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SPHERICAL ISOMETRIES ARE HYPOREFLEXIVE

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ABSTRACT. The result from the title is shown.

Let $L(H)$ denote the algebra of all bounded linear operators on a complex Hilbert space, H. If $\mathcal{M} \subset L(H)$, then we denote by $\mathcal{M}'$ the commutant of $\mathcal{M}$, $\mathcal{M}' = \{S \in L(H) : TS = ST \text{ for every } T \in \mathcal{M}\}$. The second commutant is denoted by $\mathcal{M}'' = (M')'$. Denote further by $\mathcal{W}(\mathcal{M})$ the smallest weakly closed subalgebra of $L(H)$ containing $\mathcal{M}$ and by $\operatorname{Alg}\operatorname{Lat}\mathcal{M}$ the algebra of all operators leaving invariant all subspaces which are invariant for all operators from $\mathcal{M}$. Recall that $\mathcal{M}$ is said to be *reflexive* if $\mathcal{W}(\mathcal{M}) = \operatorname{Alg}\operatorname{Lat}\mathcal{M}$. For a commutative set $\mathcal{M}$, there is also a weaker version of the reflexivity: $\mathcal{M}$ is called *hyporeflexive* if $\mathcal{W}(\mathcal{M}) = \operatorname{Alg}\operatorname{Lat}\mathcal{M} \cap \mathcal{M}'$.

Reflexivity and hyporeflexivity have been studied intensely by many authors. Deddens in [**3**] proved the reflexivity of a single isometry. The result was extended to sets of commuting isometries in [**2**], see also [**6**].

An analogy and, in some sense, a generalization of commuting N-tuples of isometries are spherical isometries. A *spherical isometry* is an N-tuple $T = (T_1, \dots, T_N)$ of mutually commuting operators on H satisfying $T_1^*T_1 + \cdots + T_N^*T_N = I_H$.

The reflexivity of doubly commuting spherical isometries was mentioned in [**7**]. The aim of the paper is to show the hyporeflexivity of spherical isometries.

If μ is a positive Borel measure on the unit sphere

$$\partial\mathbf{B}_N = \{(z_1, \dots, z_N) \in \mathbf{C}^N : |z_1|^2 + \cdots + |z_N|^2 = 1\},$$

then denote by $H^2(\mu)$ the closure of polynomials in $L^2(\mu)$. We start with the following

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Lemma 1. *Let μ be a finite positive Borel measure supported on $\partial\mathbf{B}_N$. Then, for all nonnegative functions $h \in L^1(\mu)$ and $\varepsilon > 0$, there exists a polynomial p such that*

$$\|h - |p|^2\|_1 < \varepsilon.$$

Proof. First note that there is a nonnegative continuous function g such that $\|h - g\|_1 < (\varepsilon/2)$. Moreover, since μ is finite, we can also assume that $g > 0$. By Theorem 3,5 of [**8**], there exists a sequence p_n of polynomials such that $|p_n| < \sqrt{g}$ and $|p_n(z)| \to \sqrt{g(z)}$ almost everywhere μ on $\partial\mathbf{B}_N$. Hence $g - |p_n|^2 \leq g \in L^1(\mu)$ and $g(z) - |p_n(z)|^2 \to 0$ almost everywhere μ on $\partial\mathbf{B}_N$. By the Lebesgue dominated convergence theorem, there exists an n such that $\|g - |p_n|^2\|_1 < (\varepsilon/2)$. Hence, $\|h - |p|^2\|_1 < \varepsilon$ for $p = p_n$.

