ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 2, Summer 1999

THE HYPERBOLIC TANGENT AND GENERALIZED MELLIN INVERSION

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ABSTRACT. We prove an identity that reduces a certain product of $(n-1)n/2$ hyperbolic tangent functions to a sum of products of $(n-1)/2$ or $n/2$ such factors. Using this result, we obtain a simplified expression for the Plancherel measure, and corresponding inversion formula, for $O(n,\mathbf{R})$-invariant functions on the space $\mathcal{P}_n$ of positive definite, symmetric real matrices.

1. Statement of results. The first objective of this article is to derive the following curious "hyperbolic trig identity":

Proposition 1. *Let $n \in \mathbf{Z}^+$; let $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbf{C}$ be such that $\alpha_j - \alpha_k \notin i\pi(1/2+\mathbf{Z})$ for any $1 \le j,\ k \le n$. Also let $\bar{n} = \lfloor n/2 \rfloor$, the greatest integer $\le n/2$. Then*

$$\prod_{1\le j<k\le n} \tanh(\alpha_j - \alpha_k) = \frac{1}{2^{\bar{n}}\bar{n}!} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{l=1}^{\bar{n}} \tanh(\alpha_{\sigma(2l-1)} - \alpha_{\sigma(2l)}),$$

where S_n is the symmetric group on $\{1,2,\dots,n\}$ and $\operatorname{sgn}(\sigma)$ denotes the sign of σ.

We prove this in Section 2 below, where we also note, see Proposition 1′, that, in fact, certain summands on the righthand side may be identified to yield a "shorter" sum. (In Proposition 1 and throughout this paper, an empty product is understood to equal 1.)

Our second goal is to use the above proposition to simplify the "Mellin inversion formula," whose various elements are due to Helgason [**3,**

Received by the editors on June 14, 1996, and in revised form on July 17, 1997.
Research supported in part by NSF grant #DMS-9400324.

4], Harish-Chandra [**2**] and Bhanu-Murty [**1**], for $O(n,\mathbf{R})$-invariant functions on the space $\mathcal{P}_n$ of positive definite, symmetric, real matrices. To state our result, Proposition 2 below, we briefly recall what is known concerning this inversion formula. For our discussion we follow Terras' concise exposition [**6**, Chapter 4].

To begin with, we note that $\mathcal{P}_n$ is a homogeneous space for $G = GL(n,\mathbf{R})$ under the action

$$Y \longrightarrow Y[g] \equiv {}^t g Y_g, \quad Y \in \mathcal{P}_n; g \in GL(n,\mathbf{R}).$$

A function f on $\mathcal{P}_n$ is said to be $K = O(n,\mathbf{R})$-*invariant* if $f(Y[k]) = f(Y)$ for all $Y \in \mathcal{P}_n$ and $k \in K$. The version of harmonic analysis on $\mathcal{P}_n$ that we are concerned with involves the resolution of such an f into an integral of the "power function"

$$p_r(Y) = \prod_{j=1}^{n} (|Y_j|/|Y_{j-1}|)^{r_j + j/2 - (n+1)/4}$$
$$Y \in \mathcal{P}_n, \quad r = (r_1, r_2, \dots, r_n) \in \mathbf{C}^n.$$

Here Y_j is the $j \times j$ upper lefthand minor of Y, and $|\ |$ is the determinant (we define $|Y_0| \equiv 1$).

In particular, let the Helgason-Mellin transform, also known as the Helgason-Fourier transform, $\hat{f}$ of an infinitely differentiable, compactly supported, K-invariant function $f : \mathcal{P}_n \to \mathbf{C}$ be defined by

$$\hat{f}(r) = \int_{Y \in \mathcal{P}_n} f(Y)\overline{p_r(Y)}\, d\mu_n(Y),$$

where $d\mu_n(Y)$ is a G-invariant measure on $\mathcal{P}_n$, normalized as in [**6**]. Then Mellin inversion on $\mathcal{P}_n$, which in fact amounts to the usual Mellin inversion formula on $\mathbf{R}^+$, when $n = 1$, states that

$$f(Y) = \omega_n \int_{t \in \mathbf{R}^n} \hat{f}(it) h_{it}(Y) |c_n(t)|^{-2}\, dt_1 \cdots dt_n, \tag{1.1a}$$

