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MULTIPOINT PADÉ-TYPE APPROXIMATION: AN ALGEBRAIC APPROACH

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ABSTRACT. In this paper multipoint Padé-type approximants are formally introduced by defining a linear functional on the space of certain rational functions with prescribed poles. Some expressions for the numerator, a compact formula and some error formulas are also given.

1. Introduction. In [**1**], Brezinski introduced the so-called Padé-type approximants in one point, giving an algebraic development for such a rational approximation. The basis of this approach consisted in defining a linear functional on the space of usual polynomials. Following the same procedure, Draux [**3, 4**] and González-Vera [**5**] constructed two-point (zero and infinity) Padé-type approximants, starting from a linear functional defined on the space of Laurent-polynomials. In [**12**], see also [**13**], Van Iseghem extended the former cases to the multipoint one following a different technique, namely, the Hermite polynomial interpolation.

In this paper we introduce multipoint Padé-type approximation in a similar way to Brezinski, using an idea employed by Njåstad in [**9**] to construct multipoint Padé approximants. We shall define a linear functional acting on the space of R-functions [**8**], when none of the points where the approximant is to be built on is infinity, or on the space of generalized R-functions (GR-functions), [**10**], otherwise. Since R-functions can be considered as a particular case of GR-functions, we shall restrict ourselves to this general case.

In Section 2 we shall define multipoint Padé-type approximants (MPTA) and introduce the notation to be used in the paper, recalling also the definition of GR-functions. Section 3 is concerned with the construction of MPTAs, of type $(k-1, k)$, i.e., the numerator is of degree $k-1$ and the denominator of degree k at most, and using

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the functional introduced with this purpose. We also give expressions of the numerator which allow the computation of the approximants. Illustrative numerical examples are given. The results introduced in this section are the starting point for the construction of approximants with numerators and denominators of arbitrary degrees, developed in Section 4. Compact and error formulas analogous to the one-point case are obtained in the last section.

We shall use the term "formal" in the same sense as Brezinski in [**1**], i.e., when a power series appears in one of the sides of an identity, then the corresponding function represents either the sum of the series, if it converges, or its analytic continuation (if it exists), otherwise.

For an alternative approach to multipoint Padé-type approximation via interpolation and quadrature formulas, see [**7**].

2. Definitions and notations. Let us consider $p+1$ formal power series $L_1,\dots,L_p,L_{p+1}$ that we globally denote by $\mathbf{L}$, $\mathbf{L}\equiv(L_1,\dots,L_p,L_{p+1})$,

$$L_i(t)=\sum_{j=0}^{\infty}c_{i,j}(t-a_i)^j \quad (t\to a_i), \qquad i=1,2,\dots,p, \tag{2.1}$$

$$L_{p+1}(t)=\sum_{j=1}^{\infty}c_j t^{-j} \quad (t\to\infty), \tag{2.2}$$

where $a_1,\dots,a_p$ are distinct points of the complex plane $\mathbf{C}$. Then, given a polynomial $Q_k(t)$ of degree k such that $Q_k(a_i)\neq 0$ for $i=1,2,\dots,p$, we say that the rational function $P_{k-1}(t)/Q_k(t)$, with $P_{k-1}\in\Pi_{k-1}$ a polynomial to be determined, (Π_n denotes the space of polynomials of degree at most n) is a $(k-m/k)$ multipoint Padé-type approximant, $(k-m/k)$-MPTA, of order $(k_1,\dots,k_p;m)$, k_i's and m nonnegative integers such that $\sum_{i=1}^{p}k_i=k-m$, to $\mathbf{L}$, whenever

$$L_i(t)-\frac{P_{k-1}(t)}{Q_k(t)}=O((t-a_i)^{k_i}) \quad (t\to a_i), \qquad i=1,2,\dots,p \tag{2.3}$$

and

$$L_{p+1}(t) - \frac{P_{k-1}(t)}{Q_k(t)} = O((t^{-1})^{m+1}) \quad (t \to \infty). \tag{2.4}$$

We shall denote this rational function by

$$(k-m/k)_{\mathbf{L}(k_1,\dots,k_p;m)}(t), \quad (k-m/k)_{\mathbf{L}}(t)$$

or by $(k-m/k)(t)$, for short.

Conditions (2.3) and (2.4) allow us to pose a Hermite rational interpolation problem with prescribed poles which possess a unique solution under the conditions stated above, [**8, 11, 14**]; thus, we shall not be concerned with uniqueness in what follows.

Remark 1. The notation above is an extension of the two point (zero and infinity) situation [**5**]. Here $k-m$ is the order of correspondence at the points of the finite plane and k is the degree of the denominator. When $m=0$ (infinity is not considered), we shall use a notation as in the one point case [**1**], i.e.,

$$P_{k-1}/Q_k = (k-1/k)_{\mathbf{L}(k_1,\dots,k_p;0)}.$$

Here $k-1$ is the maximum degree of the numerator. □

Now we recall how the spaces of GR-functions are defined and set some notations to be used below. $\mathcal{G}$ will denote the space of *GR*-functions, that is, of all rational functions of the form

$$R(t) = \sum_{i=1}^{p}\sum_{j=1}^{k_i} \frac{\alpha_{i,j}}{(t-a_i)^j} + \sum_{j=0}^{m} \alpha_j (t-a)^j = \frac{P(t)}{Q(t)}, \qquad \alpha_j, \alpha_{i,j} \in \mathbf{C}, \tag{2.5}$$

where a is an arbitrary complex or real point, which we shall assume to be zero in the sequel. We denote by $\mathcal{G}(k_1,\dots,k_p;m)$ the subspaces of all *GR*-functions of the form (2.5) with $\delta P \le \sum_1^p k_i + m$. We say that $R \in \mathcal{G}(k_1,\dots,k_p;m)$ is degenerate if the degree of its numerator is less than $\sum_1^p k_i + m$. The subspace of all degenerate functions is denoted by

$\mathcal{G}^0(k_1,\dots,k_p;m)$. If we take $m=0$ in (2.5), the space $\mathcal{R}$ of R-functions results. In the notation of [**9**], $\mathcal{G}(k_1,\dots,k_p;0)=\mathcal{R}(k_1,\dots,k_p)$.

