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THE AHLFORS MAP AND SZEGŐ KERNEL FOR AN ANNULUS

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1. Introduction. In the case of an annulus, it is simple to find an orthonormal basis for the Hardy space. This allows one to write both the Szegő and Garabedian kernel functions as infinite series. These series are classical. The Ahlfors map is a two-to-one branched covering map of the annulus onto the unit disk and is given by the quotient of the Szegő and Garabedian kernels. One of the two zeros of the Ahlfors map arises from the pole of the Garabedian kernel. The other zero corresponds to the zero of the Szegő kernel. In Section 5 it is shown how to use the series for the Szegő kernel to find its zero.

The boundary values of the Garabedian kernel are given in terms of those of the Szegő kernel. Hence, knowing the boundary values of the Szegő kernel is tantamount to knowing those of the Ahlfors map. A discovery of Kerzman and Stein provides an efficient numerical method for computing the boundary values of the Szegő kernel for a smoothly bounded, planar domain and hence the boundary values of the Ahlfors map. Since the Ahlfors map is a holomorphic function smooth up to the boundary, the Cauchy integral formula provides the interior values of this map. Unfortunately, this integral formula has a singular nature for interior points near the boundary. In Section 7 it is shown how to alleviate this singular behavior for the annulus, and in Section 8 graphical examples are given for the Szegő and Garabedian kernels and for the Ahlfors map.

2. Preliminaries. Suppose that Ω is a domain in $\mathbf{C}$ with C^∞ smooth boundary. Let $L^2(b\Omega)$ denote the space of square integrable functions with respect to arc length measure on the boundary $b\Omega$ of Ω, and let $H^2(b\Omega)$ denote the subspace of $L^2(b\Omega)$ consisting of functions that extend to be holomorphic on Ω. An inner product $\langle\cdot,\cdot\rangle$ is defined

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on $L^2(b\Omega)$, and hence on $H^2(b\Omega)$, via

$$\langle u, v\rangle = \int_{b\Omega} u(z)\overline{v(z)}\, ds_z.$$

A norm $\|\cdot\|$ is also defined on $L^2(b\Omega)$ via $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

Now $H^2(b\Omega)$ is a closed subspace of $L^2(b\Omega)$ and the orthogonal projection $\mathcal{P}$ of $L^2(b\Omega)$ onto $H^2(b\Omega)$ is classically known as the Szegő projection. This projection is represented by the Szegő kernel $S_\Omega(z,a) = S(z,a)$ as follows:

$$\mathcal{P}_\varphi(a) = \int_{b\Omega} \varphi(z)\overline{S(z,a)}\, ds_z$$

for all $\varphi \in L^2(b\Omega)$ and for $a \in \overline{\Omega}$. It is known that $S(z,a)$ extends to become holomorphic in z and anti-holomorphic in a on $\Omega \times \Omega$ and is of class $C^\infty((\overline{\Omega} \times \overline{\Omega})\backslash\{(z,z) \in b\Omega \times b\Omega\})$. Further, the Szegő kernel is Hermitian symmetric, i.e., $S(z,a) = \overline{S(a,z)}$ for all $z, a \in \Omega$.

The Garabedian kernel $L_\Omega(z,a) = L(z,a)$ is defined for $z \in b\Omega$ and $a \in \Omega$ via $L(z,a) = i\overline{S(z,a)T(z)}$. Here T is the complex unit tangent vector to the boundary of Ω defined by $T(z) = T(z(t)) = z'(t)/|z'(t)|$ where $z(t)$ parametrizes $b\Omega$. We note that knowing the boundary values of the Szegő kernel is equivalent to knowing the boundary values of the Garabedian kernel.

It is known that $L(z,a)$ extends meromorphically to $\Omega \times \Omega$ with a single simple pole of residue $1/2\pi$ at $z = a$ and further is of class $C^\infty((\overline{\Omega}\times\overline{\Omega})\backslash\{(z,z) \in \overline{\Omega}\times\overline{\Omega}\})$. Also known is the fact that, if Ω is simply connected and $a \in \Omega$ is fixed, then f_a is the unique biholomorphism of Ω onto the unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ such that $f_a(a) = 0$ and $f_a'(a) > 0$, i.e., the Riemann map, then $f_a = S_a/L_a$ where $S_a \equiv S_\Omega(\cdot, a)$ and $L_a \equiv L_\Omega(\cdot, a)$.

