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ON THE ESSENTIAL SPECTRA OF REGULARLY SOLVABLE OPERATORS IN THE DIRECT SUM SPACES

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ABSTRACT. The problem of investigation of the spectral properties of the operators which are regularly solvable with respect to minimal operators $T_0(M_p)$ and $T_0(M_p^+)$ generated by a general quasi-differential expression M_p and its formal adjoint M_p^+ on any finite number of intervals $I_p = (a_p, b_p)$, $p = 1, \dots, N$, are studied in the setting of the direct sums of $L^2_{wp}(a_p, b_p)$-spaces of functions defined on each of the separate intervals. These results extend those of formally symmetric expression M studied in [**1**] and [**15**] in the single-interval case, and also extend those proved in [**10**] and [**13**] in the general case.

1. Introduction. Akhiezer and Glazman [**1**] and Naimark [**15**] showed that the self-adjoint extensions of the minimal operator $T_0(M)$ generated by a formally symmetric differential expression M with maximal deficiency indices have resolvents which are Hilbert-Schmidt integral operators and consequently have a wholly discrete spectrum. In [**10**], Ibrahim extended their results for general ordinary quasi-differential expression M of nth order with complex coefficients in the single-interval case with one singular endpoint.

The minimal operators $T_0(M)$ and $T_0(M^+)$ generated by a general ordinary quasi-differential expression M and its formal adjoint M^+, respectively, form an adjoint pair of closed, densely-defined operators in the underlying L^2_w-space, that is, $T_0(M) \subset [T_0(M^+)]^*$. The operators which fulfill the role that the self-adjoint and maximal symmetric operators play in the case of a formally symmetric expression M are those which are regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$. Such an operator S satisfies $T_0(M) \subset S \subset [T_0(M^+)]^*$ and, for some $\lambda \in \mathbf{C}$, $(S - \lambda I)$ is a Fredholm operator of zero index, this means that S has the desirable Fredholm property that the equation $(S - \lambda I)u = f$ has a solution if and only if f is orthogonal to the solution space of

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$(S - \lambda I)v = 0$ and, furthermore, the solution spaces of $(S - \lambda I)u = 0$ and $(S^* - \bar{\lambda} I)v = 0$ have the same finite dimension. This notion was originally due to Visik in [**18**]. In [**6**] and [**7**], Everitt and Zettl considered the problem of characterizing all self-adjoint operators which can be generated by a formally symmetric Sturm-Liouville differential (quasidifferential) expression M_p, defined on a finite or countable number of intervals I_p, $p = 1, \dots, N$.

Our objective in this paper is to extend the results concerning the spectral properties of the operators in [**1, 10, 13** and **15**] for the case when the quasi-differential expressions M_p are arbitrary and there is any finite number of intervals I_p, $p = 1, \dots, N$ when all solutions of the equations $M_p[u] - \lambda w_p u = 0$ and $M_p^+[v] - \bar{\lambda} w_p v = 0$ are in $L^2_{wp}(a_p, b_p)$, $p = 1, \dots, N$, for some, and hence all, $\lambda \in \mathbf{C}$.

The operators involved are no longer symmetric but direct sums

$$T_0(M) = \bigoplus_{p=1}^{N} T_0(M_p)$$

and

$$T_0(M^+) = \bigoplus_{p=1}^{N} T_0(M_p^+)$$

where $T_0(M_p)$ and $T_0(M_p^+)$ form an adjoint pair of closed operators in

$$\bigoplus_{p=1}^{N} L^2_{wp}(a_p, b_p).$$

We deal throughout with a quasi-differential expression M_p of arbitrary order n defined by a general Shin-Zettl matrix, and the minimal operator $T_0(M_p)$ generated by $w_p^{-1} M_p[.]$ in $L^2_{wp}(I_p)$, $p = 1, \dots, N$, where w_p is a positive weight function on the underlying interval I_p. The endpoints of I_p may be regular or singular.

2. Preliminaries. We begin with a brief summary of adjoint pairs of operators and their associated regularly solvable operators; a full treatment may be found in [**2**, Chapter 3], [**3**] and [**13**].

The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$ and $R(T)$, respectively, and $N(T)$ will denote its null space. The *nullity* of T, written $\mathrm{nul}\,(T)$, is the dimension of $N(T)$ and the *deficiency* of T, $\mathrm{def}\,(T)$, is the codimension of $R(T)$ in H if T is densely defined and $R(T)$ is closed, then $\mathrm{def}\,(T) = \mathrm{nul}\,(T^*)$. The *Fredholm domain* of T is (in the notation of [**2**]) the open subset $\Delta_3(T)$ of $\mathbf{C}$ consisting of those values $\lambda \in \mathbf{C}$ which are such that $(T - \lambda I)$ is a Fredholm operator, where I is the identity operator in H. Thus, $\lambda \in \Delta_3(T)$ if and only if $(T - \lambda I)$ has closed range and finite nullity and deficiency. The *index* of $(T - \lambda I)$ is the number $\mathrm{ind}\,(T - \lambda I) = \mathrm{nul}\,(T - \lambda I) - \mathrm{def}\,(T - \lambda I)$, this being defined for $\lambda \in \Delta_3(T)$.

Two closed densely-defined operators, A, B in H, are said to form an *adjoint pair* if $A \subset B^*$ and consequently $B \subset A^*$, equivalently. $(Ax, y) = (x, By)$ for all $x \in D(A)$ and $y \in D(B)$, where $(. , .)$ denotes the inner-product on H.

The *joint field of regularity* $\Pi(A, B)$ of A and B is the set of $\lambda \in \mathbf{C}$ which is such that $\lambda \in \Pi(A)$, the field of regularity of A, $\bar{\lambda} \in \Pi(B)$ and $\mathrm{def}\,(A - \lambda I)$ and $\mathrm{def}\,(B - \bar{\lambda} I)$ are finite. An adjoint pair A, B is said to be *compatible* if $\Pi(A, B) \neq \varnothing$. Recall that $\lambda \in \Pi(A)$ if and only if there exists a positive constant $K(\lambda)$ such that

$$\|(A - \lambda I)x\| \geq K(\lambda)\|x\| \quad \text{for all } x \in D(A),$$

or equivalently, on using the Closed-Graph Theorem, $\mathrm{nul}\,(A - \lambda I) = 0$ and $R(A - \lambda I)$ is closed.

A closed operator S in H is said to be *regularly solvable* with respect to the compatible adjoint pair A, B if $A \subset S \subset B^*$ and $\Pi(A, B) \cap \Delta_4(S) \neq \varnothing$, where

$$\Delta_4(S) = \{\lambda : \lambda \in \Delta_3(S), \mathrm{ind}\,(A - \lambda I) = 0\}.$$

If $A \subset S \subset B^*$ and the resolvent set $\rho(S)$, see [**2**], of S is nonempty, S is said to be *well-posed* with respect to A and B. Note that, if $A \subset S \subset B^*$ and $\lambda \in \rho(S)$, then $\lambda \in \Pi(A)$ and $\bar{\lambda} \in \rho(S^*) \subset \Pi(B)$ so that if $\mathrm{def}\,(A - \lambda I)$ and $\mathrm{def}\ (B - \bar{\lambda} I)$ are finite, then A and B are compatible; in this case S is regularly solvable with respect to A and B. The terminology "regularly solvable" comes from Visik's paper

[**18**], while the notion of "well-posed" was introduced by Zhikhar in his work on J-self-adjoint operators in [**22**]. The complement of $\rho(S)$ in $\mathbf{C}$ is called the spectrum of S and written $\sigma(S)$. The point spectrum $\sigma_p(S)$, continuous spectrum $\sigma_c(S)$ and residual spectrum $\sigma_r(S)$ are the following subsets of $\sigma(S)$:

(i) $\lambda \in \sigma_p(S)$ if and only if $R(S-\lambda I) = \overline{R(S-\lambda I)} \subset H$,

(ii) $\lambda \in \sigma_c(S)$ if and only if $R(S-\lambda I) \subset \overline{R(S-\lambda I)} = H$,

(iii) $\lambda \in \sigma_r(S)$ if and only if $R(S-\lambda I) \subset \overline{R(S-\lambda I)} \subset H$.

For a closed operator S we have

$$\sigma(S) = \sigma_p(S) \cup \sigma_c(S) \cup \sigma_r(S).$$

An important subset of the spectrum of a closed densely-defined operator T in H is the *so-called essential* spectrum. The various essential spectra of T are defined as in [**2**, Chapter 9] to be the sets,

$$\sigma_{ek}(T) = \mathbf{C} \backslash \Delta_k(T), \quad k = 1, 2, 3, 4, 5, \tag{2.1}$$

$\Delta_3(T)$ and $\Delta_4(T)$ have been defined earlier.

The sets $\sigma_{ek}(T)$ are closed and $\sigma_{ek}(T) \subset \sigma_{ej}(T)$ if $k < j$, the inclusion being strict in general. We refer the reader to [**2**, Chapter 9] for further information about the sets $\sigma_{ek}(T)$.

We now turn to the quasi-differential expressions defined in terms of a Shin-Zettl matrix F_p on an open interval I_p, where I_p denotes an open interval with left end-point a_p and right end-point b_p, $-\infty \leq a_p < b_p \leq \infty$, $p = 1, \ldots, N$. The set $Z_n(I_p)$ of Shin-Zettl matrices on I_p consists of $(n \times n)$-matrices $F_p = \{f_{rs}^p\}$ whose entries are complex-valued functions on I_p which satisfy the following conditions:

$$\begin{aligned} & f_{rs}^p \in L^1_{\text{loc}}(I), \qquad 1 \leq r, s \leq n, n \geq 2; \quad p = 1, \ldots, N \\ & f_{r,r+1}^p \neq 0, \qquad \text{a.e. on } I, \quad 1 \leq r \leq n-1 \\ & f_{rs}^p = 0, \qquad \text{a.e. on } I, \quad 2 \leq r+1 < s \leq n. \end{aligned} \tag{2.2}$$

For $F_p \in Z_n(I_p)$, the *quasi-derivatives* associated with F_p are defined by

$$(2.3)\quad \begin{aligned} y^{[0]} &:= y \\ y^{[r]} &:= (f^p_{r,r+1})^{-1}\left\{(y^{[r-1]})' - \sum_{s=1}^{r} f^p_{rs} y^{[s-1]}\right\}, \quad 1 \le r \le n-1 \\ y^{[n]} &:= (y^{[n-1]})' - \sum_{s=1}^{n} f^p_{ns} y^{[s-1]}, \end{aligned}$$

where the prime $'$ denotes differentiation.