where $t = (t_1, \dots, t_n)$; it denotes $(it_1, \dots, it_n)$;

$$\omega_n = \prod_{j=1}^{n} \frac{\Gamma(j/2)}{(2\pi) j \pi^{j/2}}; \tag{1.1b}$$

$$(1.1c)\quad |c_n(t)|^{-2} = \prod_{1\le j<k\le n} \frac{\Gamma^2((k-j)/2)}{\Gamma^2((1+k-j)/2)}(t_j - t_k)\tanh \pi(t_j - t_k);$$

and h_r is the "(zonal) spherical function"

$$(1.1d)\qquad h_r(Y) = \int_{k\in K} p_r(Y[k])\,dk$$

(dk is Haar measure normalized so that K has unit volume). In stating this inversion formula, we have used essentially the notation of Terras [**6**, particularly Theorem 1 and equation 3.4, Section 4.3]. However, the latter uses, instead of our variables $t_1, t_2, \dots, t_n$, variables $s_1, s_2, \dots, s_n$ given by

$$it_j = s_j + s_{j+1} + \cdots + s_n - j/2 + (n+1)/4, \quad 1 \le j \le n.$$

(Note that $ds_1 \cdots ds_n = i^n\, dt_1 \cdots dt_n$). The advantage of the t-variables is that, as is in fact pointed out in [**6**, Theorem 3, Section 4.2, and Theorem 1, Section 4.3], both $h_{it}(Y)$ and $\hat{f}(it)$ are invariant under permutations of the t_j's.

We will exploit this last fact to simplify the above inversion formula. In particular, we see from (1.1c) that the Plancherel measure $\omega_n|c_n(t)|^{-2}$ involves a product of $(n-1)n/2$ tanh functions. By Proposition 1, we may rewrite this product as a sum of products of $\bar{n}$ tanh factors. We exchange this sum with the integral in (1.1a). A change of variable is made in each resulting integral; by the stated invariance of $\hat{f}(it)h_{it}(Y)$ we see that all the integrals are in fact the same. So they may be combined.

The end result, as we prove in Section 3 below, is:

Proposition 2. *If f is as above, then*

$$f(Y) = \frac{\pi^{-n(n+1)/4}}{2^{\bar{n}+n}\bar{n}!}\left[\prod_{j=1}^{n} \frac{1}{\Gamma(j/2)}\right] \cdot \int_{t\in \mathbf{R}^n} \hat{f}(it)h_{it}(Y)\left[\prod_{1\le j<k\le n}(t_j - t_k)\right] \cdot \left[\prod_{l=1}^{\bar{n}} \tanh \pi(t_{2l-1} - t_{2l})\right] dt_1 \cdots dt_n.$$

Remark. Clearly, Proposition 2 provides a reduction of the number of hyperbolic tangent factors required for Mellin inversion. Perhaps more important for explicit calculations, though, is the fact that these factors are "disentangled" from each other in our new inversion formula. More precisely, we note that, in formula (1.1c), each variable t_j, $1 \leq j \leq n$, appears in $n-1$ distinct tanh factors, and in each of these factors appears in combination with a different variable t_k, $k \neq j$. On the other hand, in the inversion formula of Proposition 2, each t_j (where, here, $1 \leq j \leq 2\bar{n}$) occurs in exactly one of, and in combination with only one other t_k through, the tanh factors.

Thus, Proposition 2 yields the promised simplification of formulas (1.1). We already have, with Wallace (see [**5**, especially the proof of Proposition 5.1]), observed the usefulness of such a simplification in the case of the determinant-one subspace $H^3 \equiv SL(3,\mathbf{R})/SO(3,\mathbf{R})$ of $\mathcal{P}_3$. Specifically, we have used the "Selberg trace formula," as realized by Wallace [**7**] for $SL(3,\mathbf{Z})\backslash H^3$, to study eigenvalues of the Laplacian on the latter space. This application of the trace formula requires explicit information about the behavior of a certain $SO(3,\mathbf{R})$-invariant function f, called the "heat kernel" for H^3, when initially only (the H^3 analog of) $\hat{f}$ is known. Such information is more readily obtained when the Plancherel measure is as simple as possible.

We hope that Proposition 2 will ultimately prove helpful in higher-rank generalizations of the study just cited; at the moment, though such generalizations await more explicit versions of the Selberg trace formula for cases of rank > 2.