For an n-connected domain $\Omega \subset \mathbf{C}$, and a point $a \in \Omega$, the Ahlfors map f_a is a branched n-to-one analytic function mapping Ω onto the unit disk, with $f_a(a) = 0$. Further, the Ahlfors map takes each component of the boundary of Ω one-to-one and onto the unit circle. Among all analytic functions mapping Ω into the unit disk, f_a is the unique function with maximum positive derivative at a. If Ω has C^∞-smooth boundary, then the Ahlfors map can be written explicitly as the ratio of the Szegő kernel and the Garabedian kernel, i.e., $f_a = S_a/L_a$.

All of the above facts can be found in [**1**].

3. The Szegő kernel. Another way to think about the Szegő kernel is as follows. Let $\{\varphi_k(z)\}_{k=1}^{\infty}$ be an orthonormal basis for $H^2(b\Omega)$. Then, given a point $a \in \Omega$, the Szegő kernel can be written

$$S_a(z) = \sum_{k=0}^{\infty} \varphi_k(z)\overline{\varphi_k(a)} \tag{3.1}$$

with absolute and uniform convergence on compact subsets of Ω [**1**].

Let Ω be the annulus $\{z : \rho < |z| < 1\}$. (We choose this class of conformal equivalents for annuli in Sections 3, 4 and 5, as the results there are easily translated to the case of an annulus $\{z : r_2 < |z| < r_1\}$; however, in Section 6 we use annuli of the form $\{z : r_2 < |z| < r_1\}$ as the formulas there are more complicated to translate.) A complete orthogonal set for Ω is $\{z^n\}_{n=-\infty}^{\infty}$. In order to construct an orthonormal basis for $H^2(b\Omega)$, we notice that

$$\|z^n\|^2 = \int_{b\Omega} |z|^{2n}\, ds_z = 2\pi(1 + \rho^{2n+1}).$$

Thus an orthonormal basis for $H^2(b\Omega)$ is

$$\left\{ \frac{z^n}{\sqrt{2\pi(1+\rho^{2n+1})}} \right\}_{n=-\infty}^{\infty}. \tag{3.2}$$

From this and (3.1), we get that the Szegő kernel for the annulus is

$$S_a(z) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(z\bar{a})^n}{1+\rho^{2n+1}}. \tag{3.3}$$

4. The Garabedian kernel. As with the Szegő kernel, there are many ways to think about the Garabedian kernel. For our calculations, we will use the fact [**1**] that for the Garabedian kernel, $L_a(z)$, we have

$$L_a(z) = \mathcal{P}^{\perp}\left(\frac{1}{2\pi} \frac{1}{z-a} \right).$$

Thus, if we have an orthonormal basis $\{\varphi_k(z)\}_{k=1}^{\infty}$ for $H^2(b\Omega)$, the Garabedian kernel can be written

$$L_a(z) = \frac{1}{2\pi}\frac{1}{z-a} - \frac{1}{2\pi}\sum_{k=1}^{\infty}\left\langle \frac{1}{\zeta - a}, \varphi_k(\zeta)\right\rangle \varphi_k(z).$$

Let Ω be the annulus $\{z : \rho < |z| < 1\}$. We have an orthonormal basis for $H^2(b\Omega)$ in (3.2). Each inner product is evaluated by first using the fact that $dz = T ds$ and then applying the residue theorem. Rearranging the resulting sum yields

$$L_a(z) = \frac{1}{2\pi}\frac{1}{z-a} + \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{\rho^{2n+1}(z^{2n+1} - a^{2n+1})}{(za)^{n+1}(1+\rho^{2n+1})}. \tag{4.1}$$

Using (3.3) and (4.1), we have an explicit formula for the Ahlfors map for an annulus in

$$f_a(z) = \frac{S_a(z)}{L_a(z)}.$$

Recall that f_a is a branched, two-to-one map from the annulus onto the unit disk. One zero, arising from the single, simple pole of the Garabedian kernel, lies at the point $z = a$. We shall locate the zero of the Szegő kernel, and hence the other zero of the Ahlfors map, in the next section.