The sets

$$\mathcal{C}_1(k_1,\dots,k_p;m) = \left\{1, \frac{1}{(t-a_1)},\dots,\frac{1}{(t-a_p)}, t,\dots,\frac{1}{(t-a_1)^{k_1}},\dots,\frac{1}{(t-a_p)^{k_p}}, t^m\right\},$$

and

$$\mathcal{C}_2(k_1,\dots,k_p;m) = \left\{\frac{1}{B_{k-m}(t)}, \frac{t}{B_{k-m}(t)}, \frac{t^2}{B_{k-m}(t)},\dots,\frac{t^k}{B_{k-m}(t)}\right\},$$

with $B_{k-m}(x) = (x-a_1)^{k_1}\cdots(x-a_p)^{k_p}$ are obviously bases for $\mathcal{G}(k_1,\dots,k_p;m)$. The corresponding bases for the degenerate space will be denoted by $\mathcal{C}_1^0(k_1,\dots,k_p;m)$ and $\mathcal{C}_2^0(k_1,\dots,k_p;m)$, respectively.

3. Construction of the $(k-m/k)$-MPTA. Let Φ be a linear functional on the space of the GR-functions $\mathcal{G}$, defined by

$$\begin{aligned}\Phi(x^j) &= -c_{j+1}, \quad j=0,1,2,\dots \\ \Phi((x-a_i)^{-j}) &= c_{i,j-1}, \quad j=1,2,\dots;\ i=1,\dots,p.\end{aligned} \tag{3.1}$$

Consider the generating function $(x-t)^{-1}$. Writing

$$(x-t)^{-1} = ((x-a_i)-(t-a_i))^{-1} = \sum_{j=0}^{\infty}(x-a_i)^{-j-1}(t-a_i)^j, \quad i=1,\dots,p, \tag{3.2}$$

and

$$(x-t)^{-1} = -\sum_{j=0}^{\infty} x^j t^{-j-1} \tag{3.3}$$

it is easy to prove, at least formally, the following

Lemma 1 [**7**]. *If Φ acts on x and t is a parameter, then $\Phi((x-t)^{-1}) = L_i(t)$, $i=1,2,\dots,p+1$, as $(t\to a_i)$, $i=1,\dots,p$ and $(t\to\infty)$ for $i=p+1$.*

Now let $H_k(x) = Q_k(x)/B_{k-m}(x)$ be a given GR-function belonging to $\mathcal{G}(k_1, \dots, k_p; m)$, $Q_k(t)$ being precisely the denominator of the MPTA that we want to construct and, where B_{k-m} is as above, the polynomial

$$B_{k-m}(x) = (x - a_1)^{k_1} \cdots (x - a_p)^{k_p} \tag{3.4}$$

(recall that $\sum_{i=1}^{p} k_i = k - m$). The main result of this section is the following.

Theorem 1. *The* $(k - m/k)_{\mathbf{L}(k_1,\dots,k_p;m)}$ MPTA *is given by the rational function* $P_{k-1}(t)/Q_k(t) = -H^*_{k-1}(t)/H_k(t)$, *where*

$$H^*_{k-1}(t) = \Phi\left(\frac{H_k(x) - H_k(t)}{x - t}\right). \tag{3.5}$$

Proof. In order to apply the functional Φ in (3.5), let us prove that

$$\frac{H_k(x) - H_k(t)}{x - t} \tag{3.6}$$

belongs to $\mathcal{G}^0(k_1, \dots, k_p; m)$ both in x and in t. Put

$$S_{k-1}(x, t) = \frac{Q_k(x)B_{k-m}(t) - Q_k(t)B_{k-m}(x)}{x - t}. \tag{3.7}$$

The numerator of (3.7) is a polynomial in x (and in t) of degree at most k that vanishes for $x = t$ and therefore it is divisible by $(x - t)$. Then $S_{k-1}(x, t)$ is a polynomial in x (and in t) of degree at most $k - 1$. But

$$\frac{H_k(x) - H_k(t)}{x - t} = \frac{S_{k-1}(x, t)}{B_{k-m}(x)B_{k-m}(t)}, \tag{3.8}$$

and hence the lefthand side in (3.8) is a GR-function of x (and of t) in $\mathcal{G}^0(k_1, \dots, k_p; m)$.

Now applying the functional Φ to the GR-function (3.6), with x as the variable and t as a parameter, we get a new GR-function

$H^*_{k-1}(t) \in \mathcal{G}^0(k_1, \dots, k_p; m)$ defined by (3.5), associated with $H_k(x)$. Indeed, from (3.8) and (3.5),

$$H^*_{k-1}(t) = \frac{1}{B_{k-m}(t)}\Phi\left(\frac{S_{k-1}(x,t)}{B_{k-m}(x)}\right) = -\frac{P_{k-1}(t)}{B_{k-m}(t)}, \tag{3.9}$$

where $P_{k-1}(t) \in \Pi_{k-1}$. The meaning of the minus sign will be apparent below.