Using the terminology of Bercovici [**2**], the result of Lemma 1 exactly means that the subspace $H^2(\mu) \subset L^2(\mu)$ has the *approximate factorization property.* Thus, by Corollary 1.2 of [**2**], we have

Corollary 2. *Let μ be a finite positive Borel measure on $\partial\mathbf{B}_N$ and $h \in L^1(\mu)$. Then there exist $f \in H^2(\mu)$ and $g \in L^2(\mu)$ such that $h(z) = f(z)\overline{g(z)}$ almost everywhere μ.*

Let us fix from now on a spherical isometry $T = (T_1, \dots, T_N) \subset L(H)$. Then T is jointly subnormal by Proposition 2 of [**1**]. It means that there exist a Hilbert space $K \supset H$ and an N-tuple $\tilde{T} = (\tilde{T}_1, \dots, \tilde{T}_N)$ of mutually commuting normal operators on K such that $T_i = \tilde{T}_i|_H$ for $i = 1, \dots, N$. Further, the joint spectral measure of $\tilde{T}$ is supported on $\partial\mathbf{B}_N$, equivalently $\tilde{T}_1^*\tilde{T}_1 + \dots + \tilde{T}_N^*\tilde{T}_N = I$. We can assume that $\tilde{T}$ is the minimal normal extension of T in the sense that $K = \vee_{\alpha\in\mathbf{Z}_+^N}\tilde{T}^{*\alpha}H$.

If $S \in T'$, then, by Proposition 8 of [**1**], there exists $\tilde{S} \in L(K)$ commuting with $\tilde{T}_i$, $i = 1, \dots, N$, such that $\|\tilde{S}\| = \|S\|$ and $S = \tilde{S}|_H$. In fact, $\tilde{S}$ is uniquely determined for the minimal extension $\tilde{T}$. Indeed, $\tilde{S}$ commutes also with $\tilde{T}_i^*$ for $i = 1, \dots, N$ by Fuglede's theorem and $\tilde{S}\tilde{T}^{*\alpha}x = \tilde{T}^{*\alpha}\tilde{S}x = \tilde{T}^{*\alpha}Sx$ for $\alpha \in \mathbf{Z}_+^N$ and $x \in H$. Hence, the uniqueness of $\tilde{S}$ follows from the minimality of the normal extension.

Proposition 3. *Let T be a spherical isometry. If $S \in \operatorname{Alg}\operatorname{Lat} T \cap T'$, then $\tilde{S} \in \tilde{T}''$.*

Proof. Proposition 3, formulated for commuting N-tuples of isometries, is the main result of [**5**], but it remains true with the same proof in our situation. For convenience of the reader, we indicate briefly the main steps of the proof.

Let $E(\cdot)$ be the spectral measure of the normal N-tuple $\tilde{T}$. Denote by $\mu_x = \|\mu_x(\cdot)x\|^2$ the positive scalar measure corresponding to $x \in K$.

A [**5**, Lemma 4]. For $x, y \in H$ there exists a complete number λ such that the measures $\mu_x \vee \mu_y$ and $\mu_{x+\lambda y}$ are equivalent, i.e., absolutely continuous with respect to each other.

In fact, all but countably many complex numbers satisfy the property of A.

For $x \in H$, denote by $Z_+(x)$, $Z(x)$, the smallest subspace of K containing x which is invariant, reducing, respectively, with respect to all $\tilde{T}_1, \dots, \tilde{T}_N$.

By the assumption, $\tilde{S}$ commutes with $\tilde{T}_i$ and $\tilde{T}_i^*$, $i = 1, \dots, N$. Further, $SZ_+(x) \subset Z_+(x)$. Since $Z(x) = \vee_{\alpha \in \mathbf{Z}_+^n} \tilde{T}^{*\alpha} Z_+(x)$, we have $\tilde{S}Z(x) \subset Z(x)$. Consequently, $\tilde{S}|Z(x) = t_x(\tilde{T})|Z(x)$ for some function $t_x \in L^\infty(\mu_x)$.

B [**5**, Lemma 5]. If $x, y \in H$ and $\mu_x \prec \mu_y$, then $t_x = t_y$ almost everywhere μ_x.

By induction it is easy to generalize A and B to finite families of vectors.

Proof of Proposition 3. Let $\tilde{V} \in L(K)$ commute with $\tilde{T}_1, \dots, \tilde{T}_N$, let $u \in K$ and $\varepsilon > 0$.

