $\beta_n = (n+1)^a$ for some a in $\mathbf{R}$, we denote the spaces $H^2(\beta)$ by S_a. Then $S_0 = H^2$, the classical Hardy space; $S_{1/2} = \mathbf{D}$, the Dirichlet space; and $S_{-1/2} = B$, the Bergman space.

2) If $\beta_n = \exp n^a$ for a in $[1/2, 1)$, we denote the spaces $H^2(\beta)$ by Q_a.

Every derivative of a function in the space Q_a exists and is continuous on the closed unit disc. Functions in Q_a can have only finitely many zeros in the closed unit disc, counting multiplicity, and so they form a quasi-analytic class of functions.

For ω in $\mathbf{D}$, let

$$k_\omega^\beta(z) = \sum_{n=0}^{\infty} \frac{1}{\beta_n^2} \bar{\omega}^n z^n.$$

Then $(f, k_\omega^\beta)_\beta = f(\omega)$ for all f in $H^2(\beta)$. Hence, k_ω^β is the *reproducing kernel* for ω, and it belongs to $H^2(\beta)$. This is not true in general if ω is on the unit circle. But, if the sequence β converges to infinity fast enough, for example, if $\sum_{n=0}^\infty (1/\beta_n^2) < \infty$, then all of the above is true for all ω in the closed unit disc. Note that, in this case, the spaces $H^2(\beta)$ are small and consist of functions that are continuous on the unit circle.

For a point ω in $\mathbf{D}$, we define a *derivative reproducing kernel* d_ω^β by

$$d_\omega^\beta(z) = \frac{d}{d\bar{\omega}} k_\omega^\beta(z).$$

Then $d_\omega^\beta(z) = \sum_{n=1}^\infty (n/\beta_n^2) \bar{\omega}^{n-1} z^n$. For every ω in $\mathbf{D}$, d_ω^β belongs to $H^2(\beta)$, and $(f, d_\omega^\beta)_\beta = f'(\omega)$.

Taking the sequence β to approach infinity fast enough so that $\sum_{n=1}^\infty (n^2/\beta_n^2) < \infty$, we get that the functions $d_\omega^\beta(z)$ are in $H^2(\beta)$ and that $(f, d_\omega^\beta)_\beta = f'(\omega)$ for all ω in $\overline{\mathbf{D}}$ and f in $H^2(\beta)$. Note that in this case the space $H^2(\beta)$ is very small and, for every function in it, that the first derivative exists and is continuous on the unit circle.

We shall call such spaces $H^2(\beta)$ *smooth weighted Hardy spaces.*

It is easy to see that, for a point ω in $\mathbf{D}$ and a reproducing kernel k_ω^β in $H^2(\beta)$ we have that

$$C_\phi^* k_\omega^\beta = k_{\phi(\omega)}^\beta$$

where C_ϕ^* is the adjoint of the (bounded) composition operator C_ϕ on $H^2(\beta)$. If ξ in $\mathbf{D}$ is a fixed point of ϕ, where ϕ and d_ξ^β are in $H^2(\beta)$, then

$$C_\phi^* d_\xi^\beta = \overline{\phi'(\xi)}\, d_\xi^\beta$$

for C_ϕ a (bounded) composition operator on $H^2(\beta)$. Whenever $H^2(\beta)$ is a smooth weighted Hardy space, both of the above equations are true for all ω in $\overline{\mathbf{D}}$. Note that, in this case, the derivative $\phi'(\xi)$ exists as a radial limit of ϕ' at ξ.

A necessary condition for the operator C_ϕ to be cyclic on $H^2(\beta)$ is that ϕ be univalent. The proof from [**1**, Proposition 1.2] for C_ϕ cyclic on H^2 applies. The main idea is to use the fact that if $\phi(a) = \phi(b)$ for $a \neq b$, then $k_a^\beta - k_b^\beta$ is a nonzero function in the orthogonal complement of the range of C_ϕ, and then to produce at least two pairs of such a's and b's. For details, see [**1**].