Now, from Lemma 1, (3.2) and (3.5), we can write

$$\begin{aligned} L_i(t) + \frac{H^*_{k-1}(t)}{H_k(t)} &= \Phi((x-t)^{-1}) + \frac{1}{H_k(t)}\Phi\left(\frac{H_k(x) - H_k(t)}{x-t}\right) \\ &= \Phi\left(\frac{H_k(t) + H_k(x) - H_k(t)}{H_k(t)(x-t)}\right) \\ &= \frac{1}{H_k(t)}\Phi\left(\frac{H_k(x)}{x-t}\right) \\ &= \frac{B_{k-m}(t)}{Q_k(t)}\Phi\left(\frac{Q_k(x)}{B_{k-m}(x)(x-t)}\right) \\ &= \frac{B_{k-m}(t)}{Q_k(t)}\sum_{j=0}^{\infty}\Phi\left(\frac{Q_k(x)}{B_{k-m}(x)(x-a_i)^{j+1}}\right)(t-a_i)^j \\ &= Q_k^{-1}(t)\prod_{\substack{s=1\\ s\neq i}}^{p}(t-a_s)^{k_s}\sum_{j=0}^{\infty}\Phi\left(\frac{Q_k(x)}{B_{k-m}(x)(x-a_i)^{j+1}}\right) \\ &\qquad \cdot (t-a_i)^{j+k_i} \\ &= O((t-a_i)^{k_i}), \quad i = 1, 2, \dots, p, \end{aligned}$$

where the latter equality is a consequence of the fact that $Q_k(a_i) \neq 0$ for $i = 1, 2, \dots, p$. Hence, conditions (2.3) are fulfilled. The remaining condition (2.4) can be proved using Lemma 1 and (3.3). Indeed, the following holds

$$\begin{aligned} L_{p+1}(t) - \frac{P_{k-1}(t)}{Q_k(t)} &= \Phi((x-t)^{-1}) + \frac{H^*_{k-1}(t)}{H_k(t)} \\ &= \Phi((x-t)^{-1}) + \frac{1}{H_k(t)}\Phi\left(\frac{H_k(x)-H_k(t)}{x-t}\right) \\ &= \frac{1}{H_k(t)}\Phi\left(\frac{H_k(x)}{x-t}\right) \\ &= \frac{B_{k-m}(t)}{Q_k(t)}\sum_{j=1}^{\infty}\Phi(H_k(x)x^{j-1})t^{-j} \\ &= O((t^{-1})^{m+1}), \end{aligned}$$

where the latter equality is due to the fact that $B_{k-m}(t)/Q_k(t) = O((t^{-1})^m)$. □

The result just obtained can now be used to compute the numerator of the approximant by giving two representations of the associated GR-function $H^*_{k-1}(t)$. The first one, which uses the coefficients of the partial fraction decomposition of $H^*_{k-1}(t)$, is the natural extension of that obtained in the one point case. The second representation supplies an algorithm to compute the coefficients of the polynomial $-P_{k-1}(t)$. Observe that these two sets of numbers are the components of the associated GR-function with respect to the algebraic bases $\mathcal{C}^0_1(k_1,\dots,k_p;m)$ and $\mathcal{C}^0_2(k_1,\dots,k_p;m)$, respectively.

Theorem 2. *The GR-function associated to $H_k(t)$ with respect to the functional Φ is given by*

$$H^*_{k-1}(t) = \sum_{s=0}^{m-1}\beta_s t^s + \sum_{i=1}^{p}\sum_{j=1}^{k_i}\frac{\beta_{i,j}}{(t-a_i)^j}, \tag{3.10}$$

where

$$\begin{aligned} \beta_s &= -\sum_{n=s+1}^{m}\alpha_n c_{n-s}, \quad s=0,1,\dots,m-1, \\ \beta_{i,j} &= -\sum_{n=j}^{k_i}\alpha_{i,n}c_{i,n-j}, \quad j=1,\dots,k_i;\ i=1,\dots,p, \end{aligned} \tag{3.11}$$

and

$$H_k(t) = \sum_{s=0}^{m} \alpha_s t^s + \sum_{i=1}^{p} \sum_{j=1}^{k_i} \frac{\alpha_{i,j}}{(t-a_i)^j}. \tag{3.12}$$

Proof. From (3.5) and (3.12), one has

$$H^*_{k-1}(t) = \sum_{s=1}^{m} \alpha_s \Phi\left(\frac{x^s - t^s}{x-t}\right) + \sum_{i=1}^{p}\sum_{j=1}^{k_i} \alpha_{i,j}\Phi\left(\frac{(x-a_i)^{-j} - (t-a_i)^{-j}}{x-t}\right).$$

Setting $u = (x-a_i)^{-1}$ and $v = (t-a_i)^{-1}$, or $u = x$ and $v = t$, the identity

$$\frac{u^j - v^j}{u-v} = u^{j-1} + u^{j-2}v + \cdots + uv^{j-2} + v^{j-1},$$

along with (3.1), allows us to write

$$\begin{aligned} H^*_{k-1}(t) &= \sum_{s=1}^{m} \alpha_s \Phi(x^{s-1} + x^{s-2}t + \cdots + xt^{s-2} + t^{s-1}) \\ &\quad - \sum_{i=1}^{p}\left(\sum_{j=1}^{k_i} \alpha_{i,j} \sum_{s=1}^{j} \Phi\left(\frac{1}{(x-a_i)^{j-s+1}}\right)\frac{1}{(t-a_i)^s}\right) \\ &= -\sum_{s=0}^{m-1}\left(\sum_{n=s+1}^{m} \alpha_n c_{n-s}\right) t^s \\ &\quad - \sum_{i=1}^{p}\left(\sum_{j=1}^{k_i}\left(\sum_{s=j}^{k_i} \alpha_{i,s} c_{i,s-j}\right)\frac{1}{(t-a_i)^j}\right). \end{aligned} \tag{3.13}$$

Since $H^*_{k-1}(t) \in \mathcal{G}^0(k_1, \dots, k_p; m)$, it can be expressed in the form (3.10). This fact and (3.13) easily yield (3.11). □

The coefficients for the numerator of the approximant can be obtained with the aid of the following lemma. Notation is as in (3.7) and (3.9).