5. The zeros of the Szegő kernel. We saw above that, for the annulus $\Omega = \{z : \rho < |z| < 1\}$, the corresponding Szegő kernel is

$$S_a(z) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(z\bar{a})^n}{1+\rho^{2n+1}}.$$

We know that for fixed $a \in \Omega$ this function has exactly one zero [**1**]. The question is, where is this zero?

Theorem 1. $S_a(-\rho/\bar{a}) = 0$.

Proof. First, note that

$$S_a(-\rho/\bar{a}) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-(\rho/\bar{a})\bar{a})^n}{1+\rho^{2n+1}} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-\rho)^n}{1+\rho^{2n+1}}, \tag{5.1}$$

so the problem reduces to showing that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n \rho^n}{1+\rho^{2n+1}} = 0.$$

Since (5.1) converges absolutely, we can write

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1)^n \rho^n}{1+\rho^{2n+1}} &= \sum_{n=-\infty}^{-1} \frac{(-1)^n \rho^n}{1+\rho^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \rho^n}{1+\rho^{2n+1}} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n \rho^{-n}}{1+\rho^{-2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \rho^n}{1+\rho^{2n+1}},\end{aligned}$$

and reindexing the second term gives us

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{(-1)^n \rho^{-n}}{1+\rho^{-2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \rho^n}{1+\rho^{2n+1}} \\ &\qquad= \sum_{n=1}^{\infty} \frac{(-1)^n \rho^{-n}}{1+\rho^{-2n+1}} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \rho^{n-1}}{1+\rho^{2n-1}} \\ &\qquad= \sum_{n=1}^{\infty} \frac{(-1)^n \rho^{-n}}{1+\rho^{-2n+1}} \frac{\rho^{2n-1}}{\rho^{2n-1}} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \rho^{n-1}}{1+\rho^{2n-1}} \\ &\qquad= \sum_{n=1}^{\infty} \frac{(-1)^n \rho^{n-1}}{1+\rho^{2n-1}} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \rho^{n-1}}{1+\rho^{2n-1}} \\ &\qquad= 0.\end{aligned}$$

Theorem 2. *Let $\Omega = \{z : \rho < |z| < 1\}$ be an annulus. Given a point $a \in \Omega$, the Ahlfors map f_a of Ω onto the unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ has zeros exactly at the points $z = a$ and $z = -\rho/\bar{a}$.*

6. A numerical approach. Kerzman and Stein discovered [**3**] that the Szegő kernel of a smoothly bounded, simply connected planar domain satisfies an integral equation which is particularly amenable to numerical solution. Indeed, this integral equation is a Fredholm equation of the second kind with C^{∞} smooth kernel. This allows the boundary values of the Szegő kernel, and hence the Riemann map, of Ω to be numerically approximated. Indeed, examples of using this method to calculate the Riemann map can be found in [**4, 6, 7**].

The Kerzman-Stein integral equation described above continues to be valid for multiply connected domains [**2**] and hence can be used to calculate the boundary values of the Szegő and Garabedian kernels, and hence the Ahlfors map. If the interior values of these functions are desired, then the Cauchy integral formula can be used since, for any function f holomorphic on Ω and for all $z \in \Omega$,

$$f(z) = \frac{1}{2\pi i} \int_{\zeta \in b\Omega} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

For the Garabedian kernel, we notice that the Cauchy integral formula must be applied to the function

$$L_a(z) - \frac{1}{2\pi} \frac{1}{z - a}$$

since this kernel is meromorphic on Ω with a single simple pole at $z = a$ with residue $1/2\pi$.

The only problem with evaluating the Cauchy integral formula numerically is that this integral is singular in nature whenever $z \in \Omega$ is near $b\Omega$. In the next two sections we shall show how to alleviate the singular nature of this integral in the case of multiply connected planar domains with circular boundary components. Indeed, representations for the Szegő and Garabedian kernels in terms of integral formulas whose singularities are at points lying strictly outside of Ω are given. So, in this case, it is quite easy to numerically compute these kernels and hence the Ahlfors map.