Note that the univalence of the inducing map guarantees boundedness of the composition operator on some small weighted Hardy spaces, like for example S_a spaces with $0 < a < 1/2$. For details, see [**9**].

Linear fractional maps, as a simple example of univalent functions, will be of special interest. They are maps of the form $((az+b)/(cz+d))$ where a, b, c and d are complex numbers satisfying $ad - bc \neq 0$.

Linear fractional maps have either one or two fixed points in $\mathbf{C} \cup \{\infty\}$ and, with respect to their general behavior around the fixed points, they are classified into four different groups: parabolic (only one fixed point), elliptic, hyperbolic and loxodromic (see the first chapter in Ford's book on automorphic functions [**5**]).

We shall be interested in the linear fractional self maps of $\mathbf{D}$. In that case the Denjoy-Wolff theorem says even more about the fixed point properties.

Theorem (Denjoy-Wolff). *Suppose that ϕ is analytic in $\mathbf{D}$, maps $\mathbf{D}$ into itself and is not an elliptic automorphism of $\mathbf{D}$. Then there exists a unique fixed point α in $\overline{\mathbf{D}}$ such that the sequence of iterates of ϕ, $\{\phi^{(n)}\}_{n=0}^{\infty}$, converges to the constant function α uniformly on compact subsets of δ. Moreover,*

(i) *if $|\alpha| < 1$, then $0 \leq |\phi'(\alpha)| < 1$ and*

(ii) *if $|\alpha| = 1$, then $\lim_{r \to 1} \phi'(r\alpha)$ exists and is in $(0, 1]$.*

This special fixed point α is called the *Denjoy-Wolff point* of ϕ.

To prove the cyclicity of the composition operator in some of the cases in the following sections, we shall refer to a theorem of Clancey and Rogers, see [**3**, Lemma 4 and Theorem 3].

Theorem (Clancey and Rogers). *If A is a Hilbert space operator and there exists a spanning set of eigenvectors for A corresponding to pairwise distinct eigenvalues, then A is cyclic and, moreover, A has a dense set of cyclic vectors.*

Note that, if an operator A has a hypercyclic vector f, then it has a dense set of hypercyclic vectors because every element in the orbit of f is also hypercyclic for A. This is not necessarily true for every cyclic operator. For example, the operator multiplication by z on H^2 has outer functions for cyclic vectors and they are not dense in H^2. On the other hand, composition operators on smooth weighted Hardy spaces, the same as on H^2 (see [**1**]), are either noncyclic or have a dense set of cyclic vectors.

In some cases the cyclicity of a composition operator on two different spaces can be derived one from another. If $H^2(\beta_1)$ and $H^2(\beta_2)$ are two spaces such that $H^2(\beta_1) \subset H^2(\beta_2)$, then a composition operator that is bounded and cyclic (or hypercyclic) on the smaller space must also be cyclic (or hypercyclic) on the bigger space.

This, or more precisely the part about hypercyclicity, is discussed in [**10**, p. 111] as a property called the *Comparison Principle*. The main reasons why the comparison works are that, since polynomials are dense in all of the weighted Hardy spaces, the space $H^2(\beta_1)$ is dense in $H^2(\beta_2)$ and that the convergence in $H^2(\beta_1)$ implies convergence in $H^2(\beta_2)$. A cyclic (or hypercyclic) vector of the operator from the smaller space is then a cyclic (or hypercyclic) vector of the operator on the bigger space.

Thus, exploring cyclicity on smaller spaces provides cyclic vectors for the operator on the bigger space that are nicer in the sense that they satisfy some additional properties. For details, see [**10**].

Note that, for example, Proposition 3.4 implies that, for the composition operator C_ϕ on H^2 with ϕ a nonelliptic disc automorphism, there exists a hypercyclic vector that is in the Dirichlet space.