The quasi-differential expression M_p associated with F_p is given by

$$(2.4)\qquad M_p[y] := i^n y^{[n]}, \quad p = 1, \dots, N,$$

this being defined on the set

$$(2.5)\quad V(M_p) := \{y : y^{[r-1]} \in AC_{\mathrm{loc}}(I_p),\ r = 1, \dots, n;\ p = 1, \dots, N\},$$

where $AC_{\mathrm{loc}}(I_p)$ denotes the set of functions which are absolutely continuous on every compact subinterval of I_p.

The *formal adjoint* M_p^+ of M_p is defined by the matrix $F_p^+ \in Z_n(I_p)$ given by

$$(2.6)\qquad F_p^+ = -J_{n\times n}^{-1} F_p^* J_{n\times n},$$

where F_p^* is the conjugate transpose of F_p and $J_{n\times n}$ is the nonsingular $n \times n$-matrix.

$$(2.7)\qquad J_{n\times n} = ((-1)^r \delta_{r,n+1-s}), \quad 1 \le r, s \le n,$$

δ being the Kronecker delta. If $F_p^+ = \{f^p_{rs}\}^+$, then it follows that

$$(2.8)\qquad \{f^p_{rs}\}^+ = (-1)^{r+s+1} \overline{f^p}_{n-s+1,n-r+1}.$$

The quasi-derivatives associated with F_p^+ are therefore

$$
\begin{aligned}
y_+^{[0]} &:= y,\\
y_+^{[r]} &:= (\overline{f^p}_{n-r,n-r+1})^{-1}\\
&\qquad \cdot \left\{ (y_+^{[r-1]})' - \sum_{s=1}^{r} (-1)^{r+s+1} \overline{f^p}_{n-s+1,n-r+1} y_+^{[s-1]} \right\}\\
y_+^{[n]} &:= (y_+^{[n-1]})' - \sum_{s=1}^{n} (-1)^{n+s+1} \overline{f^p}_{n-s+1,1} y_+^{[s-1]},\\
&\qquad\qquad 1 \le r \le n-1,
\end{aligned}
\tag{2.9}
$$

and

$$M_p^+[y] := i^n y_+^{[n]}, \quad p = 1, \dots, N \quad \text{for all } y \in V(M_p^+); \tag{2.10}$$

$$V(M^+) := \{ y : y_+^{[r-1]} \in AC_{\text{loc}}(I_p),\ r = 1, \dots, n;\ p = 1, \dots, N \}. \tag{2.11}$$

Note that $(F_p^+)^+ = F_p$ and so $(M_p^+)^+ = M_p$. We refer to [**2, 3, 5, 8** and **13**] for a full account of the above and subsequent results on quasi-differential expressions.

For $u \in V(M_p)$, $v \in V(M_p^+)$ and $\alpha, \beta \in I_p$, we have Green's formula,

$$\int_\alpha^\beta \{ \bar{v} M_p[u] - \overline{u M_p^+[v]} \}\, dx = [u, v]_p(\beta) - [u, v]_p(\alpha), \tag{2.12}$$

where

$$
\begin{aligned}
[u, v]_p(x) &= \left(i^n \sum_{r=0}^{n-1} (-1)^{n+r+1} u^{[r]}(x) \bar{v}_+^{[n-r-1]}(x) \right)_p,\\
&= \left((-i)^n (u, \dots, u^{[n-1]}) J_{n\times n} \begin{pmatrix} \bar{v}_+(x) \\ \vdots \\ \bar{v}_+^{[n-1]}(x) \end{pmatrix} \right)_p,
\end{aligned}
\tag{2.13}
$$

$p = 1, \dots, N$, see [**10, 12**] and [**20**, Corollary 1].

Let the interval I_p have end-points a_p, b_p, $-\infty \leq a_p < b_p \leq \infty$, and let w_p be a function which satisfies

$$w_p > 0 \quad \text{a.e. on } I_p, \quad w_p \in L^1_{\text{loc}}(I_p). \tag{2.14}$$

The equation

$$M_p[y] - \lambda w_p y = 0, \quad p = 1, \dots, N, \quad \lambda \in \mathbf{C} \tag{2.15}$$

on I_p is said to be *regular* at the left end-point a_p if a_p is finite, and for all $X \in (a_p, b_p)$,

$$\begin{gathered} a_p \in \mathbf{R}, w_p, f^p_{rs} \in L^1[a_p, X], \\ r = 1, \dots, n; \quad p = 1, \dots, N. \end{gathered} \tag{2.16}$$

Otherwise (2.15) is said to be *singular* at a_p. Similarly, we define the terms regular and singular at b_p. If (2.15) is regular on (a_p, b_p), then we have

$$\begin{gathered} a_p, b_p \in \mathbf{R}, w_p, f^p_{rs} \in L^1(a_p, b_p), \\ r, s = 1, \dots, n; \quad p = 1, \dots, N. \end{gathered} \tag{2.17}$$

Note that, in view of (2.8), an endpoint of I_p is regular for (2.15) if and only if it is regular for the equation

$$M^+_p[y] - \bar{\lambda} w_p y = 0, \quad p = 1, \dots, N, \quad \lambda \in \mathbf{C} \quad \text{on } I_p. \tag{2.18}$$

Let $H_p = L^2_{w_p}(a_p, b_p)$ denote the usual-weighted L^2-space with inner-product,

$$(f, g)_p = \int_{Ip} f(x)\overline{g(x)} w_p(x)\, dx, \quad p = 1, \dots, N, \tag{2.19}$$

and $\|f\|_p := (f, f)^{1/2}$: this is a Hilbert space on identifying functions which differ only on null sets. Set

$$\begin{gathered} D(M_p) := \{u : u \in V(M_p), u \text{ and } w_p^{-1} M_p[u] \in L^2_{wp}(a_p, b_p)\}, \\ p = 1, \dots, N, \\ D(M^+_p) := \{v : v \in V(M^+_p), v \text{ and } w_p^{-1} M^+_p[v] \in L^2_{wp}(a_p, b_p)\}, \\ p = 1, \dots, N. \end{gathered} \tag{2.20}$$

Note that, at a regular endpoint, a_p say, $u^{[r-1]}(a_p)\{v_+^{[r-1]}(a_p)\}$ is defined for all $u \in V(M_p)$ $(v \in V(M_p^+))$, $r = 1, 2, \dots, n$. The manifolds, $D(M_p)$, $D(M_p^+)$ of $L^2_{wp}(a_p, b_p)$ are the domains of the so-called *maximal operators* $T(M_p), T(M_p^+)$, respectively, defined by

$$T(M_p)u := w_p^{-1} M_p[u], \quad u \in D(M_p)$$

and

$$T(M_p^+)v = w_p^{-1} M_p^+[v], \quad v \in V(M_p^+).$$

For the regular problem, the *minimal operators* $T_0(M_p)$, $T_0(M_p^+)$ are the restrictions of $w_p^{-1}M_p[.]$ and $w_p^{-1}M_p^+[.]$ to the subspaces,

(2.21)

$$\begin{aligned} D_0(M_p) &:= \{u : u \in D(M_p),\ u^{[r-1]}(a_p) = u^{[r-1]}(b_p) = 0,\ r = 1, \dots, n, \} \\ D_0(M_p^+) &:= \{v : v \in D(M_p^+),\ v_+^{[r-1]}(a_p) = v_+^{[r-1]}(b_p) = 0, \\ &\qquad r = 1, \dots, n,\ p = 1, \dots, N, \} \end{aligned}$$

respectively. The subspaces $D_0(M_p)$ and $D_0(M_p^+)$ are dense in $L^2_{wp}(a_p, b_p)$ and $T_0(M_p)$, $T_0(M_p^+)$ are closed operators, see [**20**, Section 3]. In the singular problem we first introduce operators $T_0'(M_p)$, $T_0'(M_p^+)$, where $T_0'(M_p)$ is the restriction of $w_p^{-1}M_p[.]$ to

$$D_0'(M_p) := \{u : u \in D(M_p),\ \operatorname{supp}(u) \subset (a_p, b_p)\}, \tag{2.22}$$

and with $T_0'(M^+)$ defined similarly. These operators are densely-defined and closable in $L^2_w(a_p, b_p)$, and we define the minimal operators $T_0(M_p)$, $T_0(M_p^+)$ to be their respective closures, cf. [**13**] and [**20**, Section 5]. We denote the domains of $T_0(M_p)$ and $T_0(M_p^+)$ by $D_0(M_p)$ and $D_0(M_p^+)$, respectively. It can be shown that, if (2.15) is regular at a_p,

$$\begin{aligned} u \in D_0(M_p) &\Longrightarrow u^{[r-1]}(a_p) = 0, \quad r = 1, \dots, n;\ p = 1, \dots, N \\ v \in D_0(M_p^+) &\Longrightarrow v_+^{[r-1]}(a_p) = 0, \quad r = 1, \dots, n;\ p = 1, \dots, N. \end{aligned} \tag{2.23}$$

Moreover, in both the regular and singular problems we have

$$T_0^*(M_p) = T(M_p^+), \quad T^*(M_p) = T_0(M_p^+), \quad p = 1, \dots, N, \tag{2.24}$$

see [**20**, Section 5] in the case when $M_p = M_p^+$, and compare with the treatment in [**2**, Section 3] in the general case.

In the case of two singular endpoints, the problem on (a_p, b_p) is effectively reduced to the problems with one singular endpoint on the intervals $(a_p, c_p]$ and $[c_p, b_p)$, where $c_p \in (a_p, b_p)$. We denote by $T(M_p; a_p)$, $T(M_p; b_p)$ the maximal operators with domains $D(M_p; a_p)$ and $D(M_p; b_p)$, and denote by $T_0(M_p; a_p)$ and $T_0(M_p; b_p)$ the closures of the operators $T_0'(M; a)$ and $T_0'(M_p; b_p)$ defined in (2.22) on the intervals $(a_p, c_p]$ and $[c_p, b_p)$, respectively, see [**5, 11, 14, 15** and **19**].

Let $\tilde{T}_0'(M_p)$ be the orthogonal sum,

$$\tilde{T}_0'(M_p) = T_0'(M_p; a_p) \oplus T_0'(M_p; b_p)$$

in

$$L^2_{w_p}(a_p, b_p) = L^2_{w_p}(a_p, c_p) \oplus L^2_{w_p}(c_p, b_p),$$

$\tilde{T}_0'(M_p)$ is densely-defined and closable in $L^2_{w_p}(a_p, b_p)$ and its closure is given by

$$\tilde{T}_0(M_p) = T_0(M_p; a_p) \oplus T_0(M_p; b_p), \quad p = 1, \dots, N.$$

Also

$$\begin{aligned} \operatorname{nul}[\tilde{T}_0(M_p) - \lambda I] &= \operatorname{nul}[T_0(M_p; a_p) - \lambda I] \\ &\quad + \operatorname{nul}[T_0(M_p; b_p) - \lambda I], \\ \operatorname{def}[\tilde{T}_0(M_0) - \lambda I] &= \operatorname{def}[T_0(M_p; a_p) - \lambda I] \\ &\quad + \operatorname{def}[T_0(M_p; b_p) - \lambda I], \end{aligned}$$

and $R[\tilde{T}_0(M_p) - \lambda I]$ is closed if and only if $R[T_0(M_p; a_p) - \lambda I]$ and $R[T_0(M_p; b_p) - \lambda I]$ are both closed. These results imply in particular that

$$\Pi[\tilde{T}_0(M_p)] = \Pi[T_0(M_p; a_p)] \cap \Pi[T_0(M_p; b_p)], \quad p = 1, \dots, N.$$

We refer to [**2**, Section 3.10.4], [**7**] and [**11**] for more details.