7. The case of an annulus. Let $\Omega = \{z : r_2 < |z| < r_1\}$ be an annulus and fix a point $a \in \Omega$. For $\zeta \in b\Omega$ the Szegő kernel $S(\zeta) = S_a(\zeta)$ satisfies the Kerzman-Stein integral equation [**3**]

$$S(\zeta) = g(\zeta) - \int_{w \in b\Omega} A(\zeta, w) S(w) \, ds_w$$

where $g(\zeta) = g_a(\zeta) = \overline{H(a, \zeta)}$, $H(w, \zeta) = (1/2\pi i)(T(\zeta)/(\zeta - w))$ and

$$A(\zeta, w) = \begin{cases} \overline{H(w, \zeta)} - H(\zeta, w) & w, \zeta \in b\Omega, \, w \neq \zeta, \\ 0 & w = \zeta \in b\Omega. \end{cases}$$

Furthermore, the apparent singularities of $A(\zeta, w)$ cancel out so that $A(\zeta, w) \in C^\infty(b\Omega \times b\Omega)$. By an application of the Cauchy integral formula, it follows that

(7.1)

$$S(z) = \frac{1}{2\pi i}\int_{\zeta\in b\Omega} \frac{g(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{\zeta\in b\Omega}\int_{w\in b\Omega} \frac{A(\zeta, w)S(w)}{\zeta - z}\, ds_w\, d\zeta$$

for all $z \in \Omega$.

Let $b\Omega = \gamma_1 \cup \gamma_2$ where $\gamma_1 = \{|\zeta| = r_1\}$ and $\gamma_2 = \{|\zeta| = r_2\}$ and parametrize $b\Omega$ by $\gamma_1 : \zeta_1(t) = r_1 e^{it}$ and $\gamma_2 : \zeta_2(t) = r_2 e^{-it}$ for $0 \le t < 2\pi$. Then the complex unit tangent vectors are given by

$$T_1(\zeta) = \zeta_1'(t)/|\zeta_1'(t)| = i\zeta_1(t)/r_1$$

and

$$T_2(\zeta) = \zeta_2'(t)/|\zeta_2'(t)| = -i\zeta_2(t)/r_2.$$

The first integral in (7.1) is calculated as follows:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\zeta\in b\Omega} \frac{g(\zeta)}{\zeta - z}\, d\zeta &= \frac{1}{2\pi i}\int_{\zeta\in\gamma_1} \frac{g(\zeta)}{\zeta - z}\, d\zeta \\
&\quad + \frac{1}{2\pi i}\int_{\zeta\in\gamma_2} \frac{g(\zeta)}{\zeta - z}\, d\zeta \\
&= \frac{1}{2\pi i}\int_{\zeta\in\gamma_1} \frac{i}{2\pi}\frac{-i\bar\zeta/r_1}{(\bar\zeta - \bar a)(\zeta - z)}\, d\zeta \\
&\quad + \frac{1}{2\pi i}\int_{\zeta\in\gamma_2} \frac{i}{2\pi}\frac{i\bar\zeta/r_2}{(\bar\zeta - \bar a)(\zeta - a)}\, d\zeta \\
&= \frac{1}{2\pi i}\int_{\zeta\in\gamma_1} \frac{1}{2\pi}\frac{r_1}{(r_1^2 - \bar a\zeta)(\zeta - z)}\, d\zeta \\
&\quad - \frac{1}{2\pi i}\int_{\zeta\in\gamma_2} \frac{1}{2\pi}\frac{r_2}{(r_2^2 - \bar a\zeta)(\zeta - z)}\, d\zeta \\
&= \frac{1}{2\pi}\frac{r_1}{r_1^2 - \bar a z} - \frac{1}{2\pi}\frac{r_2}{r_2^2 - \bar a z}
\end{aligned}$$

where the last equality follows by dint of the fact that

$$\frac{1}{2\pi}\frac{r_1}{(r_1^2 - \bar a\zeta)} \quad \text{and} \quad -\frac{1}{2\pi}\frac{r_2}{(r_2^2 - \bar a\zeta)}$$

are holomorphic in $\{|\zeta| < r_1\}$ and $\{|\zeta| > r_2\}$, respectively.