3. Cyclicity of C_ϕ. In this section we shall present our results on cyclicity of composition operators on small weighted Hardy spaces contained in H^2. We give a complete characterization for the special case of linear fractional composition operators on smooth weighted Hardy spaces which are *disc-automorphism invariant*, i.e., spaces $H^2(\beta)$ for which disc automorphisms induce bounded composition operators.

The spaces S_a from Example 1 are disc-automorphism invariant, but the spaces Q_a from Example 2 are not, see [**8**].

We shall suppose that the composition operators considered in this paper are all bounded.

Let us dispose first of the cyclicity in the case of an elliptic linear fractional map. Any elliptic linear fractional self map ϕ of $\mathbf{D}$ must be an elliptic disc automorphism. That means that C_ϕ is similar to a composition operator C_ψ induced by a rotation ψ. Being an isometry, C_ψ cannot be hypercyclic. If ψ is not a finite order rotation, then any k_ω^β with $\omega \neq 0$ is a cyclic vector for C_ψ. Finally, because hypercyclicity and cyclicity are preserved under similarity, it follows that C_ϕ is never hypercyclic, and if ϕ is not of a finite order, C_ϕ is cyclic on every disc-automorphism invariant space $H^2(\beta)$. Note that no smoothness is needed for the conclusion.

Part (a) of the following result shows how drastically the situation with hypercyclicity of general composition operators changes when we move from H^2 to smaller weighted Hardy spaces. As for the cyclicity, i.e., part (b) of the theorem, the conclusion in the case when the inducing function has two fixed points in $\overline{\mathbf{D}}$ is the same as in H^2, but the proof is much simpler when dealing with smaller spaces.

Theorem 3.1. *Let the sequence β satisfy $\sum_{n=0}^{\infty}(1/\beta_n^2) < \infty$. Then*

(a) *No composition operator on the space $H^2(\beta)$ is hypercyclic.*

(b) *If the function ϕ has two fixed points in $\overline{\mathbf{D}}$, then C_ϕ is not cyclic on $H^2(\beta)$.*

Proof. (a) Let α be a fixed point of ϕ in $\overline{\mathbf{D}}$ and k_α the reproducing kernel for α. Then $(f, k_\alpha^\beta)_\beta = f(\alpha)$ for all f in $H^2(\beta)$ and $C_\phi^* k_\alpha^\beta = k_{\phi(\alpha)^\beta} = k_\alpha^\beta$. If g is a function in the orbit of some function f in $H^2(\beta)$, i.e., $g = C_\phi^n f$ for some integer n, we have that

$$g(\alpha) = (g, k_\alpha)_\beta = (C_\phi^n f, k_\alpha)_\beta = (f, (C_\phi^*)^n k_\alpha)_\beta = (f, k_\alpha) = f(\alpha).$$

So, no orbit of C_ϕ can be dense in $H^2(\beta)$ and C_ϕ is not hypercyclic on $H^2(\beta)$.

To prove (b), we use the well-known fact that the adjoint of a cyclic operator can have only simple eigenvalues. If α and γ are two different fixed points of ϕ in $\overline{\mathbf{D}}$, we have that $C_\phi^* k_\alpha^\beta = k_{\phi(\alpha)}^\beta = k_\alpha^\beta$ and $C_\phi^* k_\gamma^\beta = k_{\phi(\gamma)}^\beta = k_\gamma^\beta$. So 1 is an eigenvalue for C_ϕ^* with multiplicity at least two, and C_ϕ is not cyclic. $\square$

In the case when the spaces are even smaller, i.e., when $\sum_{n=0}^\infty (n^2/\beta_n^2) < \infty$, the following is true.