Remark 2.1. If $S_p^{a_p}$ is a regularly solvable extension of $T_0(M_p; a_p)$ and $S_p^{b_p}$ is a regularly solvable extension of $T_0(M_p; b_p)$, then $S_p = S_p^{a_p} \oplus S_p^{b_p}$

is a regularly solvable extension of $\tilde{T}_0(M_p)$. We refer to [**2**, Section 3.10.4], [**7**] and [**11**] for more details.

Next we state the following results; the proof is similar to that in [**2**, Section 3.10.4], [**11**] and [**15**].

Theorem 2.2. *$\tilde{T}_0(M_p) \subset T_0(M_p)$, $T(M_p) \subset T(M_p; a_p) \oplus T(M_p; b_p)$ and*

$$\dim\{D[T_0(M_p)]/D[\tilde{T}_0(M_p)]\} = n, \quad p = 1, \dots, N.$$

If $\lambda \in \Pi[\tilde{T}_0(M_p)] \cap \Delta_3[T_0(M_p) - \lambda I]$, then

$$\begin{aligned}\operatorname{ind}[T_0(M_p) - \lambda I] = n &- \operatorname{def}[T_0(M_p; a_p) - \lambda I] \\ &- \operatorname{def}[T_0(M_p; b_p) - \lambda I],\end{aligned}$$

and in particular, if $\lambda \in \Pi[T_0(M_p)]$,

$$\begin{aligned}\operatorname{def}[T_0(M_p) - \lambda I] = \operatorname{def}&[T_0(M_p; a_p) - \lambda I] \\ &+ \operatorname{def}[T_0(M_p; b_p) - \lambda I] - n.\end{aligned} \tag{2.25}$$

Remark 2.3. It can be shown that

$$\begin{aligned} D[\tilde{T}_0(M_p)] &= \{u : u \in D[T_0(M_p)] \text{ and } u^{[r-1]}(c_p) = 0, \\ &\qquad r = 1, \dots, n\}, \\ D[\tilde{T}_0(M_p^+)] &= \{v : v \in D[T_0(M_p^+)] \text{ and } v_+^{[r-1]}(c_p) = 0, \\ &\qquad r = 1, \dots, n\} \end{aligned} \tag{2.26}$$

$p = 1, \dots, N$, see [**2**, Section 3.10.4].

Lemma 2.4. *For $\lambda \in \Pi[T_0(M_p), T_0(M_p^+)]$, $p = 1, \dots, N$, $\operatorname{def}[T_0(M_p) - \lambda I] + \operatorname{def}[T_0(M_p^+) - \bar{\lambda} I]$ is constant and*

$$0 \le \operatorname{def}[T_0(M_p) - \lambda I] + \operatorname{def}[T_0(M_p^+) - \bar{\lambda} I] \le 2n.$$

In the problem with one singular endpoint,

$$n \le \operatorname{def}[T_0(M_p) - \lambda I] + \operatorname{def}[T_0(M_p^+) - \bar{\lambda} I] \le 2n \quad \text{for all } \lambda \in \Pi[T_0(M_p), T_0(M_p^+)].$$

In the regular problem,

$$\operatorname{def}[T_0(M_p)-\lambda I]+\operatorname{def}[T_0(M_p^+)-\bar{\lambda}I]=2n$$
$$\text{for all } \lambda\in\Pi[T_0(M_p),T_0(M_p^+)].$$

Proof. See [**4**], [**11**, Lemma 3.1].

Let H be the direct sum,

$$H=\bigoplus_{p=1}^{N}H_p=\bigoplus_{p=1}^{N}L^2_{w_p}(a_p,b_p). \tag{2.27}$$

The elements of H will be denoted by $f=\{f_1,\dots,f_N\}$ with $f_1\in H_1,\dots,f_N\in H_N$.

When $I_i\cap I_j=\varnothing$, $i\neq j$, $i,j=1,2,\dots,N$, the direct sum space $\oplus_{p=1}^{N}L^2_{w_p}(a_p,b_p)$ can be naturally identified with the space $L^2_w(\cup_{p=1}^{N}I_p)$, where $w=w_p$ on I_p, $p=1,\dots,N$. This remark is of particular significance when $\cup_{p=1}^{N}I_p$ may be taken as a single interval; see [**6**] and [**7**].

We now establish by [**2, 6**] and [**11**] some further notation.

$$\begin{aligned} D_0(M)&=\bigoplus_{p=1}^{N}D_0(M_p), & D(M)&=\bigoplus_{p=1}^{N}D(M_p),\\ D_0(M^+)&=\bigoplus_{p=1}^{N}D_0(M_p^+), & D(M^+)&=\bigoplus_{p=1}^{N}D(M_p^+), \end{aligned} \tag{2.28}$$

$$\begin{gathered} T_0(M)f=\{T_0(M_1)f_1,\dots,T_0(M_N)f_N\},\\ f_1\in D_0(M_1),\dots,f_N\in D_0(M_N)\\ T_0(M^+)g=\{T_0(M_1^+)g_1,\dots,T_0(M_N^+)g_N\},\\ g_1\in D_0(M_1^+),\dots,g_N\in D_0(M_N^+). \end{gathered} \tag{2.29}$$

Also,

$$\begin{gathered} T(M)f=\{T(M_1)f_1,\dots,T(M_N)f_N\},\\ f_1\in D(M_1),\dots,f_N\in D(M_N)\\ T(M^+)g=\{T(M_1^+)g_1,\dots,T(M_N^+)g_N\},\\ g_1\in D(M_1^+),\dots,g_N\in D(M_N^+) \end{gathered} \tag{2.30}$$

$$[f,g]=\sum_{p=1}^{N}\{[f_p,g_p]_p(b_p)-[f_p,g_p]_p(a_p)\}, \tag{2.31}$$
$$f\in D(M),\quad g\in D(M^+);$$

$$(f,g)=\sum_{p=1}^{N}(f_p,g_p)_p, \tag{2.32}$$

where $f=(f_1,\ldots,f_N)$, $g=(g_1,\ldots,g_N)$ and $(.\,,.)_p$ the inner-product defined in (2.19). Note that $T_0(M)$ is a closed densely-defined operator in H.

We summarize a few additional properties of $T_0(M)$ in the form of a lemma.

Lemma 2.5. *We have* (a)

$$[T_0(M)]^*=\bigoplus_{p=1}^{N}[T_0(M_p)]^*=\bigoplus_{p=1}^{N}T(M_p^+)$$
$$[T_0(M^+)]^*=\bigoplus_{p=1}^{N}[T_0(M_p^+)]^*=\bigoplus_{p=1}^{N}T(M_p).$$

In particular,

$$D[T_0(M)]^*=D[T(M^+)]=\bigoplus_{p=1}^{N}D[T(M_p^+)],$$
$$D[T_0(M^+)]^*=D[T(M)=\bigoplus_{p=1}^{N}D[T(M_p)].$$

(b)

$$\operatorname{nul}[T_0(M)-\lambda I]=\sum_{p=1}^{N}\operatorname{nul}[T_0(M_p)-\lambda I],$$
$$\operatorname{nul}[T_0(M^+)-\bar{\lambda} I]=\sum_{p=1}^{N}\operatorname{nul}[T_0(M_p^+)-\bar{\lambda} I].$$

(c) *The deficiency indices of* $T_0(M)$ *are given by*

$$\operatorname{def}[T_0(M) - \lambda I] = \sum_{p=1}^{N} \operatorname{def}[T_0(M_p) - \lambda I],$$
$$\textit{for all } \lambda \in \Pi[T_0(M)],$$

$$\operatorname{def}[T_0(M^+) - \bar{\lambda} I] = \sum_{p=1}^{N} \operatorname{def}\ [T_0(M_p^+) - \bar{\lambda} I],$$
$$\textit{for all } \bar{\lambda} \in \Pi[T_0(M^+)].$$

Proof. Part (a) follows immediately from the definition of $T_0(M)$ and from the general definition of an adjoint operator. The other parts are either direct consequences of part (a) or follow immediately from the definitions.

Lemma 2.6. *Let* $T_0(M) = \oplus_{p=1}^{N} T_0(M_p)$ *be a closed densely-defined operator on* H. *Then*

$$\Pi[T_0(M)] = \bigcap_{p=1}^{N} \Pi[T_0(M_p)]. \tag{2.33}$$

Proof. The proof follows from Lemma 2.4 and since $R[T_0(M) - \lambda I]$ is closed, if and only if $R[T_0(M_p) - \lambda I]$, $p = 1, \ldots, N)$ are closed.

Lemma 2.7. *If* S_p, $p = 1, \ldots, N$, *are regularly solvable with respect to* $T_0(M_p)$ *and* $T_0(M_p^+)$, *then*

$$S = \bigoplus_{p=1}^{N} S_p,$$

is regularly solvable with respect to $T_0(M)$ *and* $T_0(M^+)$.

Proof. The proof follows from Lemmas 2.4 and 2.5.

Remark 2.8. Let $S = \oplus_{i=1}^{N} S_i$ be an arbitrary closed operator on H and, since $\lambda \in \rho(S)$, if and only if $\text{nul}\,(S - \lambda I) = \text{def}\,(S - \lambda I) = 0$, see [**2**, Theorem 1.3.2], we have $\rho(S) = \cap_{i=1}^{N} \rho(S_i)$, and hence

$$\begin{aligned} &\sigma(S) = \bigcap_{i=1}^{N} \sigma(S_i), \qquad \sigma_p(S) = \bigcap_{i=1}^{N} \sigma_p(S_i) \quad \text{and} \\ &\sigma_r(S) = \bigcap_{i=1}^{N} \sigma_r(S_i). \end{aligned} \tag{2.34}$$

Also,

$$\sigma_{ek}(S) = \bigcup_{p=1}^{N} \sigma_{ek}(S_p), \quad k = 2, 3. \tag{2.35}$$

We refer to [**2**, Chapter 9] for more details.