Lemma 2. *The identity* $S_{k-1}(x,t) = s_0(x) + s_1(x)t + \cdots + s_{k-1}(x)t^{k-1}$ *holds, where*

(3.14)
$$s_{k-j}(x) = \sum_{i=0}^{k-1}\Bigg(\sum_{s=\max\{j-i-1,0\}}^{j-1}(q_{k-s}b_{i+s-j+1} - b_{k-s}q_{i+s-j+1})\Bigg)x^i, \qquad j = 1,2,\ldots,k,$$

$Q_k(x) = q_0 + \cdots + q_kx^k$ *and* $B_{k-m}(x) = b_0 + \cdots + b_{k-m}x^{k-m}$ *is the polynomial* (3.4). *If* $j > k-m$, *we take* $b_j = 0$ *in* (3.14).

Proof. Let $S_{k-1}(x,t) = T_k(x,t)/(t-x)$. Then

$$T_k(x,t) = Q_k(t)B_{k-m}(x) - Q_k(x)B_{k-m}(t) = \sum_{j=0}^{k} r_j(x)t^j,$$

where

$$r_j(x) = \sum_{i=0}^{k}(q_jb_i - q_ib_j)x^i, \quad j = 0,\ldots,k. \tag{3.15}$$

Since $T_k(x,t)$ is divisible by $(t-x)$, the Horner's algorithm furnishes the recursive formula

$$\begin{aligned} s_{k-1}(x) &= r_k(x)\\ s_{k-j}(x) &= r_{k-j+1}(x) + xs_{k-j+1}(x)\\ &\quad j = 2,3,\ldots,k. \end{aligned} \tag{3.16}$$

In order to establish (3.14), we proceed by induction on j. The result clearly holds for $j = 1$, since from (3.15) and (3.16) one has

$$s_{k-1}(x) = \sum_{i=0}^{k-1}(q_kb_i - q_ib_k)x^i.$$

Suppose now that (3.14) is true for $j = n-1 < k$. Again from (3.15) and (3.16), and by inductive hypotheses,

$$\begin{aligned} s_{k-n}(x) = &\left(\sum_{s=1}^{n-2}(q_{k-s}b_{k+s-n+1} - b_{k-s}q_{k+s-n+1})\right)x^k \\ &+ \sum_{i=1}^{k-1}\Bigg(q_{k-n+1}b_i - b_{k-n+1}q_i \\ &+ \sum_{s=\max\{n-i-1,0\}}^{n-2}(q_{k-s}b_{i+s-n+1} - b_{k-s}q_{i+s-n+1})\Bigg)x^i \\ &+ (q_{k-n+1}b_0 - b_{k-n+1}q_0). \end{aligned}$$

But

$$\sum_{s=1}^{n-2} b_{k-s}q_{k+s-n+1} = \sum_{s=1}^{n-2} q_{k-s}b_{k+s-n+1},$$

and therefore $s_{k-n}(x) \in \Pi_{k-1}$. The coefficients of x^i for $i = k-1, \dots, 1$, turn out to be

$$\begin{aligned} &(q_{k-n+1}b_i - b_{k-n+1}q_i) \\ &\qquad + \sum_{s=\max\{n-i-1,0\}}^{n-2}(q_{k-s}b_{i+s-n+1} - b_{k-s}q_{i+s-n+1}) \\ &\qquad = \sum_{s=\max\{n-i-1,0\}}^{n-1}(q_{k-s}b_{i+s-n+1} - b_{k-s}q_{i+s-n+1}), \end{aligned}$$

the constant term being $q_{k-n+1}b_0 - b_{k-n+1}q_0$. The proof is thus complete. □

The coefficients of $-P_{k-1}$ are now given by the following theorem.

Theorem 3. *The GR-function associated to $H_k(t)$ with respect to the functional Φ is given by*

$$H_{k-1}^*(t) = \sum_{j=0}^{k-1} p_j \frac{t^j}{B_{k-m}(t)},$$

where

(3.17)
$$p_{k-j} = \sum_{i=0}^{k-1} \left(\sum_{s=\max\{j-i-1,0\}}^{j-1} (b_{k-s} q_{i+s-j+1} - q_{k-s} b_{i+s-j+1} \right) \gamma_i$$
$$j = 1, 2, \dots, k,$$

(3.18)
$$\gamma_i = \Phi\left(\frac{x^i}{B_{k-m}(x)} \right) = - \sum_{j=0}^{i-k+m} \eta_j^{(i)} c_{j+1} + \sum_{s=1}^{p} \sum_{n=1}^{k_s} \gamma_{s,n}^{(i)} c_{s,n-1},$$
$$i = 0, 1, \dots, k-1.$$

The numbers $\eta_j^{(i)}$ *and* $\gamma_{s,n}^{(i)}$ *are the components of the elements of the base* $\mathcal{C}_2^0(k_1, \dots, k_p; m)$ *with respect to the base* $\mathcal{C}_1^0(k_1, \dots, k_p; m)$*, again,* $b_j = 0$ *if* $j > k$ *and* $\sum_i^j \equiv 0$ *if* $j < i$*, and*

$$H_k(t) = \sum_{j=0}^{k} q_j \frac{t^j}{B_{k-m}(t)}.$$

Proof. Write
$$P_{k-1}(t) = \sum_{j=0}^{k-1} p_j t^j.$$

From (3.9) and Lemma 2, we have

$$H_{k-1}^*(t) = \sum_{j=0}^{k-1} \Phi\left(\frac{s_j(x)}{B_{k-m}}(x) \right) \frac{t^j}{B_{k-m}(t)},$$

and therefore

$$p_j = -\Phi\left(\frac{s_j(x)}{B_{k-m}(x)} \right), \quad j = 0, \dots, k-1. \tag{3.19}$$

Formula (3.17) is a consequence of (3.14), (3.18) and (3.19). □

Remark 2. In short, for a given polynomial $Q_k(t) = \sum_{i=0}^{k} q_i t^i$ such that $Q_k(a_i) \neq 0$, for $i = 1, \ldots, p$, the corresponding multipoint Padé type approximant is

$$(k-m/k)(t) = -\sum_{j=0}^{k-1} p_j \frac{t^j}{Q_k(t)},$$

where the coefficients p_j are given by (3.17). □

Remark 3. For $p = 1$, $a_1 = 0$, $k_1 = k$ and $m = 0$, one-point Padé-type approximants, as defined by Brezinski [**1**] result. Indeed, taking

$$H_k(x) = \frac{\tilde{Q}_k(x)}{x^k} = Q_k(x^{-1}),$$

where $\tilde{Q}_k(x) = x^k Q_k(x^{-1})$, the Padé-type approximants with generating function $(1-zt)^{-1}$, here $z = 1/x$, and generating polynomial $Q_k(z)$, are obtained.