Theorem 3.2. *Let ϕ be a self-map of $\mathbf{D}$ with the Denjoy-Wolff point α on the unit circle. If $\phi'(\alpha) = 1$, then C_ϕ is not cyclic on any of the smooth weighted Hardy spaces.*

Proof. Since $\phi(\alpha) = \alpha$ and $\phi'(\alpha) = 1$, we get that

$$C_\phi^* k_\alpha = k_{\phi(\alpha)} = k_\alpha,$$

and that

$$C_\phi^* d_\alpha^\beta = \overline{\phi'(\alpha)} d_\alpha^\beta = d_\alpha^\beta,$$

where d_α^β is the derivative reproducing kernel of $H^2(\beta)$. But then, as before, 1 is an eigenvalue of C_ϕ^* with multiplicity at least two, and C_ϕ cannot be cyclic. $\square$

Note that parabolic self maps of $\mathbf{D}$ satisfy the restrictions of Theorem 3.2 and, so, they do not induce cyclic composition operators on the smooth weighted Hardy spaces. This is a striking difference from the H^2 situation, where parabolic maps induce cyclic composition operators with a dense set of cyclic vectors, see [**1**].

An interesting family of weighted Hardy spaces is the spaces S_a, Example 1. As a grows, the spaces S_a get smaller. When a is bigger than $1/2$, i.e., when the space S_a is smaller than the Dirichlet space, we have an example of a space that satisfies the restrictions of Theorem 3.1. When a is bigger than $3/2$, i.e., when the derivative of a function in S_a is in a space smaller than the Dirichlet space, we have an example of a space that satisfies the restrictions of Theorem 3.2, i.e., a space that is a smooth weighted Hardy space.

What happens with the cyclic behavior of composition operators on the spaces between H^2 and the Dirichlet space, or between the Dirichlet space and $S_{3/2}$? Where exactly is the place where the changes occur?

The following results determine the cutoff space for the hypercyclicity of composition operators. They show that the hypothesis on β needed for Theorem 3.1 is sharp.

We start with a lemma that is a generalization of a result about H^2 from [**2**, Proposition 1.1].

Lemma 3.3. *For ζ on the unit circle, let A_ζ be the set of functions analytic on the closed unit disc that vanish at ζ. Then A_ζ is dense in the space $H^2(\beta)$, where β is such that $\sum_{n=0}^{\infty}(1/\beta_n^2) = \infty$.*

Proof. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is in $H^2(\beta)$ and is orthogonal to A_ζ. Just as in [**2**], we shall use the functions $(\zeta - z)(z^n/\beta_{n+1}^2)$ which are in $H^2(\beta)$ for all n. We have that:

$$\begin{aligned} 0 &= \left(f, (\zeta - z)\frac{z^n}{\beta_{n+1}^2}\right)_\beta = \bar{\zeta}\left(f, \frac{z^n}{\beta_{n+1}^2}\right)_\beta - \left(f, \frac{z^{n+1}}{\beta_{n+1}^2}\right)_\beta \\ &= \bar{\zeta} a_n \frac{\beta_n^2}{\beta_{n+1}^2} - a_{n+1}, \end{aligned}$$

and so $|a_{n+1}|\beta_{n+1} = |a_0|(1/\beta_{n+1})$ for all n.

Then, either $a_0 = 0$ and thus $a_n = 0$ for all n, implying $f \equiv 0$ or

$$\sum_{n=0}^{\infty} \frac{1}{\beta_n^2} = \frac{1}{|a_0|^2} \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty,$$

since f is in $H^2(\beta)$. But the second case contradicts the assumption on β, and so it must be that $f \equiv 0$. Hence, A_ζ is dense in $H^2(\beta)$. ◻

Note that, if $\sum_{n=0}^{\infty}(1/\beta_n^2) < \infty$, then the point evaluation k_ζ is in $H^2(\beta)$ and is orthogonal to A_ζ. Thus, A_ζ cannot be dense in such $H^2(\beta)$.

Proposition 2.3 from [**1**] states that every nonelliptic disc automorphism induces a hypercyclic composition operator on H^2. The proof uses the fact that the sets A_ζ from above are dense in H^2. So, using Lemma 3.3, the same proof applies to the following.