Lemma 2.9. *Letting $T_0(M) = \oplus_{i=1}^{N} T_0(M_i)$ and $T_0(M^+) = \oplus_{i=1}^{N} T_0(M_i^+)$, then the point spectra $\sigma_p[T_0(M)]$ and $\sigma_p[T_0(M^+)]$ of $T_0(M)$ and $T_0(M^+)$ are empty.*

Proof. From [**12**, Theorem 4.1], we have $\sigma_p[T_0(M_i)] = \varnothing$ and $\sigma_p[T_0(M_i^+)] = \varnothing$, $i = 1, \dots, N$. Hence, by (2.34), $\sigma_p[T_0(M) = \cap_{i=1}^{N} \sigma_p[T_0(M_i)] = \varnothing$, and $\sigma_p[T_0(M^+) = \cap_{i=1}^{N} \sigma_p[T_0(M_i^+)] = \varnothing$.

3. Some technical lemmas. Let $\phi_k(t, \lambda)$ for $k = 1, \dots, n$ be the solutions of the homogeneous equation $M[u] - \lambda w u = 0$ satisfying

$$\begin{gathered} \phi_j^{[k-1]}(t_0, \lambda) = \delta_{jk}, \quad j, k = 1, \dots, n \\ \text{for fixed } t_0, \quad a < t_0 < b. \end{gathered}$$

Then $\phi_j^{[k-1]}(t, \lambda)$ is continuous in (t, λ) for $a < t < b$, $|\lambda| < \infty$, and for fixed t it is entire in λ. Let $\phi_k^+(t, \lambda)$ for $k = 1, \dots, n$ be the solutions of the homogeneous equation (2.18) satisfying

$$\begin{gathered} (\phi_k^+)^{[r]}(t_0, \lambda) = (-1)^{k+r} \delta_{k, n-r}, \\ \text{for fixed } t_0 \in [a, b), \end{gathered}$$

$k = 1, \dots, n$, $r = 1, \dots, n-1$.

Suppose $a < c < b$. According to Gilbert in [**9**, Section 3] and Ibrahim in [**13**, Section 4], a solution of $M[u] - \lambda w u = wf$, $f \in L^1_w(a,b)$ satisfying $\phi^{[r]}(c) = 0$, $r = 0, \dots, n-1$, is given by

$$(3.1)\quad \phi(t) = ((\lambda - \lambda_0)/(i^n)) \sum_{j,k=1}^{n} \xi^{jk} \phi_j(t,\lambda) \int_c^t \overline{\phi_k^+(s,\lambda)} f(s) w(s)\, ds,$$

where $\phi_k^+(t,\lambda)$ stands for the complex conjugate of $\phi_k(t,\lambda)$ and for each j and k, ξ^{jk} is a constant which is independent of t and λ (but does depend in general on t_0).

The variation of parameters formula for general ordinary quasi-differential equations is given by the following lemma:

Lemma 3.1. *For f locally integrable, the solution $\phi(t,\lambda)$ of the quasi-differential equation $M[u] - \lambda w u = wf$ satisfying*

$$\phi^{[r]}(t_0,\lambda) = \alpha_{r+1}(\lambda)$$

for all $r = 0, 1, \dots, n-1$, $t_0 \in [a,b)$ is given by

$$(3.2)\quad \begin{aligned} \phi(t,\lambda) = \sum_{j=1}^{n} \alpha_j(\lambda)\phi_j(t,\lambda_0) + ((\lambda - l_0)/(i^n)) \\ \cdot \Bigg(\sum_{j,k=1}^{n} \xi^{jk} \phi_j(t,\lambda_0) \int_a^t \overline{\phi_k^+(s,\lambda_0)} f(s) w(s)\, ds \Bigg), \end{aligned}$$

for some constants $\alpha_1(\lambda) \cdots \alpha_n(\lambda) \in \mathbf{C}$.

Proof. See [**4, 10, 13, 15**] and [**20**].

Lemma 3.1 contains the following lemma as a special case.

Lemma 3.2. *Suppose f is a locally $L^1_w(a,b)$ function and $\phi(t,\lambda)$ is the solution of $M[u] - \lambda w u = wf$ satisfying*

$$\phi^{[r]}(t_0,\lambda) = \alpha_{r+1}(\lambda)$$

for all $r = 0, 1, \dots, n-1$, $t_0 \in [a, b)$. *Then*

$$\begin{aligned}\phi^{[r]}(t,\lambda) = \sum_{j=1}^{n} \alpha_j(\lambda)\phi_j^{[r]}(t,\lambda_0) + ((\lambda-\lambda_0)/(i^n))\\ \cdot\left(\sum_{j,k=1}^{n} \xi^{jk}\phi_j^{[r]}(t,\lambda_0)\int_a^t \overline{\phi_k^+(s,\lambda_0)}f(s)w(s)\,ds\right),\end{aligned} \tag{3.3}$$

for $r = 0, 1, \dots, n-1$, *see* [**21**].

Lemma 3.3 [**10**, Proposition 3.24]. *Suppose that, for some* $\lambda_0 \in \mathbf{C}$, *all solutions of*

$$M[\phi] - \lambda_0 w\phi = 0 \quad \textit{and} \quad M^+[\phi^+] - \bar{\lambda}_0 w\phi^+ = 0$$

are in $L^2_w(a, b)$. *Then, all solutions of*

$$M[\phi] - \lambda w\phi = 0 \quad \textit{and} \quad M^+[\phi^+] - \bar{\lambda} w\phi^+ = 0$$

are in $L^2_w(a, b)$ *for every complex number* $\lambda \in \mathbf{C}$.

Lemma 3.4. *Suppose that, for some complex number* $\lambda_0 \in \mathbf{C}$, *all solutions of the equations*

$$M[\phi] - \lambda_0 w\phi = 0 \quad \textit{and} \quad M^+[\phi^+] - \bar{\lambda}_0 w\phi^+ = 0, \tag{3.4}$$

are in $L^2_w(a, b)$. *Suppose* $f \in L^2_w(a, b)$. *Then all solutions of the equation* $M[\phi] - \lambda w\phi = wf$ *are in* $L^2_w(a, b)$ *for all* $\lambda \in \mathbf{C}$.

Proof. Let $\{\phi_1(.,\lambda_0), \dots, \phi_n(.,\lambda_0)\}$ and $\{\phi_1^+(.,\lambda_0), \dots, \phi_n^+(.,\lambda_0)\}$ be two sets of linearly independent solutions of the equations in (3.4). Then, for any solution $\phi(t,\lambda)$ of $M[\phi] - \lambda w\phi = wf$ which may be written as follows $M[\phi] - \lambda_0 w\phi = (\lambda - \lambda_0)w\phi + wf$, it follows from (3.2) that

$$\begin{aligned}\phi(t,\lambda) &= \sum_{j=1}^{n} \alpha_j(\lambda)\phi_j(t,\lambda_0)\\ &\quad + \frac{1}{i^n}\left(\sum_{j,k=1}^{n} \xi^{jk}\phi_j(t,\lambda_0)\right.\\ &\quad \left.\cdot\int_a^t \overline{\phi_k^+(s,\lambda_0)}[(\lambda-\lambda_0)\phi(s,\lambda) + f(s)]w(s)\,ds\right).\end{aligned} \tag{3.5}$$

Hence,

$$
\begin{aligned}
|\phi(t,\lambda)| \le \sum_{j=1}^{n} |\alpha_j(\lambda)|\,|\phi_j(t,\lambda_0)| + \sum_{j,k=1}^{n} |\xi^{jk}|\,|\phi_j(t,\lambda_0)| \\
\cdot \int_a^t |\overline{\phi_k^{+}(s,\lambda_0)}|(|\lambda-\lambda_0|\,|\phi(s,\lambda)| + |f(s)|)w(s)\,ds.
\end{aligned} \tag{3.6}
$$

Since $f \in L_w^2(a,b)$ and $\phi_k^{+}(t,\lambda_0) \in L_w^2(a,b)$ for some $\lambda_0 \in \mathbf{C}$, $k = 1,\dots,n$, then $\phi_k^{+}(t,\lambda_0)f \in L_w^1(a,b)$ for some $\lambda_0 \in \mathbf{C}$ and $k = 1,\dots,n$. Setting

$$
\begin{aligned}
C_k(\lambda) = \int_a^b \overline{\phi_k^{+}(s,\lambda_0)} f(s,\lambda) w(s)\,ds, \\
k =, 1 \dots, n,
\end{aligned} \tag{3.7}
$$

then

$$
\begin{aligned}
|\phi(t,\lambda)| \le \sum_{j,k=1}^{n} (|\alpha_j(\lambda)| + C_k(\lambda)|\xi^{jk}|)|\phi_j(t,\lambda_0)| \\
+ |\lambda-\lambda_0| \sum_{j,k=1}^{n} |\zeta^{jk}| \\
\cdot \left(|\phi_j(t,\lambda_0)| \int_a^t |\overline{\phi_k^{+}(s,\lambda_0)}|\,|\phi(s,\lambda)|\right) w(s)\,ds.
\end{aligned} \tag{3.8}
$$

On application of the Cauchy-Schwartz inequality to the integral in (3.8), we get

$$
\begin{aligned}
|\phi(t,\lambda)| \le \sum_{j,k=1}^{n} (|\alpha_j(\lambda)| + C_k(\lambda)|\xi^{jk}|)\phi_j(t,\lambda_0) \\
+ |\lambda-\lambda_0| \sum_{j,k=1}^{n} |\xi^{jk}| \\
\cdot |\phi_j(t,\lambda_0)| \left(\int_a^t |\overline{\phi_k^{+}(s,\lambda_0)}|^2 w(s)\,ds\right)^{1/2} \\
\cdot \left(\int_a^t |\phi(s,\lambda)|^2 w(s)\,ds\right)^{1/2}.
\end{aligned}
$$

From the inequality,

$$(u+v)^2 \leq 2(u^2+v^2), \tag{3.9}$$

it follows that

$$\begin{aligned}|\phi(t,\lambda)|^2 \leq 4 \sum_{j,k=1}^{n} (|\alpha_j(\lambda)|^2 + C_k^2(\lambda)|\xi^{jk}|^2) \\ \cdot |\phi_j(t,\lambda_0)|^2 + 4|\lambda-\lambda_0|^2 \sum_{j,k=1}^{n} |\xi^{jk}|^2 \\ \cdot |\phi_j(t,\lambda_0)|^2 \left(\int_a^t \overline{|\phi_k^+(s,\lambda_0)}|^2 w(s)\,ds \right) \\ \cdot \left(\int_a^t |\phi(s,\lambda)|^2 w(s)\,ds \right).\end{aligned}$$

