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QUOTIENT NEARRINGS OF SEMILINEAR NEARRINGS

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1. Introduction. All nearrings in this paper will be right nearrings. Let R^n denote the n-dimensional Euclidean group. In [**1**], we showed that if λ is a continuous map from R^n to R and a multiplication $*$ is defined on R^n by $v * w = \lambda(w)v$, then $(R^n, +, *)$ is a topological nearring if and only if $\lambda(av) = a\lambda(v)$ for all $v \in R^n$ and $a \in \operatorname{Ran}(\lambda)$ where $\operatorname{Ran}(\lambda)$ denotes the range of λ. Any map from R^n to R with this property will be referred to as a *semilinear map*. Such maps are quite abundant. For example, let P be any homogeneous polynomial of degree m. That is, $P(tv_1, tv_2, \dots, tv_n) = t^m P(v_1, v_2, \dots, v_n)$ for all $t \in R$ and all $v \in R^n$. Define $\lambda(v) = |P(v)|^{1/m}$. Then λ is a semilinear map. If m is odd, one can also obtain a semilinear map λ by defining $\lambda(v) = (P(v))^{1/n}$. By a *semilinear nearring*, we mean a topological nearring $(R^n, +, *)$ where the multiplication $*$ is induced by a semilinear map λ, and we will denote such a nearring by $N_\lambda(R^n)$. In [**1**], we determined all the ideals (here, ideal means two-sided ideal) of a semilinear nearring. In this paper we show that every nonzero quotient nearring of a semilinear nearring is isomorphic to a semilinear nearring, and we determine precisely when two quotient nearrings of $N_\lambda(R^n)$ are isomorphic. Among other things, we will see that, although $N_\lambda(R^n)$ may have infinitely many quotient nearrings, it has, up to isomorphism, only finitely many and, in fact, this number cannot exceed $n + 1$.

2. The results. Let $N_\lambda(R^n)$ be a semilinear nearring, and let
(2.1)
$$C(\lambda) = \{w \in R^n : \lambda(v + aw) = \lambda(v) \text{ for all } a \in R \text{ and all } v \in R^n\}.$$

In [**1**], we proved the following

Theorem 2.1. *Let λ be any nonconstant semilinear map from R^n to R. Then $C(\lambda) \subseteq \lambda^{-1}(0)$, $C(\lambda)$ is a linear subspace of R^n and the*

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proper ideals of $N_\lambda(R^n)$ are precisely the linear subspaces of $C(\lambda)$. In particular, $C(\lambda)$ is the largest proper ideal of $N_\lambda(R^n)$.

Let J be an ideal of $N_\lambda(R^n)$. For $v \in N_\lambda(R^n)$, we denote by $\langle v\rangle$ the equivalence class, induced by J, which contains v. The proofs of the next two results are contained in the proofs of Lemmas 3.12 and 3.13 of [**2**], but the proofs are short so we include them here.

Lemma 2.2. *Let J be an ideal of $N_\lambda(R^n)$, and define a map λ^* from $N_\lambda(R^n)/J$ into R by $\lambda^*(\langle v\rangle) = \lambda(v)$. Then λ^* is a well-defined continuous map from $N_\lambda(R^n)/J$ into R which has the following properties:*

$$\lambda^*(a\langle v\rangle) = a\lambda^*(\langle v\rangle) \quad \text{for all } \langle v\rangle \in N_\lambda(R^n)/J \text{ and all } a \in \mathrm{Ran}\,(\lambda^*). \tag{2.2}$$

$$\langle v\rangle * \langle w\rangle = (\lambda^*(\langle w\rangle)(\langle v\rangle) \quad \text{for all } \langle v\rangle, \langle w\rangle \in N_\lambda(R^n)/J. \tag{2.3}$$