Proposition 3.4. *If ϕ is a nonelliptic disc automorphism, then C_ϕ is hypercyclic on every disc automorphism invariant space $H^2(\beta)$ with $\sum_{n=0}^{\infty}(1/\beta_n^2) = \infty$.*

The previous proposition shows that, for hypercyclicity of disc-automorphic composition operators on S_a spaces, the cut off occurs at $a = (1/2)$, i.e., at the Dirichlet space. It also confirms that the restriction on β in Theorem 3.1 is necessary and sharp. It is still an open question if the smoothness of the space is necessary for the conclusion of Theorem 3.2. Even a more specific question is the following: if ϕ is a parabolic linear fractional transformation, is C_ϕ cyclic on S_a whenever $a \leq (3/2)$? Is there any difference in the cut off for cyclicity of parabolic disc-automorphic and non disc-automorphic composition operators?

We now turn from general univalent to linear fractional self maps of $\mathbf{D}$. Since the latter ones have either one or two fixed points in $\mathbf{C} \cup \{\infty\}$, the cases left to determine the cyclicity of the corresponding composition operators on smooth weighted Hardy spaces are when the map has an external fixed point.

We shall show that, in that case, the induced composition operator is cyclic on every disc-automorphism invariant space $H^2(\beta)$ and has a dense set of cyclic vectors.

The following two examples will set the ground work for the last theorem, which completes the cyclicity classification.

Examples. (a) Exterior and interior fixed points. Let $\phi(z) = az + b$, where a and b are complex numbers such that $|b| < |1 - a|$ and ϕ maps $\mathbf{D}$ into $\mathbf{D}$. Then ϕ is a linear fractional self map of $\mathbf{D}$ with interior fixed point $(b/(1-a))$ and exterior fixed point infinity. Let

$$f_k(z) = \left(z - \frac{b}{1-a}\right)^k \quad \text{for } k = 0, 1, 2, \dots .$$

Then

$$\begin{aligned} C_\phi f_k = f_k \circ \phi &= \left(az + b - \frac{b}{1-a}\right)^k \\ &= \left(az - a\frac{b}{1-a}\right)^k = a^k f_k \\ &\quad \text{for all } k, \end{aligned}$$

and so f_k's are eigenvectors for C_ϕ on a general $H^2(\beta)$ corresponding to different eigenvalues a^k. (Note that, by the Denjoy-Wolff theorem, $|a| < 1$.) Also the functions f_k for $k = 0, 1, 2, \dots$, not only belong to any general space $H^2(\beta)$ but they even span the space. By the theorem of Clancey and Rogers, C_ϕ is then cyclic on $H^2(\beta)$ and has a dense set of cyclic vectors.

(b) Exterior and boundary fixed points. Let $\phi(z) = rz + 1 - r$ where $0 < r < 1$. Then ϕ is a linear fractional self-map of $\mathbf{D}$ with boundary fixed point 1 and exterior fixed point infinity. Let

$$f_k(z) = (1-z)^k \quad \text{for } k = 0, 1, 2, \dots .$$

As in Example 1, the f_k's are functions that belong to every space $H^2(\beta)$, they are eigenvectors for C_ϕ corresponding to the distinct eigenvalues r^k and form a spanning set for $H^2(\beta)$. Thus, by the theorem of Clancey and Rogers, C_ϕ is cyclic on every $H^2(\beta)$ and has a dense set of cyclic vectors.

Theorem 3.5. *Let ϕ be a nonelliptic linear fractional self-map of $\mathbf{D}$ with exterior fixed point. Let β be a weight such that $H^2(\beta)$ is disc-automorphism invariant, and such that C_ϕ is bounded on $H^2(\beta)$. Then C_ϕ is cyclic on $H^2(\beta)$ and has a dense set of cyclic vectors in $H^2(\beta)$.*

Proof. Let ξ_1 be the exterior fixed point of ϕ. Since ϕ is a linear fractional self map of $\mathbf{D}$, it must have another fixed point ξ_2 which is in $\overline{\mathbf{D}}$.