We proceed with our derivations.

2. Proof of Proposition 1. It will be convenient to introduce the notation

$$h(j,k) = \tanh(\alpha_j - \alpha_k)$$

for $j,k \in \mathbf{Z}^+$. In the course of our proof, we will treat $h(j,k)$ as a formal symbol, subject only to the identities

$$h(j,k) = -h(k,j) \tag{2.1}$$

and

$$h(j,k)h(j,l)h(k,l) = h(j,k) - h(j,l) + h(k,l). \tag{2.2}$$

The latter identity is equivalent to the statement that

$$\tanh(a+b) = \frac{\tanh(a)+\tanh(b)}{1+\tanh(a)\tanh(b)},$$

and is at the core of our derivations. (Note that, in fact, equation (2.2) is just Proposition 1, or more precisely, Proposition 1′ below, in the case $n=3$.) We assume throughout that the α_j's are as in Proposition 1, so that the factors $h(j,k)$ are always defined.

We proceed by induction on n: the case $n=1$ is trivial. Let us then assume the proposition true for the integer n. As the cases n odd and n even are quite different, we consider them separately.

Case 1. n odd. (So $\bar{n}=(n-1)/2$.) Note that

$$\begin{aligned}\prod_{1\le j<k\le n+1} h(j,k) &= \left[\prod_{j=1}^{n} h(j,n+1)\right]\left[\prod_{1\le j<k\le n} h(j,k)\right]\\ &= \frac{1}{2^{\bar{n}}\bar{n}!}\left[\prod_{j=1}^{n} h(j,n+1)\right]\\ &\quad\cdot\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{l=1}^{\bar{n}} h(\sigma(2l-1),\sigma(2l)),\end{aligned} \tag{2.3}$$

the latter equality following from the induction hypothesis. Let us extend the domain of S_n to $\{1,2,\ldots,n+1\}$ by putting $\sigma(n+1)=n+1$ for $\sigma\in S_n$. Then, since

$$\begin{aligned}\prod_{j=1}^{n} h(j,n+1) = h(\sigma(n),\sigma(n+1))\prod_{l=1}^{\bar{n}} &h(\sigma(2l-1),\sigma(n+1))\\ &\cdot h(\sigma(2l),\sigma(n+1))\end{aligned}$$

for any $\sigma\in S_n$, we have by equations (2.2) and (2.3),

$$\begin{aligned}\prod_{1\le j<k\le n+1} h(j,k) = \frac{1}{2^{\bar{n}}\bar{n}!}&\sum_{\sigma\in S_n}\operatorname{sgn}(s)h(\sigma(n),\sigma(n+1))\\ &\cdot\prod_{l=1}^{\bar{n}}\{h(\sigma(2l-1),\sigma(2l))h(\sigma(2l-1),\sigma(n+1))\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot h(\sigma(2l),\sigma(n+1))\} \\
&= \frac{1}{2^{\bar n}\bar n!}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)h(\sigma(n),\sigma(n+1)) \\
&\quad\cdot\prod_{l=1}^{\bar n}\{h(\sigma(2l-1),\sigma(2l)) - h(\sigma(2l-1),\sigma(n+1)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + h(\sigma(2l),\sigma(n+1))\} \\
&= \frac{1}{2^{\bar n}\bar n!}\sum_{(A,B,C)}(-1)^{|B|} \\
&\quad\cdot\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)h(\sigma(n),\sigma(n+1)) \\
&\qquad\quad\cdot\left[\prod_{l\in A}h(\sigma(2l-1),\sigma(2l))\right] \\
&\qquad\quad\cdot\left[\prod_{l\in B}h(\sigma(2l-1),\sigma(n+1))\right] \\
&\qquad\quad\cdot\left[\prod_{l\in C}h(\sigma(2l),\sigma(n+1))\right],
\end{aligned}
$$

the outer sum on the right being over all triples (A,B,C) of pairwise disjoint sets whose union is $\{1,2,3,\dots,\bar n\}$. ($|B|$ is the cardinality of B.)