Since $K = \vee_{x \in H} Z(x)$, we can find vectors $x_1, \dots, x_n, x'_1, \dots, x'_m \in H$ and $u_i \in Z(x_i)$, $i = 1, \dots, n$, $u'_j \in Z(x'_j)$, $j = 1, \dots, m$, such that

$$\left\| u - \sum_{i=1}^n u_i \right\| < \varepsilon, \qquad \left\| \tilde{V}u - \sum_{j=1}^m u'_j \right\| < \varepsilon.$$

By A and B there exists a function $f \in L^\infty(\mu)$ where $\mu = \vee_{i=1}^n \mu_{x_i} \vee$

$\vee_{j=1}^m \mu_{x'_j}$ such that $\tilde S|Z(x_i) = f(\tilde T)|Z(x_i)$ and $\tilde S|Z(x'_j) = f(\tilde T)|Z(x'_j)$ for all i, j. Then

$$\begin{aligned}\|\tilde V\tilde S u - \tilde S\tilde V u\| &\le \left\|\tilde V\tilde S u - \tilde V\tilde S\sum_{i=1}^n u_i\right\| + \left\|\tilde V\tilde S\sum_{i=1}^n u_i - \tilde S\sum_{j=1}^m u'_j\right\| \\ &\quad + \left\|\tilde S\sum_{j=1}^m u'_j - \tilde S\tilde V u\right\| \\ &\le \|\tilde V\|\,\|\tilde S\|\left\|u - \sum_{i=1}^n u_i\right\| \\ &\quad + \left\|\tilde V f(\tilde T)\sum_{i=1}^n u_i - f(\tilde T)\sum_{j=1}^m u'_j\right\| + \|\tilde S\|\left\|\sum_{j=1}^m u'_j - \tilde V u\right\| \\ &\le \|\tilde V\|\,\|S\|\varepsilon \\ &\quad + \|f(\tilde T)\|\left(\left\|\tilde V\sum_{i=1}^n u_i - \tilde V u\right\| + \left\|\tilde V u - \sum_{j=1}^m u'_j\right\|\right) + \|S\|\varepsilon \\ &\le 2\|\tilde V\|\,\|S\|\varepsilon + 2\|S\|\varepsilon.\end{aligned}$$

Since ε was arbitrary, we have $\tilde V\tilde S u = \tilde S\tilde V u$ so that $\tilde S \in (\tilde T)''$.

Recall from [**4**] that an algebra $\mathcal{W} \subset L(H)$ closed in the weak operator topology has property D if every weakly continuous functional φ on $\mathcal{W}$ can be written in the form $\varphi(A) = \langle Ax, y\rangle$, $A \in \mathcal{W}$, for certain vectors $x, y \in H$. Now we will show

Lemma 4. *Let T be spherical isometry. Then* $\operatorname{Alg}\operatorname{Lat} T \cap T'$ *has property D.*

Proof. Let φ be a weakly continuous functional on $\mathcal{B} := \operatorname{Alg}\operatorname{Lat} T \cap T'$. Then there are $x_i, y_i \in H$, $i = 1, \dots, n$, such that $\varphi(S) = \sum_{i=1}^n (Sx_i, y_i)$ for $S \in \mathcal{B}$. Let K_0 be the smallest closed subspace containing $x_i, \dots, x_n, y_1, \dots, y_n$, which reduces $\tilde T_i$, $i = 1, \dots, N$. Using the spectral theory, there exist a finite measure $\mu = \vee_{i=1}^n(\mu_{x_i} \vee \mu_{y_i})$ on $\partial\mathbf{B}_N$, sets $\partial\mathbf{B}_N = \sigma_1 \supset \sigma_2 \supset \cdots \supset \sigma_{2n}$ and a unitary operator

$$U : K_0 \longrightarrow \bigoplus_{i=1}^{2n} L^2(\mu|_{\sigma_i})$$

such that $Uv(\tilde{T}|_{K_0})U^{-1} = M_v$ for every function $v \in L^\infty(\mu)$, where M_v is the operator of multiplication by v. Note that the space $\oplus_{i=1}^{2n} L^2(\mu|_{\sigma_i})$ can be identified with a subspace of $L^2(\mu, \mathcal{F})$, where $\mathcal{F}$ is a Hilbert space with $\dim \mathcal{F} = 2n$.