If ξ_2 is interior, i.e., ξ_2 is in $\mathbf{D}$, take ψ to be the disc automorphism

$$\psi_{(z)} = \frac{z - (1/\xi_2)}{1 - (1/\xi_2)z}.$$

Then $\phi_1 = \psi \circ \phi \circ \psi^{-1}$ has an interior fixed point $\psi(\xi_1)$ and the exterior fixed point is infinity. Thus, ϕ_1 must be of the form

$$\phi_1(z) = az + b$$

for some complex numbers a and b, where $(b/(1-a)) = \psi(\xi_1)$, for details see [**5**].

From Example (a) C_{ϕ_1} is cyclic on $H^2(\beta)$ and has a dense set of cyclic vectors, and by similarity, the same is true for C_ϕ.

If ξ_2 is a boundary fixed point, i.e., ξ_2 is on the unit circle, then take the disc automorphism

$$\psi(z) = \lambda \frac{z - (1/\xi_2)}{1 - (1/\xi_2)z} \quad \text{with} \quad \lambda = \frac{1 - (1/\xi_2)\xi_1}{\xi_1 - (1/\xi_2)}.$$

Then $\psi(\xi_1) = 1$ and, if $\phi_1 = \psi \circ \phi \circ \psi^{-1}$, we have that $\phi_1(1) = 1$ and the other fixed point of ϕ_1 is ∞. In that case ϕ_1 must be of a form

$$\phi_1(z) = az + 1 - a,$$

and, because of the Denjoy-Wolff theorem, we have $0 < a \leq 1$, for details see [**5**].

But a could not be 1 for ϕ is not the identity, and so ϕ_1 is as in Example (b). We conclude by similarity that C_ϕ must be cyclic on $H^2(\beta)$ and must have a dense set of cyclic vectors. □

We summarize from the results on linear fractional transformations in this section that, on smooth weighted Hardy spaces,

(a) the nonelliptic linear fractional self-maps of $\mathbf{D}$ induce nonhypercyclic composition operators and that,

(b) except in two cases, they induce noncyclic composition operators.

The two exceptional cases are covered in Theorem 3.5.

Note that, for the cyclicity of linear fractional composition operators on the class of disc-automorphism invariant spaces $H^2(\beta)$, the smooth spaces provide an example of an extreme possible situation.

4. Cyclicity on small spaces that are not disc-automorphism invariant. Recall that, by Theorem 3.5, every linear fractional transformation with exterior fixed point induces a cyclic composition operator on disc-automorphism invariant spaces $H^2(\beta)$. What happens with the same class of operators if the space is not disc-automorphism invariant? Does the result still hold?

The operators from Examples (a) and (b) from the previous section are bounded, cyclic and have a dense set of cyclic vectors on any space $H^2(\beta)$. But not all of the linear fractional transformations (automorphic or not) induce bounded composition operators on all of the spaces.

One of the results from [**7**] states that if β has an exponential growth, i.e., $\lim_{n\to\infty}(n^A/\beta_n) = 0$ for all $A > 0$, then the space $H^2(\beta)$ is not disc-automorphism invariant. Note that the spaces Q_a are such, i.e., have the defining sequence with an exponential growth, and that all of the above spaces are smooth weighted Hardy spaces.

The map $\phi(z) = (z/(z-2))$, which is not a disc-automorphism and fixes the points 0 and 3, induces the operator C_ϕ which is unbounded on, for example, the space Q_a. The reason is that $\phi(1) = -1$ and $|\phi'(1)| = 2$, and so ϕ has a derivative by modulus greater than one at a point from the unit circle that is mapped onto the unit circle. For details, see [**7**].

Is it then true that every bounded C_ϕ, where ϕ is a linear fractional map with exterior fixed point, is a cyclic composition operator on a small space $H^2(\beta)$ even though the space is not disc-automorphism invariant? We will show that the answer is yes in the case when the other fixed point of ϕ is 0.