By hypothesis there exist positive constants K_0 and K_1 such that

$$\|\phi_j(.,\lambda_0)\|_{L_w^2(a,b)} \leq K_0 \tag{3.10}$$

and

$$\|\phi_k^+(.,\lambda_0)\|_{L_w^2(a,b)} \leq K_1,$$

$j,k = 1,\dots,n$. Hence,

$$\begin{aligned}|\phi(t,\lambda)|^2 \leq 4 \sum_{j,k=1}^{n} (|\alpha_j(\lambda)|^2 + C_k^2(\lambda)|\xi^{jk}|^2)|\phi_j(t,\lambda_0)|^2 \\ + 4K_1^2|\lambda-\lambda_0|^2 \\ \cdot \sum_{j,k=1}^{n} |\xi^{jk}|^2 \left(|\phi_j(t,\lambda_0)|^2 \int_a^t |\phi(s,\lambda)|^2 w(s)\,ds \right).\end{aligned} \tag{3.11}$$

Integrating the inequality in (3.11) between a and t, we obtain

$$\begin{aligned}&\int_a^t |\phi(s,\lambda)|^2 w(s)\,ds \\ &\quad \leq K_2 + \left(4K_1^2|\lambda-\lambda_0|^2 \sum_{j,k=1}^{n} |\xi^{jk}|^2 \right) \\ &\qquad \cdot \int_a^t |\phi_j(s,\lambda_0)|^2 \left[\int_a^s |\phi(\tau,\lambda)|^2 w(\tau)\,d\tau \right] w(s)\,ds,\end{aligned}$$

where

$$K_2 = 4K_0^2 \sum_{j,k=1}^{n} \{|\alpha_j(\lambda)|^2 + C_k^2(\lambda)|\xi^{jk}|^2).$$

Now, on using Gronwall's inequality, it follows that

$$\int_a^t |\phi(s,\lambda)|^2 w(s)\, ds \\ \leq K_2 \exp\bigg(4K_1^2|\lambda-\lambda_0|^2 \sum_{j,k=1}^{n} |\xi^{jk}|^2 \int_a^t |\phi_j(s,\lambda_0)|^2 w(s)\, ds\bigg).$$

Since $\phi_j(.,\lambda_0) \in L_w^2(a,b)$ for some $\lambda_0 \in \mathbf{C}$ and for $j = 1,\dots,n$, then, $\phi(t,\lambda) \in L_w^2(a,b)$.

Remark. Lemma 3.4 also holds if the function f is bounded on $[a,b]$.

Lemma 3.5. *Let $f \in L_w^2(a,b)$. Suppose, for some $\lambda_0 \in \mathbf{C}$, that*

(i) *all solutions of $M^+[\phi] - \bar{\lambda} w\phi = 0$ are in $L_w^2(a,b)$,*

(ii) *$\phi_j^{[r]}(t,\lambda_0)$, $j = 1,\dots,n$, are bounded on $[a,b)$ for some $r = 0,1,\dots,n-1$. Then $\phi^{[r]}(t,\lambda) \in L_w^2(a,b)$ for any solution $\phi(t,\lambda)$ of the equation $M[\phi] - \lambda w\phi = wf$ for all $\lambda \in \mathbf{C}$.*

Proof. On using Lemma 3.2, the proof is similar to that in Lemma 3.4 and therefore omitted.

Lemma 3.6 [**10**, Proposition 3.23]. *Suppose that, for some complex number $\lambda \in \mathbf{C}$, all solutions of $M^+[v] - \bar{\lambda}_0 wv = 0$ are in $L_w^2(a,c)$, where $a < c < b$. Suppose $f \in L_w^2(a,b)$. Then*

$$\int_a^t \overline{\phi_j^+(s,\lambda)}\, w(s) f(s)\, ds, \quad j = 1,\dots,n,$$

is continuous in (t,λ) for $a < t < b$, for all λ.

4. The case of intervals with one singular end-point. We see from (2.24) that $T_0(M_p) \subset T(M_p) = [T_0^+(M_p^*)]$ and hence $T_0(M_p)$

and $T_0(M_p^+)$ form an adjoint pair of closed, densely-defined operators in $L_w^2(a_p, b_p)$. By Lemma 2.4, $\operatorname{def}(T_0, M_p) - \lambda I] + \operatorname{def}[T_0(M_p^+) - \bar{\lambda} I]$ is constant on the joint field of regularity $\Pi[T_0(M_p), T_0(M_p^+)]$, $p = 1, \ldots, N$, and we have that, for $\lambda \in \Pi[T_0(M_p), T_0(M_p^+)]$

$$n \leq \operatorname{def}[T_0(M_p) - \lambda I] + \operatorname{def}[T_0(M_p^+) - \bar{\lambda} I] \leq 2n. \tag{4.1}$$

For $\lambda \in \Pi[T_0(M), T_0(M^+)]$, we define r, s and m as follows.

$$\begin{aligned} r = r(\lambda) &= \operatorname{def}[T_0(M) - \lambda I] \\ &= \sum_{p=1}^{N} \operatorname{def}[T_0(M_p) - \lambda I] \\ &= \sum_{p=1}^{N} \operatorname{nul}[T(M_p^+) - \bar{\lambda} I] = \sum_{p=1}^{N} r_p \\ s = s(\lambda) &= \operatorname{def}[T_0(M^+) - \bar{\lambda} I] \\ &= \sum_{p=1}^{N} \operatorname{def}[T_0(M_p^+) - \bar{\lambda} I] \\ &= \sum_{p=1}^{N} \operatorname{nul}[T(M_p) - \lambda I] = \sum_{p=1}^{N} s_p \end{aligned} \tag{4.2}$$

and

$$m := r + s = \sum_{p=1}^{N} (r_p + s_p) = \sum_{p=1}^{N} m_p.$$

By Lemma 2.4, m is constant on $\Pi[T_0(M), T_0(M^+)]$ and

$$nN \leq m \leq 2nN. \tag{4.3}$$

For $\Pi[T_0(M), T_0(M^+)] \neq \varnothing$ the operators which are regularly solvable with respect to $T_0(M)$ and $T_0^+(M)$ are characterized by the following theorem.

Theorem 4.1. *For $\lambda \in \Pi[T_0(M), T_0(M^+)]$, let r and m be defined by (4.2) and let ψ_j, $j = 1, \dots, r$, ϕ_k, $k = r+1, \dots, m$, be arbitrary functions satisfying*

(i) $\{\psi_j : j = 1, \dots, r\} \subset D(M)$ *is linearly independent modulo $D_0(M)$ and $\{\phi_k : k = r+1, \dots, m\} \subset D(M^+)$ is linearly independent modulo $D_0(M^+)$.*

(ii)

$$([\psi_j, \phi_k])_a^b = \sum_{p=1}^{N} ([\psi_{jp}, \phi_{kp}](b_p) - [\psi_{jp}, \phi_{kp}](a_p)) = 0,$$
$$k = 1, \dots, r, \quad k = r+1, \dots, m.$$

Then the set

$$\text{(4.4)} \quad \Big\{ u : u \in D(M), ([u, \phi_k])_a^b = \sum_{p=1}^{N} [u_p, \phi_{kp}](b_p) - [u_p, \phi_{kp}](a_p) = 0, \quad k = r+1, \dots, m \Big\}$$

is the domain of an operator S which is regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ and

$$\text{(4.5)} \quad \Big\{ v : v \in D(M^+), ([\psi_j, v])_a^b = \sum_{p=1}^{N} [\psi_{jp}, v_p](b_p) - [\psi_{jp}, v_p](a_p) = 0, \quad j = 1, \dots, r \Big\}$$

is the domain of S^*, moreover, $\lambda \in \Delta_4(S)$.

Conversely, if S is regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ and $\lambda \in \Pi[T_0(M), T_0(M^+)] \cap \Delta_4(S)$, then with r and m defined by (4.2)*, there exist functions ψ_j, $j = 1, \dots, r$, ϕ_k, $k = r+1, \dots, m$, which satisfy* (i) *and* (ii) *and are such that* (4.4) *and* (4.5) *are the domains of S and S^*, respectively.*

S is self-adjoint if and only if $M = M^+$, $r = s$ and $\phi_k = \psi_{k-r}$, $k = r+1, \dots, m$; S is J-self-adjoint if and only if $M = JM^+J$, $r = s$ and $\phi_k = \bar{\psi}_{k-r}$, $k = r+1, \dots, m$.

Proof. The proof is entirely similar to that in [**4**, Theorem 3.2] and [**11**, Theorem 3.2] and is therefore omitted.

We shall now investigate in the case of the intervals I_p, $p = 1, \dots, N$, with one singular end-points that the resolvents which are direct sums of resolvents of all well-posed extensions of the minimal operators $T_0(M_p)$ and we show that, in the maximal case, i.e., when $r_p = s_p = n$, $p = 1, \dots, N$ in (4.2), these resolvents are integral operators; in fact, they are Hilbert-Schmidt integral operators by considering that the function f to be in $L^2_w(a, b)$, i.e., is quadratically integrable over the interval $[a, b)$.

The following theorem is an extension of that proved in Akhiezer and Glazman [**1**, Vol. 2] and in Naimark [**15**, Vol. 2], namely, the case of self-adjoint extensions of the minimal operator and the function f has compact support interior to the interval $[a, b)$ and also extends that proved in [**10**, Theorem 3.27] for the general case with compact support of the function f, to the case of finite number of intervals $[a_p, b_p)$, $p = 1, \dots, N$.

Theorem 4.2. *Suppose, for an operator $T_0(M)$ with one singular endpoint that* $\operatorname{def}[T_0(M) - \lambda I] = $ *def* $[T_0(M^+) - \bar{\lambda} I] = nN$ *for all* $\lambda \in \Pi[T_0(M), T_0(M^+)]$*, and let S be an arbitrary closed operator which is a well-posed extension of the minimal operator $T_0(M)$ and $\lambda \in \rho(S)$. Then the resolvents R_λ and R^*_λ of S and S^*, respectively, are Hilbert-Schmidt integral operators, i.e., for $\lambda \in \rho(S)$,*

$$(S - \lambda I)^{-1} f(x) = \int_a^b K(x, t, \lambda) w(t) f(t)\, dt, \qquad \text{a.e. } x \in [a, b), \tag{4.6}$$

$$(S^* - \bar{\lambda} I)^{-1} g(x) = \int_a^b K^+(t, x, \bar{\lambda}) w(x) g(x)\, dx, \qquad \text{a.e. } t \in [a, b), \tag{4.7}$$

where the kernels $K(x, t, \lambda) = \{K_1(x, t, \lambda), \dots, K_N(x, t, \lambda)\}$ and $K^+(t, x, \bar{\lambda}) = \{K_1^+(t, x, \bar{\lambda}), \dots, K_N^+(t, x, \bar{\lambda})\}$ are continuous functions on $[a, b) \times [a, b)$ and satisfy

$$K(x, t, \lambda) = \overline{K^+(t, x, \bar{\lambda})}, \quad \textit{for all } x, t \in [a, b),$$

and

$$\int_a^b \int_a^b |K(x,t,\lambda)|^2 w(x)w(t)\,dx\,dt < \infty.$$

Remark. An example of a closed operator which is well-posed with respect to a compatible adjoint pair is given by the Visik extension, see [**2**, Theorem 3.3.3] and [**18**, Theorem 1]. Note that if S is well-posed then $T_0(M)$ and $T_0(M^+)$ are a compatible adjoint pair and S is regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$.