Next the second integral in (7.1) is calculated. We recall [**3**] that the Kerzman-Stein kernel vanishes whenever ζ and w both lie on a circular boundary component implying that $A(\zeta, w) \equiv 0$ for $(\zeta, w) \in \gamma_j \times \gamma_j$, $j = 1, 2$. Therefore,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\zeta\in b\Omega}\int_{w\in b\Omega}\frac{A(\zeta,w)S(w)}{\zeta - z}\,ds_w\,d\zeta \\
&\qquad = \frac{1}{2\pi i}\int_{\zeta\in\gamma_1}\int_{w\in\gamma_2}\frac{A(\zeta,w)S(w)}{\zeta - z}\,ds_2\,d\zeta \\
&\qquad\quad + \frac{1}{2\pi i}\int_{\zeta\in\gamma_2}\int_{w\in\gamma_1}\frac{A(\zeta,w)S(w)}{\zeta - z}\,ds_w\,d\zeta \\
&\qquad = \frac{1}{2\pi i}\int_{w\in\gamma_2} S(w)\bigg(\int_{\zeta\in\gamma_1}\frac{A(\zeta,w)}{\zeta - z}\,d\zeta\bigg)\,ds_w \\
&\qquad\quad + \frac{1}{2\pi i}\int_{w\in\gamma_1} S(w)\bigg(\int_{\zeta\in\gamma_2}\frac{A(\zeta,w)}{\zeta - z}\,d\zeta\bigg)\,ds_w.
\end{aligned}$$

For $w \in \gamma_2$, it follows that

$$\begin{aligned}
&\int_{\zeta\in\gamma_1}\frac{A(\zeta,w)}{\zeta - z}\,d\zeta \\
&= \int_{\zeta\in\gamma_1}\frac{-(1/(2\pi i))((-ir_1/\bar{\zeta})/(\bar{\zeta}-\bar{w})) - (1/(2\pi i))((-ir_2/w)/(w-\zeta))}{\zeta - z}\,d\zeta \\
&= -\frac{i(r_1 + r_2)}{\bar{w}}\frac{1}{2\pi i} \\
&\quad\cdot \int_{\zeta\in\gamma_1}\frac{(\zeta - r_1 r_2/\bar{w})}{(\zeta - z)(\zeta - w)(\zeta - r_1^2/\bar{w})}\,d\zeta.
\end{aligned}$$

So we have the integral around γ_1 of a function meromorphic on $\{|\zeta| < r_1\}$ with simple poles at $\zeta = z$ and $\zeta = w$. Notice that the apparent pole at $\zeta = r_1^2/\bar{w}$ lies outside of $\{|\zeta| < r_1\}$. Indeed, it is interesting to note that $r_1^2/\bar{w}$ is the reflection of w across the circle

$\{|\zeta| = r_1\}$. Therefore, by the residue theorem,

$$\int_{\zeta\in\gamma_1} \frac{A(\zeta,w)}{\zeta - z}\,d\zeta = -\frac{i(r_1+r_2)}{\bar{w}} \cdot \left(\frac{(z - r_1r_2/\bar{w})}{(z-w)(z-r_1^2/\bar{w})} + \frac{(w - r_1r_2/\bar{w})}{(w-z)(w-r_1^2/\bar{w})}\right) = \frac{ir_1}{(r_1^2 - \bar{w}z)}$$

for $w \in \gamma_2$. Similarly, we have

$$\int_{\zeta\in\gamma_2} \frac{A(\zeta,w)}{\zeta - z}\,d\zeta = \frac{-ir_2}{(r_2^2 - \bar{w}z)}$$

for $w \in \gamma_1$. Therefore,

$$\frac{1}{2\pi i}\int_{\zeta\in b\Omega}\int_{w\in b\Omega} \frac{A(\zeta,w)S(w)}{\zeta - z}\,ds_w\,d\zeta = \int_{w\in\gamma_1} -\frac{1}{2\pi}\frac{r_2}{(r_2^2-\bar{w}z)}S(w)\,ds_w + \int_{w\in\gamma_2}\frac{1}{2\pi}\frac{r_1}{(r_1^2-\bar{w}z)}S(w)\,ds_w.$$

Hence, the following theorem has been proved.