We now claim that the only triple (A,B,C) contributing to the righthand side of (2.4) is the one where B and C are both empty. Indeed, we claim that, for any other triple (A,B,C), the terms in the corresponding sum on S_n cancel pairwise. To see that these claims hold, suppose B, respectively C, is nonempty, and let $l_1\in B$, respectively $l_1\in C$. Now let $\tau\in S_n$ be the transposition of $2l_1-1$, respectively of $2l_1$, and n. Note that the product

$$
\begin{aligned}
h(\sigma(n),\sigma(n+1))\cdot\left[\prod_{l\in A}h(\sigma(2l-1),\sigma(2l))\right]\left[\prod_{l\in B}h(\sigma(2l-1),\sigma(n+1))\right] \\
\cdot\left[\prod_{l\in C}h(\sigma(2l),\sigma(n+1))\right]
\end{aligned}
$$

is invariant under the substitutions of $\sigma\tau$ for σ; indeed, this substitution interchanges $h(\sigma(n),\sigma(n+1))$ with $h(\sigma(2l_1-1),\sigma(n+1))$, respectively with $h(\sigma(2l_1), \sigma(n+1))$, and leaves the other factors in the product fixed. So, if A_n is the alternating group, so that S_n is the disjoint union of A_n and $A_n\tau$, then

$$\begin{aligned}
\sum_{\sigma\in S_n} &\operatorname{sgn}(\sigma) h(\sigma(n), \sigma(n+1)) \\
&\cdot \left[\prod_{l\in A} h(\sigma(2l-1), \sigma(2l))\right] \\
&\cdot \left[\prod_{l\in B} h(\sigma(2l-1), \sigma(n+1))\right] \\
&\cdot \left[\prod_{l\in C} h(\sigma(2l), \sigma(n+1))\right] \\
&= \sum_{\sigma\in A_n\cup A_n\tau} \operatorname{sgn}(\sigma) h(\sigma(n), \sigma(n+1)) \\
&\qquad \cdot \left[\prod_{l\in A} h(\sigma(2l-1), \sigma(2l))\right] \\
&\qquad \cdot \left[\prod_{l\in B} h(\sigma(2l-1), \sigma(n+1))\right] \\
&\qquad \cdot \left[\prod_{l\in C} h(\sigma(2l), \sigma(n+1))\right] \\
&= \sum_{\sigma\in A_n} (\operatorname{sgn}(\sigma) + \operatorname{sgn}(\sigma\tau)) h(\sigma(n), \sigma(n+1)) \\
&\qquad \cdot \left[\prod_{l\in A} h(\sigma(2l-1), \sigma(2l))\right] \\
&\qquad \cdot \left[\prod_{l\in B} h(\sigma(2l-1), \sigma(n+1))\right] \\
&\qquad \cdot \left[\prod_{l\in C} h(\sigma(2l), \sigma(n+1))\right] = 0,
\end{aligned}$$

since $\operatorname{sgn}(\tau) = -1$. This proves our claims. □

Therefore, (2.4) becomes

$$\begin{aligned}\prod_{1\le j<k\le n+1} h(j,k) &= \frac{1}{2^{\bar n}\bar n!}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)h(\sigma(n),\sigma(n+1))\\ &\qquad\cdot\prod_{l=1}^{\bar n} h(\sigma(2l-1),\sigma(2l))\\ &= \frac{1}{2^{\bar n}\bar n!}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{l=1}^{\bar n+1} h(\sigma(2l-1),\sigma(2l)).\end{aligned} \tag{2.5}$$

This is not quite Proposition 1; we want a sum not over S_n but over S_{n+1}. (In particular, the latter will be necessary for the induction below in the case of even n, and is more consistent with the identity obtained in that case. However, see Proposition 1′, where a formula more concise than Proposition 1 or equation (2.5), and valid for n of either parity, is given.)

To get such a sum we need the following

Claim 1. *If, for $1 \le k \le n+1$ and $\sigma \in S_n$, $\sigma_k \in S_{n+1}$ denotes the transposition of k and $n+1$ followed by the permutation σ, then*

$$\begin{aligned}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{l=1}^{\bar n+1} h(\sigma(2l-1),\sigma(2l))\\ = \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma_k)\prod_{l=1}^{\bar n+1} h(\sigma_k(2l-1),\sigma_k(2l)).\end{aligned}$$