Proof. Let

$$\begin{gathered}\operatorname{def}[T_0(M_p)-\lambda I] = \operatorname{def}[T_0(M_p^+)-\bar{\lambda} I] = n \\ \text{for all } \lambda \in \Pi[T_0(M_p), T_0(M_p^+)],\end{gathered}$$

then we choose a fundamental system of solutions $\{\phi_{1p}(t,\lambda),\dots,\phi_{np}(t,\lambda)\}$, $\{\psi_{1p}(t,\lambda),\dots,\psi_{np}(t,\lambda)\}$, $p=1,\dots,N$, of the equations

$$\begin{gathered}M_p[\phi_{jp}] - \lambda\phi_{jp}w = 0, \quad M_p^+[\psi_{jp}] - \bar{\lambda}\psi_{jp}w = 0, \\ j=1,\dots,n \quad \text{on} \quad [a_p,b_p),\end{gathered} \tag{4.8}$$

so that $\{\phi_{1p}(t,\lambda),\dots,\phi_{np}(t,\lambda)\}$ and $\{\psi_{1p}(t,\lambda),\dots,\psi_{np}(t,\lambda)\}$ belong to $L_w^2(a_p,b_p)$, i.e., they are quadratically integrable in the intervals $[a_p,b_p)$, $p=1,\dots,N$.

Let $R_{\lambda p} = (S_p-\lambda I)^{-1}$ be the resolvent of any well-posed extensions S_p of the minimal operator $T_0(M_p)$, $p=1,\dots,N$. For $f_p \in L_w^2(a_p,b_p)$, we put $\phi_p(t,\lambda) = R_{\lambda p} f_p$. Then $M_p[\phi_p] - \lambda w\phi_p = wf_p$, $p=1,\dots,N$, and consequently has a solution $\phi_p(t,\lambda)$, $p=1,\dots,N$, in the form

$$\begin{aligned}\phi_p(t,\lambda) = \sum_{j=1}^n \alpha_{jp}(\lambda)\phi_{jp}(t,\lambda_0) \\ + ((\lambda-\lambda_0)/i^n)\bigg(\sum_{j,k=1}^n \zeta_p^{jk}\phi_{jp}(t,\lambda_0) \\ \cdot \int_a^t \overline{\phi_{kp}^+(s,\lambda_0)} f_p(s)w(s)\,ds\bigg),\end{aligned} \tag{4.9}$$

for some constants $\alpha_{1p}(\lambda),\dots,\alpha_{np}(\lambda)\in\mathbf{C}$, see Lemma 3.1. Since $f_p\in L^2_w(a_p,b_p)$ and $\phi^+_{kp}(.,\lambda_0)\in L^2_w(a_p,b_p)$, $k=1,\dots,n$, for some $\lambda_0\in\mathbf{C}$, then $\phi^+_{kp}(.,\lambda_0)f_p\in L^1_w(a_p,b_p)$, $k=1,\dots,n$, for some $\lambda_0\in\mathbf{C}$, and hence the integral in the righthand side of (4.9) will be finite.

To determine the constants $\alpha_{jp}(\lambda)$, $j=1,\dots,n$, let $\psi^+_{kp}(t,\lambda)$, $k=1,\dots,n$, be a basis for $\{D(S^*_p)/D_0(M^+_p)\}$, then because $\phi_p(t,\lambda)\in D(S_p)\subset\rho(S_p)\subset\Delta_4(S_p)$, $p=1,\dots,N$, we have from Theorem 4.1 that

$$([\phi,\psi^+_k])^b_a=\sum_{p=1}^{N}([\phi_p,\psi^+_{kp}](b_p)-[\phi^+_p,\psi_{kp}](a_p))=0,\quad\text{on }(a,b),\tag{4.10}$$

$k=1,\dots,n$, and hence, from (4.9), (4.10) and using Lemma 3.2, we have

$$\begin{aligned}[][\phi_p,\psi^+_{kp}](b_p)=\Bigg(&\sum_{j=1}^{n}\alpha_{jp}(\lambda)+((\lambda-\lambda_0)/i^n)\sum_{j,k=1}^{n}\xi^{jk}_p\\&\cdot\int_{a_p}^{b_p}\overline{\phi^+_{kp}(s,\lambda_0)}f_p(s)w(s)\,ds\Bigg)[\phi_{jp},\psi^+_{kp}](b_p),\end{aligned}$$

$[\phi_p,\psi^+_{kp}](a_p)=\sum_{j=1}^{n}\alpha_{jp}(\lambda)[\phi^+_{jp},\psi_{kp}](a_p)$, $k=1,\dots,n$, $p=1,\dots,N$. By substituting these expressions into the conditions (4.10), we get

$$\begin{aligned}&\sum_{p=1}^{N}\Bigg(\sum_{j=1}^{n}\alpha_{jp}(\lambda)+((\lambda-\lambda_0)/i^n)\\&\qquad\cdot\sum_{j,k=1}^{n}\xi^{jk}_p\int_{a_p}^{b_p}\overline{\phi^+_{kp}(s,\lambda_0)}f_p(s)w(s)\,ds\Bigg)[\phi_{jp},\psi^+_{kp}](b_p)\\&\qquad\qquad=\sum_{p=1}^{N}\Bigg(\sum_{j=1}^{n}\alpha_{jp}(\lambda)[\phi_{jp},\psi^+_{kp}](a_p)\Bigg),\end{aligned}$$

$p = 1, \dots, N$. This implies the system

$$\sum_{p=1}^{N}\left(\sum_{j=1}^{n}\alpha_{jp}(\lambda)([\phi_{jp},\psi_{kp}^{+}]_{a_p}^{b_p}\right) = -((\lambda-\lambda_0)/i^n)\sum_{p=1}^{N}\left(\sum_{j,k=1}^{n}\xi_p^{jk}[\phi_{jp},\psi_{kp}^{+}](b) \cdot \int_{a_p}^{b_p}\overline{\phi_{kp}^{+}(s,\lambda)}f_p(s)w(s)\,ds\right), \tag{4.11}$$

in the variables $\alpha_{jp}(\lambda)$, $j = 1, \dots, n$. The determinant of this system does not vanish, see [**10**, Theorem 3.27] and [**15**]. If we solve the system (4.11), we obtain

$$\alpha_{jp}(\lambda) = ((\lambda-\lambda_0)/i^n)\left(\int_{a_p}^{b_p}h_{jp}(s,\lambda)f_p(s)w(s)\,ds\right), \quad j = 1,\dots,n, \qquad p = 1,\dots,N,$$

where $h_{jp}(s,\lambda)$, $p = 1, \dots, N$, is a solution of the system

$$\sum_{p=1}^{N}\left(\sum_{j=1}^{n}h_{jp}(s,\lambda)([\phi_{jp},\psi_{kp}^{+}]_{a_p}^{b_p}\right) = -\sum_{p=1}^{N}\left(\sum_{j,k=1}^{n}\xi_p^{jk}[\phi_{jp},\psi_{kp}^{+}](b_p)\overline{\phi_{kp}^{+}(s,\lambda_0)}\right). \tag{4.12}$$

Since the determinant of the above system (4.12) does not vanish, and the functions $\phi_{kp}^{+}(s,\lambda_0)$, $k = 1, \dots, n$, are continuous in the intervals $[a_p, b_p)$, $p = 1, \dots, N$, then the functions $h_{jp}(s,\lambda)$ are also continuous in these intervals. By substituting in formula (4.9) for the expressions α_{jp}, $j = 1, \dots, n$; $p = 1, \dots, N$, we get

$$\begin{aligned} R_\lambda f &= \phi(t,\lambda) \\ &= ((\lambda-\lambda_0)/i^n)\sum_{p=1}^{N}\left(\sum_{j,k=1}^{n}\phi_{jp}(t,\lambda_0)\right. \\ &\qquad \cdot\int_{a_p}^{t}[\xi_p^{jk}\overline{\phi_{kp}^{+}(s,\lambda_0)} + h_{jp}(s,\lambda)]f_p(s)w(s)\,ds \\ &\qquad \left. + \sum_{j=1}^{n}\phi_{jp}(t,\lambda_0)\int_{t}^{b_p}h_{jp}(s,\lambda)f_p(s)w(s)\,ds\right). \end{aligned} \tag{4.13}$$

Now we put
(4.14)

$$K(t,s,\lambda)=\begin{cases} ((\lambda-\lambda_0)/i^n)\sum_{p=1}^{N}\left(\sum_{j=1}^{n}\phi_{jp}(t,\lambda_0)h_{jp}(s,\lambda)\right) & \text{for } t<s,\\ ((\lambda-\lambda_0)/i^n)\sum_{p=1}^{N}\left(\sum_{j,k=1}^{n}\phi_{jp}(t,\lambda_0)\cdot(\xi_p^{jk}\overline{\phi_{jp}^{+}(s,\lambda_0)}+h_{jp}(s,\lambda))\right) & \text{for } t>s,\end{cases}$$

formula (4.13) then takes the form

$$R_\lambda f(t)=\int_a^b K(t,s,\lambda)f(s)w(s)\,ds \quad \text{for all } t\in[a,b), \tag{4.15}$$

i.e., R_λ is an integral operator with the kernel $K(t,s,\lambda)$ operating on the function $f\in L_w^2(a,b)$. Similarly, the solution $\phi_p^{+}(s,\lambda)$ of the equation $M_p^{+}[\psi_p]-\bar{\lambda}w\psi_p=wg_p$, $p=1,\dots,N$, has the form

$$\begin{aligned}\phi_p^{+}(s,\lambda)&=\sum_{j=1}^{n}\beta_{jp}(\lambda)\phi_{jp}^{+}(s,\lambda_0)+((\bar{\lambda}-\bar{\lambda}_0)/i^n)\\ &\quad\cdot\left(\sum_{j,k=1}^{n}\xi_p^{jk}\phi_{jp}^{+}(s,\lambda_0)\int_a^s\overline{\phi_{kp}(t,\lambda_0)}g_p(t)w(t)\,dt\right),\end{aligned} \tag{4.17}$$