Before we present the result, let us mention a few other general remarks.

If the linear fractional transformation ϕ that is not a disc automorphism has an exterior and interior fixed points, then either $\overline{\phi(\mathbf{D})} \subset \mathbf{D}$, i.e. $\|\phi\|_\infty < 1$, or if $\|\phi\|_\infty = 1$, then $\phi_2 = \phi \circ \phi$ is such that $\|\phi_2\|_\infty < 1$.

Maps that are analytic on $\overline{\mathbf{D}}$, such as linear fractional transformations, and have sup norm strictly smaller than one, induce compact composition operators on any space $H^2(\beta)$, see [**12**].

Note that, if C_ϕ is bounded and C_{ϕ_2} is cyclic, then C_ϕ must be cyclic too.

We will combine these few remarks later, in the proof of the proposition.

We need one more result due to Hurst [**6**] which originates from Cowen's study of linear fractional composition operators on H^2, see [**4**], and is extended and used also in [**7**].

For the linear fractional map $\phi(z) = ((az+b)/(cz+d))$ with $ad-bc \neq 0$, let $\sigma(z) = ((\bar{a}z - \bar{c})/(-\bar{b}z + \bar{d}))$. The pair $[\phi, \sigma]$ is called a *dual pair.* If ϕ maps $\mathbf{D}$ into $\mathbf{D}$, then so does σ. If z_0 is a fixed point of ϕ in $\mathbf{C} \cup \{\infty\}$, then $(1/z_0)$ is a fixed point of σ and $\sigma'(1/z_0) = (1/\phi'(z_0))$. Define ν to be the map $\nu(z) = (\overline{ad-bc}/(-\bar{b}z+\bar{d})^2)$, and let M_ν be the operator of multiplication by ν.

For a positive sequence β, define the sequence γ by $\gamma(0) = 1$ and $\gamma(n) = (1/\beta_{n+1})$ for $n \geq 1$.

Lemma [**7**, Lemma 3.1] and [**6**, Theorem 5]. *Let the sequence β be eventually monotonically increasing. Let the maps ϕ, σ, ν and the sequence γ be as above. Then the operator M_ν is bounded on $H^2(\gamma)$ and, if C_ϕ is bounded on $H^2(\beta)$, then C_σ is bounded on $H^2(\gamma)$. The restriction of C_ϕ^* onto the invariant subspace $zH^2(\beta)$ is unitarily equivalent to $M_\nu C_\sigma$ acting on $H^2(\gamma)$ via the unitary map U from $H^2(\gamma)$ onto $zH^2(\beta)$ defined by $Uz^n = (1/\beta_{n+1}^2)z^{n+1}$.*

Proposition 4.1. *Let ϕ be a linear fractional map with exterior fixed point that is not an automorphism and such that $\phi(0) = 0$. If the sequence β is eventually monotonically increasing and if C_ϕ is bounded on $H^2(\beta)$, then C_ϕ is cyclic on $H^2(\beta)$ with a dense set of cyclic vectors.*

Proof. Let z_0 be the external fixed point of ϕ. If z_0 is ∞, then $\phi(z) = az$ and C_ϕ is cyclic, by Example (a) in Section 3. Hence, we shall assume that z_0 is finite.

We shall also assume that $\|\phi\|_\infty < 1$ since, if not, then $\|\phi_2\|_\infty < 1$ and the argument below applies to C_{ϕ_2}. But if C_{ϕ_2} is cyclic, then so is C_ϕ.