In a similar way, two-point Padé-type approximants, $m > 0$, result if the generating GR-function is now given by

$$H_k(x) = \frac{\tilde{Q}_k(x)}{x^{k-m}} = x^m Q_k(x^{-1}).$$

In other words, $H_k(x)$ is now a Laurent polynomial. □

To illustrate, next we give two numerical examples. Comparison with one point Padé-type approximation (1PTA) is provided. For a connection with quadrature formulas, see [**6**].

Example 1. Consider $f(t) = e^{-t}$. Taking its Taylor expansions $L_i(t)$ at several points a_i and the denominator

$$Q_k(t) = \left(1 + \frac{t}{k}\right)^k, \tag{3.20}$$

for which the 1PTA sequence $(k-1/k)$ at the origin converges to $f(t)$, see [**12**], we have the following numerical results. In Table 1, E_{1PTA}

TABLE 1.

t	E_{1PTA}	E_{2PTA}
-1.1	3.210^{-2}	5.210^{-2}
-0.9	1.010^{-2}	1.910^{-2}
-0.1	4.510^{-7}	1.410^{-5}
0.1	3.410^{-7}	5.910^{-6}
0.5	1.210^{-4}	1.210^{-5}
0.9	7.110^{-4}	3.810^{-7}
1.1	1.210^{-3}	1.310^{-6}
1.5	2.410^{-3}	9.710^{-5}
2.0	3.610^{-3}	7.510^{-4}
3.0	3.110^{-3}	5.310^{-3}

TABLE 2.

t	E_{1PTA}	E_{2PTA}	E_{3PTA}
-1.1	3.510^{-5}	2.210^{-7}	2.810^{-7}
-0.9	1.310^{-5}	8.810^{-8}	3.710^{-7}
-0.1	2.210^{-11}	2.310^{-8}	1.710^{-7}
0.1	2.010^{-11}	3.0110^{-8}	1.610^{-7}
0.5	2.610^{-7}	4.810^{-6}	2.010^{-6}
0.9	7.010^{-6}	2.610^{-5}	3.310^{-7}
1.1	2.110^{-5}	4.210^{-5}	5.410^{-7}
1.5	1.010^{-4}	6.010^{-5}	2.810^{-5}
2.0	4.210^{-4}	4.810^{-5}	2.110^{-4}
3.0	2.610^{-3}	1.510^{-3}	1.910^{-3}

are the errors for the (3/4)-1PTA at the origin, and E_{2PTA} are the errors for the (3/4)-2PTA with $a_1 = 0$, $a_2 = 1$ and $k_1 = k_2 = 2$, both with denominator $Q_4(t)$ given by (3.20). The data in Table 2 represent the errors for the (5/6)-1PTA at the origin, (5/6)-2PTA with $a_1 = 0$, $a_2 = -1$, $k_1 = k_2 = 3$, and (5/6)-3PTA with $a_1 = 0$, $a_2 = -1$, $a_3 = 1$

and $k_1 = k_2 = k_3 = 2$. Such approximants have the same denominator $Q_6(t)$, again given by (3.20).

Example 2. Now let $f(t)$ be the function $t^{-1}\log(1+t)$, and let its corresponding Taylor expansions near $t = 0$ and $t = 2$ be

$$L_1(t) = 1 - \frac{1}{2}t + \frac{1}{3}t^2 - \frac{1}{4}t^3 + \cdots \quad (t \to 0)$$

$$L_2(t) = \frac{1}{3}\log 2 + \left(\frac{1}{6} - \frac{1}{4}\log 3\right)(t-2) - \left(\frac{1}{9} - \frac{1}{8}\log 3\right)(t-2)^2 + \cdots \quad (t \to 2).$$

The (3/4)-1PTA at the origin to $f(t)$ with denominator $Q_4(t) = t^4 + 10t^3 + 35t^2 + 50t + 24$ yields better results even than that obtained by using the [1/2] one-point Padé approximant for the same function, see [**2**]. As before, in Table 3, we present the results for the (3/4)-1PTA and the (3/4)-2PTA, with $a_1 = 0$, $a_2 = 2$ and $k_1 = k_2 = 2$. In both approximants, the denominator has been taken to be $Q_4(t)$.

TABLE 3.

t	E_{1PTA}	E_{2PTA}
-0.8	5.910^{-2}	5.310^{-2}
-0.5	1.010^{-3}	8.610^{-3}
-0.1	2.210^{-7}	1.010^{-4}
0.1	8.710^{-8}	5.710^{-5}
0.6	9.210^{-7}	5.010^{-5}
1.1	1.410^{-4}	3.610^{-4}
1.5	4.710^{-4}	1.310^{-4}
1.9	1.010^{-3}	5.510^{-6}
2.1	1.310^{-3}	5.510^{-6}
3.0	2.010^{-3}	5.210^{-4}

4. Construction of approximants of arbitrary degrees. In view of the formal series (2.2), it is apparent that when MPTA including the point at infinity are considered, the rational approximants should have degrees $k-1$ and k in the numerator and denominator, respectively. In this section we shall be concerned with the construction of approximants of arbitrary degrees and, therefore, we shall only deal with approximants at points of the finite plane, i.e., with $m = 0$. Accordingly, the corresponding term will be omitted in the notations, see Remark 1. The definitions of Section 2 extend easily for (r/s)-MPTA in the following sense. Given the p formal power series $\mathbf{L}$ (2.1), and a polynomial $Q_s(t)$ of degree s with $Q_s(a_i) \neq 0$ for $i = 1, 2, \dots, p$, we say that the rational function $P_r(t)/Q_s(t)$ with $P_r \in \Pi_r$ is an (r/s) multipoint Padé-type approximant of order $(r_1, \dots, r_p)$ (r_i, nonnegative integers such that $\sum_{i=1}^{p} r_i = r+1$) to $\mathbf{L}$, if

$$L_i(t) - \frac{P_r(t)}{Q_s(t)} = O((t-a_i)^{r_i}) \quad (t \to a_i), \quad i = 1, 2, \dots, p.$$

In this section such approximants will be constructed following the procedure given in [**1**] for the one point case.