Proof. Suppose $\langle u\rangle = \langle v\rangle$. Then $u = v + w$ for some $w \in J \subseteq C(\lambda)$ and, from (2.1), we get $\lambda(u) = \lambda(v+w) = \lambda(v)$ which means λ^* is well defined. Note that $\mathrm{Ran}\,(\lambda^*) = \mathrm{Ran}\,(\lambda)$. For any $a \in \mathrm{Ran}\,(\lambda^*)$ and $v \in R^n$, we have

$$\lambda^*(a\langle v\rangle) = \lambda^*(\langle av\rangle) = \lambda(av) = a\lambda(v) = a\lambda^*(\langle v\rangle)$$

and we see that (2.2) is satisfied. Furthermore, we have

$$\langle v\rangle * \langle w\rangle = \langle v * w\rangle = \langle \lambda(w)v\rangle = \lambda(w)\langle v\rangle = \lambda^*(\langle w\rangle)\langle v\rangle$$

which means (2.3) is satisfied. □

The dimension of a subspace V of the vector space R^n will be denoted by $\dim V$. We recall that, according to Theorem 2.1, every ideal J of $N_\lambda(R^n)$ is a linear subspace of the vector space R^n and, consequently, $N_\lambda(R^n)/J$, in addition to being a nearring, is also a real vector space of dimension $n - \dim\ J$. In the next result, we show that every quotient nearring of $N_\lambda(R^n)$ is isomorphic to a semilinear nearring.

Theorem 2.3. *Let J be a proper ideal of $N_\lambda(R^n)$. Then $N_\lambda(R^n)/J$ is isomorphic to a semilinear nearring. Specifically, let $\dim J = m$, let α be any linear isomorphism from R^{n-m} onto $N_\lambda(R^n)/J$, and define a map μ from R^{n-m} to R by $\mu = \lambda^* \circ \alpha$. Then μ is a semilinear map and α is a topological isomorphism from $N_\mu(R^{n-m})$ onto $N_\lambda(R^n)/J$.*

Proof. It is immediate from Lemma (2.2) that μ is a semilinear map from R^{n-m} onto R and it is also immediate that α is a continuous additive isomorphism from $N_\mu(R^{n-m})$ onto $N_l(R^n)/J$. We need only show that it is a multiplicative homomorphism, so let $v, w \in N_\mu(R^{n-m})$, and we get

$$\begin{aligned}\alpha(v * w) &= \alpha(\mu(w)v) = \mu(w)\alpha(v)\\ &= (\lambda^*(\alpha(w)))\alpha(v) = \alpha(v) * \alpha(w).\end{aligned}$$

This completes our proof. □

In our next result, we determine precisely when two quotient nearrings of $N_\lambda(R^n)$ are isomorphic.

Theorem 2.4. *Let J_1 and J_2 be two ideals of $N_\lambda(R^n)$. Then the quotient nearrings $N_\lambda(R^n)/J_1$ and $N_\lambda(R^n)/J_2$ are isomorphic if and only if* $\dim J_1 = \dim J_2$.

Proof. Since the dimensions of the vector spaces $N_\lambda(R^n)/J_1$ and $N_\lambda(R^n)/J_2$ are $n - \dim J_1$ and $n - \dim\ J_2$, respectively, it is immediate that $\dim J_1 = \dim J_2$ whenever $N_\lambda(R^n)/J_1$ and $N_\lambda(R^n)/J_2$ are isomorphic. Suppose, conversely, that $\dim J_1 = \dim\ J_2 = m$, and let φ be any linear automorphism of R^n such that $\varphi[J_1] = J_2$ and $\varphi[J_2] = J_1$. For any $v \in R^n$, we denote by $\langle v\rangle_1$ and $\langle v\rangle_2$ the equivalence classes containing v which are induced by the ideals J_1 and J_2, respectively. Define a map $\hat{\varphi}$ from $N_\lambda(R^n)/J_1$ to $N_\lambda(R^n)/J_2$ by

$$\hat{\varphi}\langle v\rangle_1 = \langle \varphi(v)\rangle_2. \tag{2.4}$$

One verifies, in a straightforward manner, that $\hat{\varphi}$ is a linear isomorphism from the linear space $N_\lambda(R^n)/J_1$ onto the linear space