Theorem 3. *The Szegő kernel* $S(z) = S_a(z)$ *for the annulus* $\Omega = \{z : r_2 < |z| < r_1\}$ *satisfies*

$$S(z) = \frac{1}{2\pi}\frac{r_1}{(r_1^2 - \bar{a}z)} - \frac{1}{2\pi}\frac{r_2}{(r_2^2-\bar{a}z)} + \int_{w\in\gamma_1}\frac{1}{2\pi}\frac{r_2}{(r_2^2-\bar{w}z)}S(w)\,ds_w - \int_{w\in\gamma_2}\frac{1}{2\pi}\frac{r_1}{(r_1^2-\bar{w}z)}S(w)\,ds_w.$$

Notice that in this representation of the Szegő kernel the integrals do not have a singular behavior for $z \in \Omega$ near $b\Omega$. Indeed, the only singularity in the first integral occurs at $z = r_2^2/\bar{w}$ which is the reflection of $w \in \gamma_1$ across the circle $\{|\zeta| = r_2\}$ and thus this singularity occurs strictly outside $\bar{\Omega}$. A similar remark holds for the second integral. The utility of this representation is the following. Once the boundary values of the Szegő kernel are computed using the Kerzman-Stein integral equation, the interior values are given in terms of integrals which do not exhibit the singular behavior of the Cauchy integral. Therefore the computational difficulty associated with the Cauchy integral is avoided.

Since the Ahlfors map of a multiply connected domain is given by $f_a = S_a/L_a$, it would be nice to have a similar formula for the Garabedian kernel of the annulus $\Omega = \{z : r_2 < |z| < r_1\}$. Indeed, it is now very easy to find such a formula for L_a. We recall [**1**] that $L = L_a$ satisfies

$$L(z) + \int_{w\in b\Omega} A(z,w)L(w)\,ds_w = h(z) \tag{7.2}$$

for $z \in b\Omega$ where $h(z) = h_a(z) = (1/2\pi)(1/(z-a))$. Now

$$L(z) - \frac{1}{2\pi}\frac{1}{z-a} = L(z) - h(z)$$

extends to be holomorphic on Ω and continuous on $\overline{\Omega}$ so that the Cauchy integral formula and (7.2) imply that

$$\begin{aligned} L(z) - h(z) &= \frac{1}{2\pi i}\int_{\zeta\in b\Omega} \frac{L(\zeta)-h(\zeta)}{\zeta - z}\,d\zeta \\ &= \frac{-1}{2\pi i}\int_{\zeta\in b\Omega}\int_{w\in b\Omega} \frac{A(\zeta,w)L(w)}{\zeta - z}\,ds_w\,d\zeta \\ &= \frac{-1}{2\pi i}\int_{w\in b\Omega} L(w)\left(\int_{\zeta\in b\Omega}\frac{A(\zeta,w)}{\zeta-z}\,d\zeta\right) ds_w. \end{aligned}$$

Using the calculations for the Szegő kernel and the fact that $L_a = i\overline{S_a T}$ on $b\Omega$, we have the following theorem.

Theorem 4. *The Garabedian kernel $L(z) = L_a(z)$ for the annulus $\Omega = \{z : r_2 < |z| < r_1\}$ satisfies*

$$\begin{aligned} L(z) = \frac{1}{2\pi}\frac{1}{z-a} &+ \frac{1}{2\pi}\int_{w\in\gamma_1}\frac{r_2\bar{w}}{r_1(r_2^2 - \bar{w}z)}\overline{S(w)}\,ds_w \\ &+ \frac{1}{2\pi}\int_{w\in\gamma_2}\frac{r_1\bar{w}}{r_2(r_1^2-\bar{w}z)}\overline{S(w)}\,ds_w. \end{aligned}$$