The procedure consists in obtaining a new series $\tilde{\mathbf{L}}$ by means of "shifting" the series $\mathbf{L}$, and calculating a multipoint Padé type approximant to those series, following the techniques of Section 3.

Of course, only one of the series $\mathbf{L}$ can be shifted; however, this requires the knowledge of more of its coefficients, so the necessary information will be distributed among several or all of the series. For simplicity, we shall assume that all the series are taken into account in this process.

4.1. $(k+n/k)$-MPTA. Let $k_1, \dots, k_p$ be nonnegative integers such that $\sum_{i=1}^{p} k_i = k+n+1$, and

$$m_i = k_i - n_i, \quad i = 1, 2, \dots, p, \tag{4.1}$$

where the nonnegative integers n_i are such that $\sum_{i=1}^{p} n_i = n+1$ and $m_i > 0$ for every i. Now let us consider the formal power series $L_i^{(m)}$, $i = 1, \dots, p$, $m = 1, 2, \dots$, constructed by the recursive formula

$$L_i^{(m)}(t) = \frac{L_i^{(m-1)}(t) - d_{s,0}^{(m-1)}}{t - a_s}, \quad N_{s-1} < m \leq N_s, \tag{4.2}$$
$$s = 1, 2, \dots, p, \quad i = 1, \dots, p,$$

with the convention that $L_i^{(0)}(t) = L_i(t)$ for $i = 1, \dots, p$, and where

$$N_0 = 0, \quad N_s = \sum_{i=1}^{s} n_i, \quad s = 1, 2, \dots, p. \tag{4.3}$$

These series are obtained by shifting n_1 times the series L_1, n_2 times the series $L_2^{(n_1)}$, n_3 times the series $L_3^{(n_1+n_2)} = L_3^{N_2}$, etc.

Writing

$$L_i^{(m)}(t) = \sum_{j=0}^{\infty} d_{i,j}^{(m)} (t - a_i)^j, \qquad i = 1, \dots, p, \quad m = 0, 1, 2, \dots, \tag{4.4}$$

for $m = 1, 2, \dots$, we have

$$\begin{cases} d_{s,j}^{(m)} = d_{s,j+1}^{(m-1)} & j = 0, 1, \dots, \\ d_{i,0}^{(m)} = ((d_{i,0}^{(m-1)} - d_{s,0}^{(m-1)})/(a_i - a_s)) & i \neq s, \\ d_{i,j}^{(m-1)} = (a_i - a_s) d_{i,j}^{(m)} + d_{i,j-1}^{(m)} & i \neq s;\ j = 1, 2, \dots . \end{cases} \tag{4.5}$$

Now, put

$$\tilde{L}_i(t) = L_i^{(n+1)}(t) = \frac{L_i(t) - W_n(t)}{\prod_{j=1}^{p} (t - a_j)^{n_j}}, \quad i = 1, \dots, p, \tag{4.6}$$

where

$$W_n(t) = \sum_{i=1}^{p} \left(\prod_{r=1}^{i-1} (t - a_i)^{n_r} \sum_{j=0}^{n_i - 1} d_{i,0}^{(N_{i-1}+j)} (t - a_i)^j \right), \tag{4.7}$$

is a polynomial of degree n. Here we assume that $\prod_n^m \equiv 1$ and $\sum_n^m \equiv 0$ if $m < n$. Consider the rational function

$$R(t) = W_n(t) + \prod_{j=1}^{p} (t - a_j)^{n_j} (k - 1/k)_{\tilde{\mathbf{L}}}(t), \tag{4.8}$$

where $(k-1/k)_{\tilde{\mathbf{L}}}(t)$ is the $(k-1/k)$-ATPM of order $(m_1, \ldots, m_p)$, m_i given by (4.1) to $\tilde{\mathbf{L}} = (\tilde{L}_1, \ldots, \tilde{L}_p)$ and generating R-function

$$H_k(x) = \frac{Q_k(x)}{B_k(x)},$$

where

$$B_k(x) = (x-a_1)^{m_1} \cdots (x-a_p)^{m_p}.$$

Then we have

Theorem 4. *The function $R(t)$, given by (4.8), is the $(k+n/k)$-MPTA of order $(k_1, \ldots, k_p)$ to the series $\mathbf{L}$, with $\sum_{i=1}^{p} k_i = k+n+1$ and $n = 0, 1, 2, \ldots$.*

Proof. From (4.1), (4.6), (4.8) and the definition of MPTA, for $i = 1, \ldots, p$, we obtain

$$\begin{aligned} L_i(t) - R(t) &= \prod_{j=1}^{p} (t-a_j)^{n_j} (\tilde{L}_i(t) - (k-1/k)_{\tilde{\mathbf{L}}}(t)) \\ &= (t-a_i)^{n_i} \prod_{\substack{j=1 \\ j\neq i}}^{p} (t-a_j)^{n_j} O((t-a_i)^{m_i}) \\ &= O((t-a_i)^{k_i}). \end{aligned}$$

This and the fact that the numerator of $R(t)$ is a polynomial of degree at most $k+n$, yield the desired result. □