$N_\lambda(R^n)/J_2$. It is appropriate to remark at this point that, if φ would happen to be a nearring automorphism of $N_\lambda(R^n)$, then we would be finished because $\hat{\varphi}$ would then be a nearring isomorphism from $N_\lambda(R^n)/J_1$ onto $N_\lambda(R^n)/J_2$. However, we know only that φ is a linear automorphism of R^n where $\varphi[J_1] = J_2$ and $\varphi[J_2] = J_1$ and $\hat{\varphi}$ is a linear isomorphism from $N_\lambda(R^n)/J_1$ onto $N_\lambda(R^n)/J_2$ so we must proceed by other means. With this in mind, let α_1 be any linear isomorphism from R^{n-m} onto $N_\lambda(R^n)/J_1$, and define

$$\alpha_2 = \hat{\varphi} \circ \alpha_1. \tag{2.5}$$

Evidently, α_2 is a linear isomorphism from R^{n-m} onto $N_\lambda(R^n)/J_2$. Next, define maps λ_1^* and λ_2^* from $N_\lambda(R^n)/J_1$ and $N_\lambda(R^n)/J_2$, respectively, into R by

$$\lambda_1^*(\langle v\rangle_1) = \lambda(v) \quad \text{and} \quad \lambda_2^*(\langle v\rangle_2) = \lambda(v) \tag{2.6}$$

and then define

$$\mu_1 = \lambda_1^* \circ \alpha_1 \quad \text{and} \quad \mu_2 = \lambda_2^* \circ \alpha_2. \tag{2.7}$$

According to Theorem 2.3, μ_1 and μ_2 are semilinear maps from R^{n-m} into R and the semilinear nearring $N_{\mu_1}(R^{n-m})$ and $N_{\mu_2}(R^{n-m})$ are isomorphic to $N_\lambda(R^n)/J_1$ and $N_\lambda(R^n)/J_2$, respectively. Next define a self-map ψ of R^{n-m} by

$$\psi(v) = \alpha_1^{-1}\langle\varphi(u)\rangle_1 \quad \text{where } \alpha_1(v) = \langle u\rangle_1. \tag{2.8}$$

Let $a, b \in R$ and $u, v \in R^{n-m}$. Then there exist $w, z \in R^n$ such that $\alpha_1(u) = \langle w\rangle_1$ and $\alpha_1(v) = \langle z\rangle_1$. Then

$$\alpha_1(au + bv) = a\alpha_1(u) + b\alpha_1(v) = a\langle w\rangle_1 + b\langle z\rangle_1 = \langle aw + bz\rangle_1 \tag{2.9}$$

and we have

$$\begin{aligned} \psi(au + bv) &= \alpha_1^{-1}\langle\varphi(aw + bz)\rangle_1 \\ &= \alpha_1^{-1}\langle a\varphi(w) + b\varphi(z)\rangle_1 \\ &= \alpha_1^{-1}(a\langle\varphi(w)\rangle_1 + b\langle\varphi(z)\rangle_1) \\ &= a\alpha_1^{-1}\langle\varphi(w)\rangle_1 + b\alpha_1^{-1}\langle\varphi(z)\rangle_1 \\ &= a\psi(u) + b\psi(v). \end{aligned} \tag{2.10}$$

This verifies the fact that ψ is a linear endomorphism of R^{n-m}. Suppose $\psi(v) = 0$. Then $\alpha_1(v) = \langle u\rangle_1$ for some $u \in R^n$ and $\alpha_1^{-1}\langle\varphi(u)\rangle_1 = 0$ which means $\langle\varphi(u)\rangle_1 = 0$. Consequently, $\varphi(u) \in J_1$ which implies $u \in J_2$. From this, we get $\alpha_2(v) = \hat{\varphi}\circ\alpha_1(v) = \hat{\varphi}\langle u\rangle_1 = \langle u\rangle_2 = 0$ which implies $v = 0$ since α_2 is a linear isomorphism. Thus, ψ is a linear automorphism of R^{n-m}.