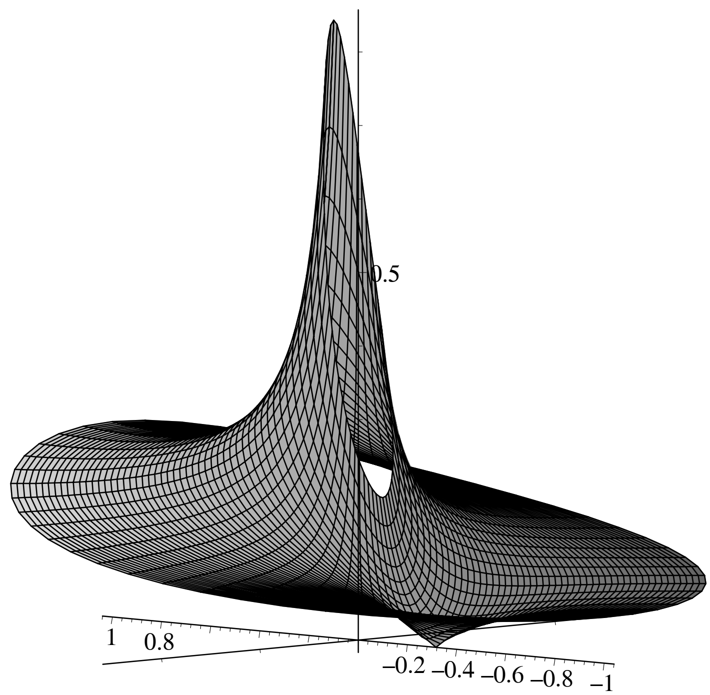

FIGURE 1. Absolute value of the Szegő kernel.

Again this shows that, once the boundary values of the Szegő kernel are calculated via the Kerzman-Stein integral equation, the interior values of the Garabedian kernel are easily computed without the problem of the singular behavior of the Cauchy kernel. Hence, it follows that both the interior and the boundary values of the Ahlfors map $f_a = S_a/L_a$ of an annulus are easily computed.

8. Numerical examples. Here we use the Kerzman-Stein integral equation followed by the formulas of Theorems 3 and 4 to compute the Szegő and the Garabedian kernels for an annulus $\Omega = \{z : \rho < |z| < 1\}$. The algorithm used for solving the Kerzman-Stein integral equation is outlined in [**1**]. The algorithm used to calculate the integrals in Theorems 3 and 4 is Simpson's Rule. Maple V [**5**] was used to do

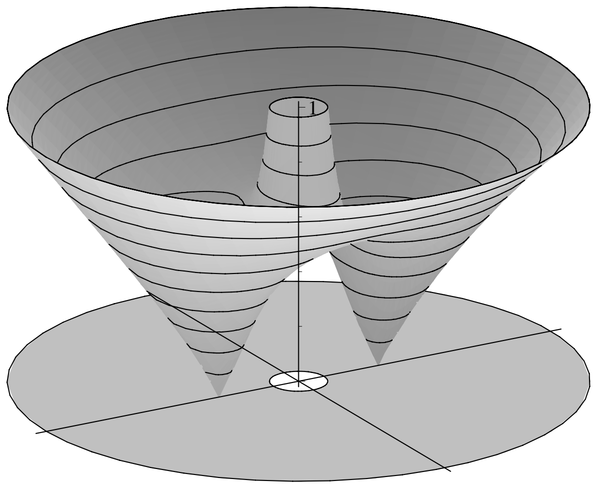

FIGURE 2. Absolute value of the Ahlfors map.

the computations and graphics. Thanks go to Matt Richey, St. Olaf College, for helping to streamline our Maple V program.

Using the formula $f_a = S_a/L_a$ we have now computed the Ahlfors map f_a. Theorem 2 shows that the Ahlfors map has zeros exactly at $z = a$ and $z = -\rho/\bar{a}$. In what follows we take $a \in \Omega \cap \mathbf{R}^+$ since if a is a complex number in Ω, then we can use a rotation to move a to the positive real axis. In this case the zeros of the Ahlfors map are exactly at $z = a$ and $z = -\rho/a$. Further, we note that there is a case where the zeros of the Ahlfors map are symmetric about the origin. Namely, if we choose $a > 0$ in Ω such that $a = -(-\rho/a)$. This says that if $a = \sqrt{\rho}$ then the zeros of the Ahlfors map are exactly at $z = \pm\sqrt{\rho}$.