Setting $(k-1/k)_{\tilde{\mathbf{L}}}(t) = P_{k-1}(t)/Q_k(t)$, and defining the linear functional

$$\Phi^{(n+1)}((x-a_i)^{-j}) = d_{i,j-1}^{(n+1)}, \quad i = 1, \ldots, p; \; j = 1, 2, \ldots \tag{4.9}$$

where $d_{i,j}^{(n+1)}$ are the coefficients of $\tilde{L}_i$, we can write for the associated R-function

$$H_{k-1}^{*}(t) = -\frac{P_{k-1}(t)}{B_k(t)} = \Phi^{(n+1)}\left(\frac{H_k(x) - H_k(t)}{x-t}\right).$$

Because of the uniqueness of the approximant, the same $(k+n/k)$-MPTA of order $(k_1,\dots,k_p)$ to $\mathbf{L}$ can be obtained for different choices of the numbers n_i, provided that $\sum_{i=1}^{p} n_i = n+1$ (or different m_i). Therefore, both the functional $\Phi^{(n+1)}$ and the R-functions H_k and H_{k-1}^* depend on this decomposition of $n+1$ (or of k).

Remark 4. From (4.5), it can be seen that the only coefficients of $\mathbf{L}$ used in the construction of $R(t)$ in Theorem 4 are $c_{i,j}$ with $j=0,1,\dots,k_i-1$, $i=1,\dots,p$. □

Remark 5. The polynomial $W_n(t)$ defined by (4.7) is the Hermite interpolation polynomial that solves the problem: "Find $P\in\Pi_n$ such that

$$\frac{d^jP(t)}{dt^j}\Big|_{t=a_i} = j!c_{i,j}, \quad j=0,1,\dots,n_i-1;\ i=1,\dots,p.\text{"}$$

Indeed, setting $\pi(t)=\prod_{j=1}^{p}(t-a_j)^{n_j}$, and taking into account that $R(t)$ is the $(k+n/k)$-MPTA of order $(k_1,\dots,k_p)$ to $\mathbf{L}$, we have

$$W_n(t)-L_i(t)+\pi(t)(k-1/k)_{\tilde{\mathbf{L}}}(t) = O((t-a_i)^{k_i}), \quad i=1,\dots,p.$$

But

$$(k-1/k)_{\tilde{\mathbf{L}}}(t) = \sum_{j=0}^{m_i-1} d_{i,j}^{(n+1)}(t-a_i)^j + O((t-a_i)^{m_i}), \quad i=1,\dots,p,$$

and consequently,

$$\begin{aligned} W_n(t)-L_i(t) &= -\pi(t)\sum_{j=0}^{m_i-1} d_{i,j}^{(n+1)}(t-a_i)^j \\ &\quad -\pi(t)O((t-a_i)^{m_i}) + O((t-a_i)^{k_i}) \\ &= O((t-a_i)^{n_i}) + O((t-a_i)^{m_i+n_i}) + O((t-a_i)^{k_i}) \\ &= O((t-a_i)^{n_i}), \quad i=1,\dots,p, \end{aligned}$$

because $m_i+n_i=k_i$ and $\pi(t)=O((t-a_i)^{n_i})$ for $i=1,\dots,p$. □

4.2. $(k/k+n)$-MPTA. Now we start the construction by setting $n-1=\sum_{i=1}^{p} n_i$, so that

$$m_i = k_i + n_i, \quad i = 1, \dots, p. \tag{4.10}$$

Next the following formal power series can be iteratively computed

$$L_i^{(m)}(t) = (t-a_s)L_i^{(m-1)}(t), \quad i = 1, 2, \dots, p,$$

with $L_i^{(0)} = L_i$, and where $s = r$ if $N_{r-1} < m \le N_r$, N_j being as in (4.3). In the notation of (4.4), the coefficients can be computed from the relations

$$\begin{cases} d_{s,0}^{(m)} = 0, d_{s,j}^{(m)} = d_{s,j-1}^{(m-1)} & j = 1, 2, \dots \\ d_{i,0}^{(m)} = (a_i - a_s)d_{i,0}^{(m-1)} & i \ne s \\ d_{i,j}^{(m)} = d_{i,j-1}^{(m-1)} + (a_i - a_s)d_{i,j}^{(m-1)} & i \ne s;\ j = 1, 2, \dots . \end{cases} \tag{4.11}$$

Take

$$\tilde{L}_i(t) = L_i^{(n-1)}(t) = \prod_{j=1}^{p}(t-a_j)^{n_j} L_i(t), \quad i = 1, \dots, p, \tag{4.12}$$

and consider the rational function

$$R(t) = \prod_{j=1}^{p}(t-a_j)^{-n_j}(k+n-1/k+n)_{\tilde{\mathbf{L}}}(t) \tag{4.13}$$

where $(k+n-1/k+n)_{\tilde{\mathbf{L}}}(t)$ is the $(k+n-1/k+n)$-MPTA of order $(m_1, \dots, m_p)$ to $\tilde{\mathbf{L}}$ and generating R-function $H_{k+n}(x) = Q_{k+n}(x)/B_{k+n}(x)$, with $B_{k+n}(x) = (x-a_1)^{m_1}\cdots(x-a_p)^{m_p}$, m_i given by (4.10). The $(k/k+n)$-MPTA is obtained from the uniqueness of the approximant and the following two results.

Theorem 5. *The function $R(t)$ given by* (4.13) *satisfies* $L_i(t) - R(t) = O((t-a_i)^{k_i})$, $i = 1, 2, \dots, p$.

Proof. From (4.10), (4.12), (4.13) and the definition of MPTA, for $i = 1, \dots, p$, we have

$$L_i(t) - R(t) = \prod_{j=1}^{p} (t - a_j)^{-n_j} (\tilde{L}_i(t) - (k + n - 1/k + n)_{\tilde{\mathbf{L}}}(t))$$
$$= (t - a_i)^{-n_i} \prod_{\substack{j=1 \\ j \neq i}}^{p} (t - a_j)^{-n_j} O((t - a_i)^{m_i})$$
$$= O((t - a_i)^{k_i}). \quad \square$$

Theorem 6. *Let the coefficients* $d_{i,j}^{(n-1)}$ *be as in* (4.4). *Then* $d_{i,j}^{(n-1)} = 0$, $j = 0, 1, \dots, n_i - 1$, $i = 1, \dots, p$.