Since $\|\phi\|_\infty < 1$ and ϕ is analytic on $\overline{\mathbf{D}}$, we get that C_ϕ must be compact on $H^2(\beta)$. The restriction of C_ϕ on the invariant subspace $zH^2(\beta)$ is also compact. If $f(0) = 0$ and if $C_\phi f = \lambda f$, then $f\circ(\phi_n(z)) = \lambda^n f(z)$ for every z in $\mathbf{D}$. But $\phi_n \to 0$ as $n \to \infty$, and if $f \neq 0$, it must be that $|\lambda| < 1$. Thus, the spectral radius $\rho(C_\phi/zH^2(\beta)) < 1$ and so $\|C_{\phi_n} g\| \to 0$ for every g in $zH^2(\beta)$. We shall use this to prove that $Uk_w^\gamma + 1$ is a cyclic vector for C_ϕ, where U is the unitary defined above and k_w^γ is a point evaluation function from $H^2(\gamma)$ with $w \in \mathbf{D}$ and $w \neq (1/z_0)$.

Let h from $H^2(\beta)$ be such that

$$(C\phi_n(U_{k_w}^\gamma + 1), h)_\beta = 0$$

for all n. The proof will be completed when we prove that h must be the zero function.

Since Ukw^γ is in $zH^2(\beta)$, we have

$$0 \le \lim_{n\to\infty} |(C_{\phi_n} Ukw^\gamma, h)_\beta| \le \lim_{n\to\infty} \|C_{\phi_n} Uk_w^\gamma\| \cdot \|h\| = 0,$$

i.e.,

$$\lim_{n\to\infty} (C_{\phi_n} Uk_w^\gamma, h)_\beta = 0.$$

But then

$$\begin{aligned} 0 &= \lim_{n\to\infty} (C\phi_n(Uk_w^\gamma + 1), h)_\beta \\ &= \lim_{n\to\infty} ((C_{\phi_n} Uk_w^\gamma, h)_\beta + (1, h)_\beta) \\ &= (1, h)_\beta \\ &= h(0), \end{aligned}$$

and so h is in $zH^2(\beta)$.

We use the formula for C_ϕ^* on $zH^2(\beta)$ from the lemma to get that

$$\begin{aligned} 0 &= (C_{\phi_n}(Uk_w^\gamma + 1), h)_\beta = (C_{\phi_n} Uk_w^\gamma, h)_\beta \\ &= (Uk_w^\gamma, C_{\phi_n}^* h)_\beta = (Uk_w^\gamma, UM_{\nu_n} C_{\sigma_n} U^* h)_\beta \\ &= (k_w^\gamma, M_{\nu_n} C_{\sigma_n} U^* h)_\gamma = \nu_n(w) \cdot U^* h(\sigma_n(w)) \end{aligned}$$

where $[\phi_n, \sigma_n]$ are a dual pair, and ν_n and U are as above.

Note that the maps ν_n are never 0 on $\mathbf{D}$ and that $\{\sigma_n(w)\}$ is a sequence of points in $\mathbf{D}$ converging to the point $(1/z_0)$. The map U^*h from $H^2(\gamma)$ is analytic and zero on a sequence which converges to a point in $\mathbf{D}$, and thus U^*h must be the zero map. Since U^* is unitary, h must also be the zero map. □

The maps $\phi(z) = rz + 1 - r$, $0 < r < 1$, from Example (b) which fix a boundary and an exterior point are cyclic on all spaces $H^2(\beta)$. It is known that they are also compact on a large class of $H^2(\beta)$ spaces, where β is of exponential growth, see [**7**] and [**8**]. Since the compactness played an important role in Proposition 4.1, it would be of interest to see if there is any connection between the compactness and cyclicity of composition operators on spaces $H^2(\beta)$ with β of exponential growth. For example, if ϕ is a nonelliptic linear fractional transformation and if C_ϕ is bounded on $H^2(\beta)$ with β of exponential growth, is C_ϕ cyclic whenever a power of C_ϕ is compact?

Note that the parabolic maps, or linear fractional maps with interior and boundary fixed point, always induce noncyclic composition operators C_ϕ on spaces $H^2(\beta)$ by Theorems 3.1 and 3.2, and no power of C_ϕ is compact on such spaces, [**7**, Theorems 2.1 and 2.2].

The possible connection between cyclicity and compactness of some composition operators is also suggested in [**2**, p. 87] where the space considered is the space H^2.

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