Let $\Omega = \{z : (1/10) < |z| < 1\}$. We start with the symmetric case $a = 1/\sqrt{10}$. In Figures 1 and 2 we sketch the graph of the absolute value of the Szegő kernel and the absolute value of the Ahlfors map, respectively. The zero of the Szegő kernel at $z = -1/\sqrt{10}$ and the zeros of the Ahlfors map at $z = \pm 1/\sqrt{10}$ are illustrated in these figures.

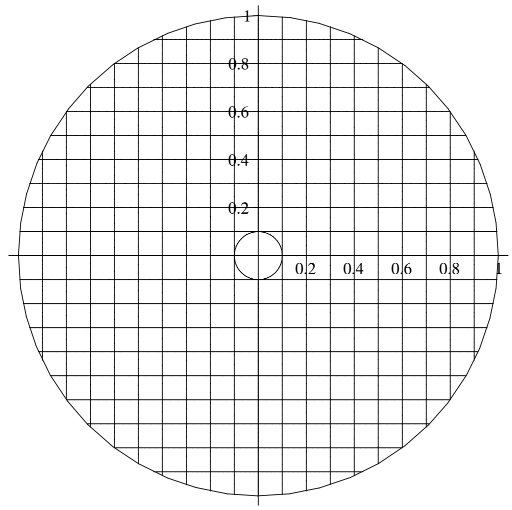

FIGURE 3. Square grid over an annulus.

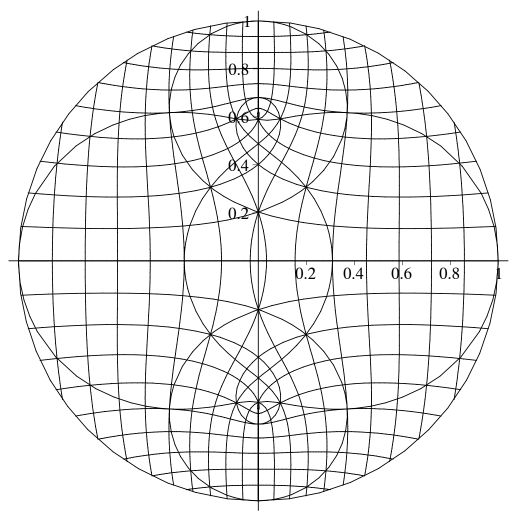

FIGURE 4. Symmetric case.

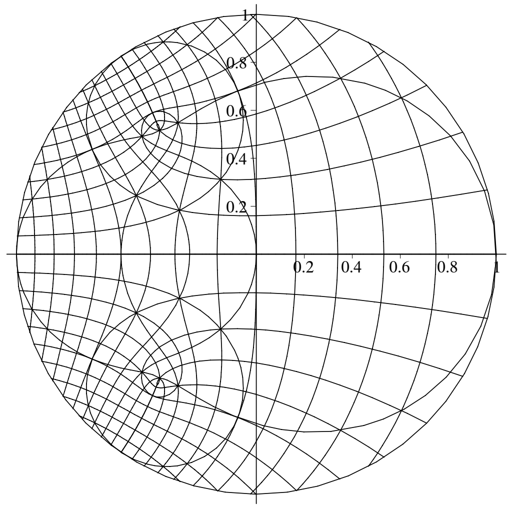

FIGURE 5. Non-symmetric case.

Next we look at the image in the unit disk of a square grid over the annulus. In Figure 3 the grid over the annulus is sketched. Figure 4 shows the image of this grid in the symmetric case $a = 1/\sqrt{10}$. Figure 5 shows the image of this grid in the case $a = 0.5$. Notice how the images of the branch points appear in the unit disk. Also notice how the grid lines either wrap or deflect around the images of the branch points as the grid lines in the annulus pass above/below or left/right of the branch points. Further, these pictures show that the Ahlfors map preserves the orientation of the outer boundary of the annulus and reverses the orientation of the inner boundary of the annulus. This fact can also be seen in a color version of Figure 2 where color represents the argument of the Ahlfors map.

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