Proof of Claim 1. First consider even k. The claim is obviously true if $k = n+1$, so we may assume that this is not the case. Note then that

$$(2.6)\qquad \sum_{\sigma\in S_n} (\sigma_k) \prod_{l=1}^{\bar{n}+1} h(\sigma_k(2l-1), \sigma_k(2l))$$
$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma_k) h(\sigma(k-1), \sigma(n+1))$$
$$\cdot\, h(\sigma(n), \sigma(k)) \prod_{\substack{l=1\\ l\neq k/2}}^{\bar{n}} h(\sigma(2l-1), \sigma(2l)).$$

Now, as σ ranges over S_n, so does the permutation ${}^k\sigma \in S_n$ that, for a given $2 \leq k \leq n-1$, is defined to be σ preceded by the transposition of $k-1$ and n. So (2.6) yields

$$\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma_k) \prod_{l=1}^{\bar{n}+1} h(\sigma_k(2l-1), \sigma_k(2l))$$
$$= \sum_{\sigma\in S_n} \operatorname{sgn}(({}^k\sigma)_k) h({}^k\sigma(k-1), {}^k\sigma(n+1))$$
$$\cdot\, h({}^k\sigma(n), {}^k\sigma(k)) \prod_{\substack{l=1\\ l\neq k/2}}^{\bar{n}} h({}^k\sigma(2l-1), {}^k\sigma(2l))$$
$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) h(\sigma(n), \sigma(n+1))$$
$$\cdot\, h(\sigma(k-1), \sigma(k)) \prod_{\substack{l=1\\ l\neq k/2}}^{\bar{n}} h(\sigma(2l-1), \sigma(2l))$$
$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{l=1}^{\bar{n}+1} h(\sigma(2l-1), \sigma(2l));$$

the second equality relies on the assumption $k \neq n+1$, so that $\operatorname{sgn}(({}^k\sigma)_k) = (-1)^2\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma)$. This proves the claim for k even.

The case k odd is similar; we omit the details. □

Returning to Proposition 1, we find that (2.5) and Claim 1 combine to give

$$(2.7)\quad \prod_{1\le j<k\le n+1} h(j,k) = \frac{1}{(n+1)2^{\bar{n}}\bar{n}!}\sum_{k=1}^{n+1}\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma_k)\prod_{l=1}^{\bar{n}+1} h(\sigma_k(2l-1),\sigma_k(2l)).$$

But $S_{n+1} = \{\sigma_k \mid \sigma \in S_n, 1 \le k \le n+1\}$; moreover,

$$(n+1)2^{\bar{n}}\bar{n}! = 2^{\bar{n}+1}((n+1)/2)(\bar{n})! = 2^{\bar{n}+1}(\bar{n}+1)!,$$

and $\bar{n}+1 = \overline{n+1}$, for n odd. So (2.7) yields

$$\prod_{1\le j<k\le n+1} h(j,k) = \frac{1}{2^{\overline{n+1}}\,\overline{n+1}!}\sum_{\sigma\in S_{n+1}} \operatorname{sgn}(\sigma)\prod_{l=1}^{\overline{n+1}} h(\sigma(2l-1),\sigma(2l)),$$

which completes our induction in the case of odd n.

Case 2. n even. (So $\bar{n} = n/2$.) We begin in a fashion similar to that of Case 1. We have

$$\begin{aligned}
\prod_{1\le j<k\le n+1} h(j,k) &= \left[\prod_{j=1}^{n} h(j,n+1)\right]\left[\prod_{1\le j<k\le n} h(j,k)\right] \\
&= \frac{1}{2^{\bar{n}}\bar{n}!}\left[\prod_{j=1}^{n} h(j,n+1)\right] \\
&\quad \cdot \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\prod_{l=1}^{\bar{n}} h(\sigma(2l-1),\sigma(2l)) \\
&= \frac{1}{2^{\bar{n}}\bar{n}!}\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\prod_{l=1}^{\bar{n}}\{h(\sigma(2l-1),\sigma(2l)) \\
&\quad \cdot h(\sigma(2l-1),\sigma(n+1))h(\sigma(2l),\sigma(n+1))\} \\
&= \frac{1}{2^{\bar{n}}\bar{n}!}\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\prod_{l=1}^{\bar{n}}\{h(\sigma(2l-1),\sigma(2l)) \qquad (2.8)\\
&\quad - h(\sigma(2l-1),\sigma(n+1)) + h(\sigma(2l),\sigma(n+1))\} \\
&= \frac{1}{2^{\bar{n}}\bar{n}!}\sum_{(A,B,C)} (-1)^{|B|}\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \\
&\qquad \cdot \left[\prod_{l\in A} h(\sigma(2l-1),\sigma(2l))\right] \\
&\qquad \cdot \left[\prod_{l\in B} h(\sigma(2l-1),\sigma(n+1))\right] \\
&\qquad \cdot \left[\prod_{l\in C} h(\sigma(2l),\sigma(n+1))\right].
\end{aligned}$$