We next assert that

$$\mu_1 \circ \psi = \mu_2. \tag{2.11}$$

Let $v \in R^{n-m}$. Then $\alpha_1(v) = \langle u\rangle_1$ for some $u \in R^n$ and, from (2.6)–(2.8), we get

$$\mu_1 \circ \psi(v) = \lambda_1^* \circ \alpha_1 \circ \alpha_1^{-1}\langle\varphi(u)\rangle_1 = \lambda_1^*(\langle\varphi(u)\rangle_1) = \lambda(\varphi(u)). \tag{2.12}$$

On the other hand, from (2.4)–(2.8), we get

$$\begin{aligned}\mu_2(v) &= \lambda_2^* \circ \alpha_2(v) = \lambda_2^* \circ \hat{\varphi} \circ \alpha_1(v)\\ &= \lambda_2^* \circ \hat{\varphi}(\langle u\rangle_1) = \lambda_2^*(\langle\varphi(u)\rangle_2 = \lambda(\varphi(u))\end{aligned} \tag{2.13}$$

and (2.12) and (2.13) together provide a verification of (2.11). It now follows from Theorem 3.14 of [**1**] that the semilinear nearrings $N_{\mu_1}(R^{n-m})$ and $N_{\mu_2}(R^{n-m})$ are isomorphic, and we have now shown that the quotient nearrings $N_\lambda(R^n)/J_1$ and $N_\lambda(R^n)/J_2$ are isomorphic.

Corollary 2.5. *Up to isomorphism, the nearring $N_\lambda(R^n)$ has at most $n+1$ quotient nearrings.*

Proof. The nearring $N_\lambda(R^n)$ has the maximal number of ideals whenever the map λ is such that $\dim C(\lambda) = n - 1$. According to Theorem 2.4, this results in, up to isomorphism, exactly $n+1$ quotient rings including the one produced by the zero ideal and the one produced by $N_\lambda(R^n)$ itself.

Example 2.6. Define a map λ from R^3 into R by $\lambda(v) = |v_3|$ where $v = (v_1, v_2, v_3)$. It is easily checked that λ is a semilinear map. We proceed to determine, up to isomorphism, all the quotient nearrings of the semilinear nearring $N_\lambda(R^3)$. Let us begin by determining the largest proper ideal $C(\lambda)$ of $N_\lambda(R^3)$. From (2.1) we see that $w \in C(\lambda)$ if and only if $\lambda(v+aw) = \lambda(v)$ for all $v \in R^e$ and all $a \in R$. Hence, $w \in C(\lambda)$ if and only if $|v_3 + aw_3| = |v_3|$ for all $v \in R^3$ and $a \in R$. It readily follows that $C(\lambda) = \{w \in N_\lambda(R^3) : w_3 = 0\}$. Therefore, the proper nonzero ideals of $N_\lambda(R^3)$ are $C(\lambda)$ together with all its nonzero linear subspaces and, according to the previous theorem this means that, up to isomorphism, $N_\lambda(R^3)$ has exactly four different quotient nearrings. Thus, in addition to the zero ring and $N_\lambda(R^3)$, there are two other quotient nearrings and, according to Theorem 2.3, each of these is isomorphic to a semilinear nearring. We determine these two semilinear nearrings. Define a semilinear map λ_1 from R into R by $\lambda_1(x) = |x|$, and define a map φ from $N_\lambda(R^3)$ to $N_{\lambda_1}(R)$ by $\varphi(v) = v_3$. One readily verifies that φ is an epimorphism from $N_\lambda(R^3)$ onto $N_{\lambda_1}(R)$ with kernel $C(\lambda)$. Consequently, $N_\lambda(R^3)/C(\lambda)$ is isomorphic to $N_{\lambda_1}(R^2)$.

Now define a semilinear map λ_2 from R^2 into R by $\lambda_2(v) = |v_2|$, and define a map φ from $N_\lambda(R^3)$ to $N_{\lambda_2}(R^2)$ by $\varphi(v) = (v_2, v_3)$. In this case, one verifies that φ is an epimorphism from $N_\lambda(R^3)$ onto $N_{\lambda_2}(R^2)$ with kernel $\operatorname{Ker}\varphi = \{v \in N_\lambda(R^3) : v_2 = v_3 = 0\}$ so that $N_\lambda(R^3)/\operatorname{Ker}\varphi$ is isomorphic to $N_{\lambda_2}(R^2)$. According to Theorem 2.4, all the other nonzero ideals properly contained in $C(\lambda)$ produce quotient rings which are also isomorphic to $N_{\lambda_2}(R^2)$. Thus, up to isomorphism, the four quotient nearrings of $N_\lambda(R^3)$ are the zero ring, $N_{\lambda_1}(R)$, $N_{\lambda_2}(R^2)$ and $N_\lambda(R^3)$ itself.

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