where $\phi_{kp}(t,\lambda_0)$ and $\phi_{jp}^{+}(s,\lambda_0)$, $j,k=1,\dots,n$, $p=1,\dots,N$, are solutions of the equations in (4.8). The argument, as before, leads to

$$R_\lambda^{*}g=\int_a^b K^{+}(s,t,\bar{\lambda})g(t)w(t)\,dt \quad \text{for } g\in L_w^2(a,b), \tag{4.19}$$

i.e., R_λ^{*} is an integral operator with the kernel $K^{+}(s,t,\bar{\lambda})$ operating on the function $g\in L_w^2(a,b)$, where
(4.20)

$$K^{+}(s,t,\bar{\lambda})=\begin{cases} ((\bar{\lambda}-\bar{\lambda}_0)/i^n)\sum_{p=1}^{N}\left(\sum_{j=1}^{n}\phi_{jp}^{+}(s,\lambda_0)h_{jp}^{+}(t,\lambda)\right) & \text{for } s<t,\\ ((\bar{\lambda}-\bar{\lambda}_0)/i^n)\sum_{p=1}^{N}\left(\sum_{j,k=1}^{n}\phi_{jp}^{+}(s,\lambda_0)\cdot(\xi_p^{jk}\overline{\phi_{kp}(t,\lambda_0)}+h_{jp}^{+}(t,\lambda))\right) & \text{for } s>t,\end{cases}$$

and $h^+_{jp}(t,\lambda)$, $p=1,\dots,N$, is a solution of the system

$$\sum_{p=1}^{N}\left(\sum_{j=1}^{n}\overline{h^+_{jp}(t,\lambda)}([\psi_{kp},\phi^+_{jp}])_{a_p}^{b_p}\right) = -\sum_{p=1}^{N}\left(\sum_{j,k=1}^{n}\zeta_p^{jk}[\psi_{kp},\phi^+_{jp}](b_p)\phi_{kp}(t,\lambda_0)\right). \tag{4.21}$$

From definitions of R_λ and R^*_λ, it follows that

$$\begin{aligned}(R_\lambda,f,g) &= \int_a^b\left\{\int_a^b K(t,s,\lambda)f(s)w(s)\,ds\right\}\overline{g(t)}w(t)\,dt\\ &= \int_a^b\left\{\int_a^b K(t,s,\lambda)\overline{g(t)}w(t)\right\}f(s)w(s)\,ds\\ &= (f,R^*_\lambda g)\end{aligned} \tag{4.22}$$

for any continuous functions $f,g\in H$, and by construction, see (4.14) and (4.20), $K(t,s,\lambda)$ and $K^+(s,t,\bar\lambda)$ are continuous functions on $[a,b)\times[a,b)$, and (4.22) gives us

$$K(t,s,\lambda)=\overline{K^+(s,t,\bar\lambda)} \quad \text{for all } t,s\in[a,b)\times[a,b). \tag{4.23}$$

Since $\phi_j(t,\lambda)=\{\phi_{j1}(t,\lambda),\dots,\phi_{jN}(t,\lambda)\}$ and $\phi^+_k(s,\lambda)=\{\phi^+_{k1}(s,\lambda),\dots,\phi^+_{kN}(s,\lambda)\}$ are in $L^2_w(a,b)$ for $j,k=1,\dots,n$, and for fixed s, $K(t,s,\lambda)$ is a linear combination of $\phi_j(t,\lambda)$ while, for fixed t, $K^+(s,t,\bar\lambda)$ is a linear combination of $\phi^+_k(s,\lambda)$. Then we have

$$\int_a^b|K(t,s,\lambda)|^2w(t)\,dt<\infty,$$
$$\int_a^b|K^+(s,t,\bar\lambda)|^2w(s)\,ds<\infty,$$
$$a<s,\quad t<b$$

and (4.23) implies that

$$\int_a^b|K(t,s,\lambda)|^2w(s)\,ds=\int_a^b|K^+(s,t,\bar\lambda)|^2w(s)\,ds<\infty,$$
$$\int_a^b|K^+(s,t,\bar\lambda)|^2w(t)\,dt=\int_a^b|K(t,s,\lambda)|^2w(t)\,dt<\infty.$$

Now it is clear from (4.12) that the functions $h_j(s,\lambda) = \{h_{j1}(s,\lambda), \dots, h_{jN}(s,\lambda)\}$, $j = 1,\dots,n$, belong to $L_w^2(a,b)$, since $h_{jp}(s,\lambda)$ is a linear combination of the functions $\phi^+_{jp}(s,\lambda)$ which lie in $L_w^2(a_p,b_p)$ and hence $h_j(s,\lambda)$ belong to $L_w^2(a,b)$. Similarly, $h_j^+(t,\lambda)$ belong to $L_w^2(a,b)$. By the upper half of formulas (4.14) and (4.20), we have

$$\int_a^b w(t)\,dt \int_a^b |K(t,s,\lambda)|^2 w(s)\,ds < \infty,$$

for the inner integral exists and is a linear combination of products $\phi_{jp}(t,\lambda)\overline{\phi^+_{kp}(s,\lambda)}$, $j,k = 1,\dots,n$, $p = 1,\dots,N$, and these products are integrable because each of the factors belongs to $L_w^2(a_p,b_p)$. Then, by (4.23), and by the upper half of (4.14),

$$\int_a^b w(t)\,dt \int_a^t |K(t,s,\lambda)|^2 w(s)\,ds = \int_a^b w(t)\,dt \int_a^t |K^+(s,t,\bar{\lambda})|^2 w(s)\,ds < \infty.$$

Hence, we also have

$$\int_a^b \int_a^b |K(t,s,\lambda)|^2 w(t)w(s)\,dt\,ds < \infty,$$

and the theorem is completely proved for any well-posed extension.

Remark 4.3. It follows immediately from Theorem 4.2 that, if for an operator $T_0(M_p)$, $p = 1,\dots,N$, with one singular endpoint, $\operatorname{def}[T_0(M_p) - \lambda I] = \operatorname{def}[T_0(M_p^+) - \bar{\lambda}I] = n$ for all $\lambda \in \Pi[T_0(M_p), T_0(M_p^+)]$, and S_p is well-posed with respect to $T_0(M_p)$ and $T_0(M_p^+)$ with $\lambda \in \rho(S_p)$, then $R_{\lambda p} = (S_p - \lambda I)^{-1}$ is a Hilbert-Schmidt integral operator. Thus, it is a completely continuous operator, and, consequently, its spectrum is discrete and consists of isolated eigenvalues having finite algebraic (so geometric) multiplicity with zero as the only possible point of accumulation. Hence, the spectra of all well-posed operators S_p are discrete, i.e.,

$$\sigma_{ek}(S_p) = \varnothing, \quad p = 1,\dots,N \quad \text{for } k = 1,2,3,4,5, \tag{4.24}$$

and hence, by (2.35) and (4.24),

$$\sigma_{ek}(S) = \bigcup_{p=1}^{N} \sigma_{ek}(S_p) = \varnothing, \quad \text{for } k = 2, 3, \tag{4.25}$$

for any well-posed extension S of the minimal operator $T_0(M)$. We refer to [**2**, Theorem 9.3.1] for more details.

5. The case of intervals with two singular endpoints. For the case of two singular endpoints, we consider our interval to be $I = (a, b)$ and denote by $T_0(M)$ and $T(M)$ the minimal and maximal operators. We see from (2.24) and (2.28) that $T_0(M) \subset T(M) \subset [T_0(M^+)]^*$, and hence, $T_0(M)$ and $T_0(M^+)$ form an adjoint pair of closed densely-defined operators in $L^2_w(a, b)$.

From (2.25) and (4.2) we have that, for $\lambda \in \Pi[T_0(M), T_0(M^+)]$,

$$\begin{aligned} r = r(\lambda) &:= \operatorname{def}[T_0(M) - \lambda I] \\ &= \sum_{p=1}^{N} r_p = \sum_{p=1}^{N} (r_p^1 + r_p^2 - n) \\ s = s(\lambda) &:= \operatorname{def}[T_0(M^+) - \bar{\lambda} I] \\ &= \sum_{p=1}^{N} s_p = \sum_{p=1}^{N} (s_p^1 + s_p^2 - n) \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} m := r + s &= \sum_{p=1}^{N} \left\{ (r_p^1 + s_p^1) + (r_p^2 + s_p^2) - 2n \right\} \\ &= \sum_{p=1}^{N} (m_p^1 + m_p^2 - 2n). \end{aligned}$$

Since $n \leq m_p^i \leq 2n$, $i = 1, 2$, then $0 \leq m \leq 2nN$.

For an operator $T_0(M_p)$, $p = 1, \dots, N$, with two singular endpoints, Theorem 4.2 remains true in its entirety, that is, all well-posed extensions of the minimal operator $T_0(M_p)$ in the maximal case, i.e., when $r_p^1 = r_p^2 = n$ and $s_p^1 = s_p^2 = n$ in (5.1) have resolvents which are Hilbert-Schmidt integral operators and consequently have a wholly discrete spectrum and hence Remark 4.3 also remains valid. This implies as in Corollary 5.1 below that all the regularly solvable operators have standard essential spectra to be empty. We refer to [**1, 2, 15**] and [**18**] for more details. Now we prove Theorem 4.2 in the case of the intervals (a_p, b_p), $p = 1, \ldots, N$ with singular endpoints.

Proof. Let

$$\begin{gathered}\operatorname{def}[T_0(M_p) - \lambda I] = \operatorname{def}[T_0(M_p^+) - \bar{\lambda} I] = n \\ \text{for all } \lambda \in \Pi[T_0(M_p), T_0(M_p^+)],\end{gathered}$$

$p = 1, \ldots, N$, then we choose a fundamental system of solutions

$$\phi_{jp}(t,\lambda) = \begin{cases} \phi_{jp}^{a_p}(t,\lambda) \text{ on } (a_p, c_p] \\ \phi_{jp}^{b_p}(t,\lambda) \text{ on } [c_p, b_p) \end{cases}$$

and

$$\psi_{jp}(t,\lambda) = \begin{cases} \psi_{jp}^{a_p}(t,\lambda) \text{ on } (a_p, c_p] \\ \psi_{jp}^{b_p}(t,\lambda) \text{ on } [c_p, b_p) \end{cases}$$

$p = 1, \ldots, N$, of the equations

$$\begin{gathered} M_p[\phi_{jp}] - \lambda w \phi_{jp} = 0, \\ M_p^+[\psi_{jp}] - \bar{\lambda} w \psi_{jp} = 0, \\ j = 1, \ldots, n \quad \text{on} \quad (a_p, b_p), \end{gathered} \tag{5.2}$$

so that $\{\phi_{1p}(t,\lambda), \ldots, \phi_{np}(t,\lambda)\}$ and $\{\psi_{1p}(t,\lambda), \ldots, \psi_{np}(t,\lambda)\}$ belong to $L_w^2(a_p, b_p)$, i.e., they are quadratically integrable in the intervals (a_p, b_p), $p = 1, \ldots, N$.