The second equality in (2.8) is the induction hypothesis; the fourth equality is equation (2.2); on the righthand side of (2.8), A, B, C are as in Case 1.

In Case 1, we saw that a summand corresponding to a particular triple (A, B, C) dropped out if B or C was nonempty. In the present case, we claim that the terms that drop out are those for which $|B \cup C| > 1$. To see this, note that, if $k_1, k_2 \in B \cup C$ and $\tau \in S_n$ is the transposition of

$$\begin{cases} 2k_1-1 \text{ and } 2k_2-1 & \text{if } k_1,k_2\in B;\\ 2k_1-1 \text{ and } 2k_2 & \text{if } k_1\in B \text{ and } k_2\in C;\\ 2k_1 \text{ and } 2k_2-1 & \text{if } k_1\in C \text{ and } k_2\in B;\\ 2k_1 \text{ and } 2k_2 & \text{if } k_1,k_2\in C, \end{cases}$$

(and A_n is, as before, the alternating group), then:

$$\begin{aligned} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\Big[\prod_{l\in A} h(\sigma(2l-1),\sigma(2l))\Big] &\\ \cdot\Big[\prod_{l\in B} h(\sigma(2l-1),\sigma(n+1))\Big] &\\ \cdot[\prod_{l\in C} h(\sigma(2l),\sigma(n+1))\Big] &\\ = \sum_{\sigma\in A_n\cup A_n\tau} \operatorname{sgn}(\sigma)\Big[\prod_{l\in A} h(\sigma(2l-1),\sigma(2l))\Big] &\\ \cdot\Big[\prod_{l\in B} h(\sigma(2l-1),\sigma(n+1))\Big] &\\ \cdot\Big[\prod_{l\in C} h(\sigma(2l),\sigma(n+1))\Big] &\\ = \sum_{\sigma\in A_n} (\operatorname{sgn}(\sigma)+\operatorname{sgn}(\sigma\tau)) &\\ \cdot\Big[\prod_{l\in A} h(\sigma(2l-1),\sigma(2l))\Big] &\\ \cdot\Big[\prod_{l\in B} h(\sigma(2l-1),\sigma(n+1))\Big] &\\ \cdot\Big[\prod_{l\in C} h(\sigma(2l),\sigma(n+1))\Big] &\\ = 0, & \end{aligned}$$

since $\operatorname{sgn}(\tau) = -1$.

So, from (2.8),

$$\begin{aligned}\prod_{1\le j<k\le n+1} h(j,k) = \frac{1}{2^{\bar{n}}\bar{n}!} \sum_{\substack{(A,B,C)\\|B\cup C|\le 1}} (-1)^{|B|} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \\ \cdot \left[\prod_{l\in A} h(\sigma(2l-1),\sigma(2l))\right] \\ \cdot \left[\prod_{l\in B} h(\sigma(2l-1),\sigma(n+1))\right] \\ \cdot \left[\prod_{l\in C} h(\sigma(2l),\sigma(n+1))\right].\end{aligned} \tag{2.9}$$

To complete our induction in the case of n even, we then need the following

Claim 2. *Let $\sigma \in S_n$. With all the notation as above, and σ_k as in Claim 1,*

$$\begin{aligned}\operatorname{sgn}(\sigma) \sum_{\substack{(A,B,C)\\|B\cup C|\le 1}} (-1)^{|B|} \left[\prod_{l\in A} h(\sigma(2l-1),\sigma(2l)\right] \\ \cdot \left[\prod_{l\in B} h(\sigma(2l-1),\sigma(n+1))\right] \\ \cdot \left[\prod_{l\in C} h(\sigma(2l),\sigma(n+1))\right] \\ = \sum_{k=1}^{n+1} \operatorname{sgn}(\sigma_k) \prod_{l=1}^{\bar{n}} h(\sigma_k(2l-1),\sigma_k(2l)).\end{aligned} \tag{2.10}$$