Let $S \in \mathcal{B}$. Since $S \in \operatorname{Alg}\operatorname{Lat} T$, the space $H_0 := K_0 \cap H$ is invariant with respect to S. Since $\tilde{S} \in \tilde{T}'$, we also have $\tilde{S}K_0 \subset K_0$. Moreover, by Proposition 3, we have $\tilde{S} \in \tilde{T}''$, and one can easily show that $\tilde{S}|_{K_0} \in (\tilde{T}|_{K_0})''$. Thus, there is a function $u_S \in L^\infty(\mu)$ such that $\tilde{S}|_{K_0} = u_S(\tilde{T})|_{K_0}$, and consequently $S|_{H_0} = u_S(\tilde{T})|_{H_0}$.

Now we have

$$\begin{aligned}
\varphi(S) &= \sum_{i=1}^n \langle Sx_i, y_i\rangle = \sum_{i=1}^n \langle USx_i, Uy_i\rangle \\
&= \sum_{i=1}^n \langle Uu_S(\tilde{T})x_i, Uy_i\rangle = \sum_{i=1}^n \langle M_{u_S}Ux_i, Uy_i\rangle \\
&= \sum_{i=1}^n \int \langle u_S(z)(Ux_i)(z), (Uy_i)(z)\rangle_{\mathcal{F}}\, d\mu(z) \\
&= \int u_S(z) \sum_{i=1}^n \langle (Ux_i)(z), (Uy_i)(z)\rangle_{\mathcal{F}}\, d\mu(z) \\
&= \int u_S(z) f(z)\, d\mu(z),
\end{aligned}$$

where $f(z) = \sum_{i=1}^n \langle (Ux_i)(z), (Uy_i)(z)\rangle_{\mathcal{F}} \in L^1(\mu)$.

By Lemma 4 of [**5**] there is a linear combination w of $x_1, \dots, x_n$, $y_1, \dots, y_n$ such that μ_w and μ are absolutely continuous with respect to each other. Hence, $d\mu = v\, d\mu_w$ for some function $v \in L^1(\mu_w)$ and $v > 0$ almost everywhere μ_w. Thus, $fv \in L^1(\mu_W)$ and, by Corollary 2, there exist $g \in H^2(\mu_w)$ and $h \in L^2(\mu_w)$ such that $fv = g\bar{h}$ almost everywhere μ_w.

Denote by K_1 the smallest subspace of K containing w, which reduces $\tilde{T}_i$, $i = 1, \dots, N$. Let $V : K_1 \to L^2(\mu_w)$ be the unitary operator such that $Vr(\tilde{T})w = r$ for all $r \in L^\infty(\mu_w)$. Then we also have $M_r V = Vr(\tilde{T})$ for $r \in L^\infty(\mu_w) = L^\infty(\mu)$. Denote $x = V^{-1}g \in V^{-1}H^2(\mu_w) \subset H$ and

$\tilde{y} = V^{-1}h \in K_1$. Then, for $S \in \mathcal{B}$, we have

$$\begin{aligned}\varphi(S) &= \int u_S f \, d\mu = \int u_S f v \, d\mu_w \\ &= \int u_S g \bar{h} \, d\mu_w = \langle M_{u_S} g, h \rangle_{L^2(\mu_w)} \\ &= \langle V^{-1} M_{u_S} g, V^{-1} h \rangle = \langle u_S(\tilde{T}) V^{-1} g, V^{-1} h \rangle \\ &= \langle u_S(\tilde{T}) x, \tilde{y} \rangle = \langle Sx, \tilde{y} \rangle = \langle Sx, P_H \tilde{y} \rangle = \langle Sx, y \rangle,\end{aligned}$$

where $y = P_H \tilde{y}$. Hence $\mathcal{B}$ has property D.

Recall that, by Theorem 6.2 of [**4**], a unital commutative weakly closed algebra $\mathcal{W} \subset L(H)$ is hyporeflexive if $\operatorname{Alg}\operatorname{Lat}\mathcal{W} \cap \mathcal{W}'$ has property D. Hence we have proved the following main result of the paper. □

Theorem 5. *Let T be a spherical isometry. Then T is hyporeflexive, i.e.,* $\mathcal{W}(T) = \operatorname{Alg}\operatorname{Lat} T \cap T'$.

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