Let $R_{\lambda p} = (S_p - \lambda I)^{-1}$ be the resolvent of any well-posed extension $S_p = S_p^{a_p} \oplus S_p^{b_p}$ of the minimal operator $T_0(M_p)$. For $f \in L_w^2(a_p, b_p)$, we put $\phi_p(t,\lambda) = R_{\lambda p} f(t)$, then $M_p[\phi_p] - \lambda w \phi_p = wf$, $p = 1, \ldots, N$, and hence as in (4.9), we have

$$(5.3)\quad R_{\lambda p}f(t) = \phi_p(t,\lambda) = \sum_{j=1}^{n} \alpha_{jp}(\lambda)\phi_{jp}(t_0,\lambda) + ((\lambda-\lambda_0)/i^n)\bigg(\sum_{j,k=1}^{n} \xi_p^{jk}\phi_{jp}(t_0,\lambda) \cdot \int_a^t \overline{\phi_{kp}^{+}(s,\lambda_0)} f(s)w(s)\,ds\bigg),$$

for some constants $\alpha_{1p}(\lambda)\cdots\alpha_{np}(\lambda) \in \mathbf{C}$, $p = 1,\dots,N$, where

$$\phi_p(t,\lambda) = \begin{cases} \phi_p^{a_p}(t,\lambda) \text{ on } (a_p, c_p] \\ \phi_p^{b_p}(t,\lambda) \text{ on } [c_p, b_p), \end{cases}$$

and

$$\alpha_{jp}(\lambda) = \begin{cases} \alpha_{jp}^{a_p}(t,\lambda) \text{ on } (a_p, c_p] \\ \alpha_{jp}^{b_p}(t,\lambda) \text{ on } [c_p, b_p), \end{cases}$$

$j = 1,\dots,n$, $p = 1,\dots,N$. By proceeding as in Theorem 4.2, we find that $\alpha_{jp}(\lambda) = ((\lambda-\lambda_0)/i^n)(\int_{a_p}^{b_p} h_{jp}(s,\lambda)f(s)w(s)\,ds)$, $j = 1,\dots,n$, $p = 1,\dots,N$, where $h_{jp}(t,\lambda)$ are continuous functions on the intervals (a_p, b_p),

$$h_{jp}(t,\lambda) = \begin{cases} h_{jp}^{a_p}(t,\lambda) \text{ on } (a_p, c_p] \\ h_{jp}^{b_p}(t,\lambda) \text{ on } [c_p, b_p), \end{cases} \quad j = 1,\dots,n, \quad p = 1,\dots,N.$$

By substituting in (5.3) for the constants $\alpha_{jp}(\lambda)$, $j = 1,\dots,n$, we get

$$R_\lambda f = \int_a^b K(t,s,\lambda)f(s)w(s)\,ds,$$

where

$$K(t,s,\lambda) = \begin{cases} K^a(t,s,\lambda) \text{ on } (a,c] \\ K^b(t,s,\lambda) \text{ on } [c,b), \end{cases}$$

and $K^{(\cdot)}(t,s,\lambda)$ can be obtained as in (4.14). Similarly,

$$R_\lambda^* g = \int_a^b K^+(s,t,\bar{\lambda}) g(s,\lambda) w(s)\, ds,$$
$$K^+(s,t,\bar{\lambda}) = \begin{cases} K^{+(a)}(s,t,\bar{\lambda}) \text{ on } (a,c], \\ K^{+(b)}(s,t,\bar{\lambda}) \text{ on } [c,b), \end{cases}$$

and $K^{+(\cdot)}(s,t,\bar{\lambda})$ can be obtained as in (4.20).

From (4.14) and (4.20), we have that

$$\int_a^b |K(t,s,\lambda)|^2 w(t)\, dt < \infty,$$
$$\int_a^b |K^+(s,t,\bar{\lambda})|^2 w(s)\, ds < \infty,$$
$$a < s, \quad t < b$$

and (4.23) implies that

$$\int_a^b |K(t,s,\lambda)|^2 w(s)\, ds = \int_a^b |K^+(s,t,\bar{\lambda})|^2 w(s)\, ds < \infty,$$
$$\int_a^b |K^+(s,t,\bar{\lambda})|^2 w(t)\, dt = \int_a^b |K(t,s,\lambda)|^2 w(t)\, dt < \infty.$$

The rest of the proof is entirely similar to the corresponding part of the proof of Theorem 4.2. We refer to [**1, 13**] and [**15**] for more details.

Corollary 5.1. *Let* $\lambda \in \Pi[T_0(M), T_0(M^+)]$ *with*

$$\operatorname{def}[T_0(M) - \lambda I] = \operatorname{def}[T_0(M^+) - \bar{\lambda}] = nN.$$

Then

$$\sigma_{ek}(S) = \varnothing, \quad k = 2, 3, \tag{5.4}$$

of all regularly solvable extensions S *with respect to the compatible adjoint pair* $T_0(M)$ *and* $T_0(M^+)$.

Proof. Since $\operatorname{def}[T_0(M_p) - \lambda I] = \operatorname{def}[T_0(M_p^+) - \bar{\lambda} I] = n$, for all $\lambda \in \Pi[T_0(M), T_0(M^+)]$, $p = 1, \dots, N$. Then we have, from [**2**, Theorem 3.3.5] that

$$\begin{gathered}\dim\{D(S_p)/D_0(M_p)\} = \operatorname{def}[T_0(M_p) - \lambda I] = n,\\ \dim\{D(S^*)/D_0(M_p^+)\} = \operatorname{def}[T_0(M_p^+) - \bar{\lambda} I] = n,\\ p = 1, \dots, N.\end{gathered}$$

Thus, S_p is an n-dimensional extension of $T_0(M_p)$ and so, by [**2**, Corollary 9.4.2],

$$\begin{gathered}\sigma_{ek}(S_p) = \sigma_{ek}[T_0(M_p)], \quad p = 1, \dots, N,\\ \text{for } k = 1, 2, 3.\end{gathered} \tag{5.5}$$

In particular, if S_p, $p = 1, \dots, N$, is well-posed (say the Visik extension), we get from (4.24) and (5.5) that

$$\begin{gathered}\sigma_{ek}[T_0(M_p)] = \varnothing, \quad p = 1, \dots, N\\ \text{for } k = 1, 2, 3.\end{gathered}$$

On applying (5.5) again to any of the regularity solvable operators S_p, $p = 1, \dots, N$, under consideration, we have that

$$\begin{gathered}\sigma_{ek}(S_p) = \varnothing, \quad p = 1, \dots, N\\ \text{for } k = 1, 2, 3.\end{gathered}$$

Hence, by (2.35)

$$\sigma_{ek}(S) = \bigcup_{p=1}^{N} \sigma_{ek}(S) = \varnothing, \quad \text{for } k = 2, 3.$$

Corollary 5.2. *If, for some $\lambda_0 \in \mathbf{C}$, there are n linearly independent solutions of $M_p[u] - \lambda_0 w u = 0$ and $M_p^+[v] - \bar{\lambda}_0 w v = 0$ in $L^2_w(a_p, b_p)$, $p = 1, \dots, N$, then $\lambda_0 \in \Pi[T_0(M_p), T_0(M_p^+)]$, and hence $\Pi[T_0(M), T_0(M^+)] = \mathbf{C}$ and $\sigma_{ek}[T_0(M), T_0(M^+)] = \varnothing$, $k = 2, 3$, where $\sigma_{ek}[T_0(M), T_0(M^+)]$ is the joint essential spectra of $T_0(M)$ and $T_0(M^+)$ defined as $\Pi[T_0(M), T_0(M^+)]$.*

Proof. Since all solutions of $M_p[u]-\lambda_0 wu=0$ and $M_p^+[v]-\bar{\lambda}_0 wv=0$ are in $L_w^2(a_p,b_p)$ for some $\lambda_0\in\mathbf{C}$, $p=1,\ldots,N$, then

$$\operatorname{def}[T_0(M_p)-\lambda_0 I]+\operatorname{def}[T_0(M_p^+)-\bar{\lambda}_0 I]=2n \quad \text{for some } \lambda_0\in\Pi[T_0(M_p),T_0(M^+)].$$

From Lemma 2.9, we have that $T_0(M_p)$ has no eigenvalues and so $[T_0(M_p)-\lambda_0 I]^{-1}$ exists and its domain $R[T_0(M_p)-\lambda_0 I]$ is a closed subspace of $L_w^2(a_p,b_p)$. Hence, since $T_0(M_p)$ is a closed operator, then $[T_0(M_p)-\lambda_0 I]^{-1}$ is bounded and hence $\Pi[T_0(M_p)]=\mathbf{C}$. Similarly $\Pi[T_0(M_p^+)]=\mathbf{C}$, $p=1,\ldots,N$. From (2.33) we get

$$\Pi[T_0(M)]=\bigcap_{p=1}^{N}\Pi[T_0(M_p)]=\mathbf{C}$$

and

$$\Pi[T_0(M^+)]=\bigcap_{p=1}^{N}\Pi[T_0(M_p^+)]=\mathbf{C}.$$

Hence, $\Pi[T_0(M),T_0(M^+)]=\mathbf{C}$ and, from Lemma 2.5,

$$\operatorname{def}[T_0(M)-\lambda I]+\operatorname{def}[T_0(M^+)-\bar{\lambda} I]=2nN \quad \text{for all } \lambda\in\Pi[T_0(M),T_0(M^+)].$$

From Corollary 5.1 we have for any regularly solvable extension S of $T_0(M)$ that $\sigma_{ek}(S)=\varnothing$ for $k=2,3$ and by [**2**, Corollary 9.4.2], we get $\sigma_{ek}[T_0(M)]=\varnothing$, for $k=2,3$. Similarly, $\sigma_{ek}[T_0(M^+)]=\varnothing$, for $k=2,3$. Hence, $\sigma_{ek}[T_0(M),T_0(M^+)]=\varnothing$ for $k=2,3$.

Remark 5.3. If there are n linearly independent solutions of the equations $M[u]-\lambda wu=0$ and $M^+[v]-\bar{\lambda} wv=0$ in $L_w^2(a,b)$ for some $\lambda_0\in\mathbf{C}$, then the complex plane can be divided into two disjoint sets:

$$\mathbf{C}=\Pi[T_0(M),T_0(M^+)]\cup\sigma_{ek}[T_0(M),T_0(M^+)], \quad \text{for } k=2,3.$$

We refer to [**16**] and [**17**] for more details.

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