Proof of Claim 2. The lefthand side of (2.10) equals

$$\begin{aligned}\operatorname{sgn}(\sigma)\Bigg\{&\prod_{l=1}^{\bar n} h(\sigma(2l-1),\sigma(2l))\\ &-\sum_{j=1}^{\bar n} h(\sigma(2j-1),\sigma(n+1))\prod_{\substack{l=1\\ l\neq j}}^{\bar n} h(\sigma(2l-1),\sigma(2l))\\ &+\sum_{j=1}^{\bar n} h(\sigma(2j),\sigma(n+1))\prod_{\substack{l=1\\ l\neq j}}^{\bar n} h(\sigma(2l-1),\sigma(2l))\Bigg\}.\end{aligned}\tag{2.11}$$

Now by (2.1), and because a nontrivial transposition has sign -1, we find that

$$-\operatorname{sgn}(\sigma)h(\sigma(2j-1),\sigma(n+1)) = \operatorname{sgn}(\sigma_{2j})h(\sigma_{2j}(2j-1),\sigma_{2j}(2j));$$
$$\operatorname{sgn}(\sigma)h(\sigma(2j),\sigma(n+1)) = \operatorname{sgn}(\sigma_{2j-1})h(\sigma_{2j-1}(2j-1),\sigma_{2j-1}(2j))$$

for $1\le j\le \bar n = n/2$. So (2.11) is equal to

$$\begin{aligned}\operatorname{sgn}(\sigma_{n+1})&\prod_{l=1}^{\bar n} h(\sigma_{n+1}(2l-1),\sigma_{n+1}(2l))\\ &+\sum_{j=1}^{\bar n}\operatorname{sgn}(\sigma_{2j})h(\sigma_{2j}(2j-1),\sigma_{2j}(2j))\\ &\cdot\prod_{\substack{l=1\\ l\neq j}}^{\bar n} h(\sigma_{2j}(2l-1),\sigma_{2j}(2l))\\ &+\sum_{j=1}^{\bar n}\operatorname{sgn}(\sigma_{2j-1})h(\sigma_{2j-1}(2j-1),\sigma_{2j-1}(2j))\\ &\cdot\prod_{\substack{l=1\\ l\neq j}}^{\bar n} h(\sigma_{2j-1}(2l-1),\sigma_{2j-1}(2l))\\ &=\sum_{k=1}^{n+1}\operatorname{sgn}(\sigma_k)\prod_{l=1}^{\bar n} h(\sigma_k(2l-1),\sigma_k(2l)),\end{aligned}$$

and our claim is proved. □

The above claim, together with equation (2.9), give

$$
\begin{aligned}
\prod_{1\le j<k\le n+1} h(j,k) &= \frac{1}{2^{\bar n}\bar n!}\sum_{\sigma\in S_n}\sum_{k=1}^{n+1}\operatorname{sgn}(\sigma_k) \\
&\qquad\cdot\prod_{l=1}^{\bar n} h(\sigma_k(2l-1),\sigma_k(2l)) \\
&= \frac{1}{2^{\bar n}\bar n!}\sum_{\sigma\in S_{n+1}}\operatorname{sgn}(\sigma)\prod_{l=1}^{\bar n} h(\sigma(2l-1),\sigma(2l))
\end{aligned}
\tag{2.12}
$$

since, as before, $S_{n+1} = \{\sigma_k \mid \sigma\in S_n, 1\le k\le n+1\}$.

Note that $\bar n = \overline{n+1}$ for n even. So (2.12) completes our induction for such n; this in turn completes the proof of Proposition 1. □

Remark. Because of (2.1) it is clear that, if J is the subgroup of S_n consisting of all permutations that preserve the partition

$$P = \{\{1,2\},\{3,4\},\dots,\{2\bar n-1, 2\bar n\}\}$$

of $\{1,2,\dots,2\bar n\}$, then for all $\sigma,\tau\in S_n$,

$$
\begin{aligned}
\sigma\tau^{-1}\in J \iff \operatorname{sgn}(\sigma)\prod_{l=1}^{\bar n} h(\sigma(2l-1),\sigma(2l)) \\
= \operatorname{sgn}(\tau)\prod_{l=1}^{\bar n} h(\tau(2l-1),\tau(2l)).
\end{aligned}
$$

Now an element of J may be viewed as a permutation of the $\bar n$ elements of P, followed by an application, to the pair of numbers within each element of P, of either the identity permutation or the transposition of this pair. Therefore, J has $\bar n!\cdot 2^{\bar n}$ elements. So Proposition 1 may be rewritten using a "shorter" sum, whose indexing set is, however, somewhat less elementary:

Proposition 1′. *With all notation as above,*

$$\prod_{1\le j<k\le n}\tanh(\alpha_j-\alpha_k)=\sum_{\sigma\in S_n/J}\operatorname{sgn}(\sigma)\prod_{l=1}^{\bar n}\tanh(\alpha_{\sigma(2l-1)}-\alpha_{\sigma(2l)}),$$

the sum being over a complete set of coset representatives for S_n/J.

3. Proof of Proposition 2. Combining equations (1.1) with Proposition 1, we find

$$\begin{aligned}f(Y)=\frac{\omega_n}{2^{\bar n}\bar n!}&\Bigg[\prod_{1\le j<k\le n}\frac{\Gamma^2((k-j)/2)}{\Gamma^2((1+k-j)/2)}\Bigg]\\ &\cdot\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\int_{t\in\mathbf{R}^n}\hat f(it)h_{it}(Y)\\ &\cdot\Bigg[\prod_{1\le j<k\le n}(t_j-t_k)\Bigg]\\ &\cdot\Bigg[\prod_{l=1}^{\bar n}\tanh\pi(t_{\sigma(2l-1)}-t_{\sigma(2l)})\Bigg]\,dt_1\cdots dt_n.\end{aligned}\tag{3.1}$$

Now let us, for each σ, replace $t_{\sigma(q)}$ by t_q for all $1\le q\le n$, in the integral corresponding to that σ. Such a substitution, as previously noted, leaves $\hat f(it)h_{it}(Y)$ unchanged; also this change of variables clearly has the effect of multiplying

$$\Bigg[\prod_{1\le j<k\le n}(t_j-t_k)\Bigg]$$

by $\operatorname{sgn}(\sigma)$. So equation (3.1) becomes

$$\begin{aligned}f(Y)=\frac{n!\omega_n}{2^{\bar n}\bar n!}&\Bigg[\prod_{1\le j<k\le n}\frac{\Gamma^2((k-j)/2)}{\Gamma^2((1+k-j)/2)}\Bigg]\\ &\cdot\int_{t\in\mathbf{R}^n}\hat f(it)h_{it}(Y)\Bigg[\prod_{1\le j<k\le n}(t_j-t_k)\Bigg]\\ &\qquad\cdot\Bigg[\prod_{l=1}^{\bar n}\tanh\pi(t_{2l-1}-t_{2l})\Bigg]\,dt_1\cdots dt_n.\end{aligned}$$

To get the conclusion of Proposition 2, we merely note by equation (1.1b), and since $\Gamma(1/2) = \sqrt{\pi}$, that

$$\begin{aligned} \frac{n!\omega_n}{2^{\bar{n}}\bar{n}!}\left[\prod_{1\le j<k\le n}\frac{\Gamma^2((k-j)/2)}{\Gamma^2((1+k-j)/2)}\right] \\ &= \frac{n!}{2^{\bar{n}}\bar{n}!}\left[\frac{\pi^{-n(n+1)/4}}{(2\pi)^n n!}\prod_{j=1}^{n}\Gamma(j/2)\right]\left[\prod_{j=2}^{n}\frac{\Gamma^2(1/2)}{\Gamma^2(j/2)}\right] \\ &= \frac{1}{2^{\bar{n}}\bar{n}!}\left[\frac{\pi^{-n(n+1)/4}}{(2\pi)^n}\prod_{j=1}^{n}\Gamma(j/2)\right]\left[\Gamma^{2n}(1/2)\prod_{j=1}^{n}\frac{1}{\Gamma^2(j/2)}\right] \\ &= \frac{\pi^{-n(n+1)/4}}{2^{\bar{n}+n}\bar{n}!}\prod_{j=1}^{n}\frac{1}{\Gamma(j/2)}, \end{aligned}$$

and we are done. □

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