Proof. By induction on k, we shall prove that

$$d_{i,j}^{(N_k)} = 0, \quad i = 1, 2, \dots, k; \; j = 0, 1, \dots, n_i - 1.$$

If $k = 1$, then $N_1 = n_1$. By construction and (4.11),

$$d_{1,0}^{(N_1)} = 0, \quad d_{1,j}^{(N_1)} = d_{1,0}^{(N_1 - j)} = 0 \quad \text{if } j \leq n_1 - 1.$$

Assume this holds for $k = m$, i.e.,

$$d_{i,j}^{(N_m)} = 0, \quad i = 1, \dots, m; \; j = 0, 1, \dots, n_i - 1. \tag{4.14}$$

Let us prove it for $k = m + 1$. If $i < m + 1$, then, by inductive hypotheses,

$$d_{i,0}^{(N_{m+1})} = (a_i - a_{m+1}) d_{i,0}^{(N_{m+1} - 1)} = (a_i - a_{m+1})^{n_{m+1}} d_{i,0}^{(N_m)} = 0,$$
$$i = 1, \dots, m.$$

On the other hand, again by (4.11),

$$d_{i,j}^{(N_{m+1})} = d_{i,j-1}^{(N_{m+1} - 1)} + (a_i - a_{m+1}) d_{i,j}^{(N_{m+1} - 1)}$$
$$= \sum_{s=0}^{\min\{j, n_{m+1}\}} \binom{n_{m+1}}{s} (a_i - a_{m+1})^{n_{m+1} - s} d_{i,j-s}^{(N_m)},$$
$$j = 0, 1, \dots .$$

This and (4.14) prove that $d_{i,j}^{(N_{m+1})} = 0$ for $j = 0, 1, \ldots, n_i - 1$ and $i = 1, \ldots, m$. By construction and (4.11), for $i = m + 1$ the following holds

$$d_{m+1,0}^{(N_{m+1})} = 0, \qquad d_{m+1,j}^{(N_{m+1})} = d_{m+1,0}^{(N_{m+1}-j)} = 0 \quad \text{if } j \leq n_{m+1} - 1. \qquad \square$$

The numerator of the approximant can also be obtained by defining the linear functional

$$\Phi^{(1-n)}((x - a_i)^{-j}) = d_{i,j-1}^{(n-1)}, \quad i = 1, \ldots, p;\ j = 1, 2, \ldots, \tag{4.15}$$

and by observing that the associated R-function to H_{k+n} can be determined by

$$H_{k+n-1}^*(t) = -\frac{P_{k+n-1}(t)}{B_{k+n}(t)} = \Phi^{(1-n)}\left(\frac{H_{k+n}(x) - H_{k+n}(t)}{x - t}\right).$$

Remark 6. The only coefficients in the series $\mathbf{L}$ required to carry out this process are $c_{i,j}$, with $j = 0, 1, \ldots, k_i - 1$, $i = 1, \ldots, p$. $\square$

In short, the multipoint Padé-type approximant (r/s) of order $(r_1, \ldots, r_p)$ to $\mathbf{L}$ can be written, in general (compare with [**1**, p. 12] for the one point case), as

$$(r/s)_{\mathbf{L}}(t) = W_{r-s}(t) + \prod_{j=1}^{p}(t - a_j)^{n_j}(s - 1/s)_{\tilde{\mathbf{L}}}(t), \tag{4.16}$$

provided that the formal series $\tilde{\mathbf{L}}$ are conveniently constructed and the approximant on the righthand side is of order $(m_1, \ldots, m_p)$. If $r \geq s$, we decompose $r - s + 1$ in a sum of nonnegative integers n_i, such that $r - s + 1 = \sum_{i=1}^{p} n_i$ and W_{r-s} is the polynomial of degree less than or equal to $r - s$, given by (4.7) with $n = r - s$. If $r < s$, we take $r - s + 1 = \sum_{i=1}^{p} n_i$, where n_i are nonpositive integers and $W_{r-s} \equiv 0$.

Formula (4.16) holds true also for $(r/0)$-MPTA if $(s - 1/s) \equiv 0$ when $s = 0$ since, in this case, W_{r-s} is the Hermite polynomial that solves the interpolation problem stated in Remark 5.

The generating R-function for the (r/s)-MPTA is of the form

$$H_s(x) = \frac{Q_s(t)}{(x-a_1)^{s_1}\cdots(x-a_p)^{s_p}},$$

with $s_i = r_i - n_i$, $\sum_{i=1}^p r_i = r+1$, $\sum_{i=1}^p n_i = r-s+1$, $n_i \geq 0$ for all i if $r \geq s$, $n_i \leq 0$ for all i if $r < s$. The functional $\Phi^{(r-s+1)}$ defined by (4.9) or (4.15), with $n = r-s$, allows us to write

$$H^*_{s-1}(t) = \Phi^{(r-s+1)}\left(\frac{H_s(x)-H_s(t)}{x-t}\right),$$

whence

$$(r/s)_{\mathbf{L}}(t) = W_{r-s}(t) + \prod_{j=1}^p (t-a_j)^{n_j}\frac{H^*_{s-1}(t)}{H_s(t)}. \qquad \square$$

5. Compact formula and error expressions. As in the one point case, it is possible to obtain a compact formula for representing the $(k-m/k)$-MPTA of order $(k_1,\dots,k_p;m)$ to the formal series $\mathbf{L}$. This is done in the following.

Theorem 7. *Let $\{r_n(x)\}_{n=0}^k \equiv \{q_n(x)/B_{k-m}(x)\}_{n=0}^k$ be $k+1$ GR-functions of $\mathcal{G}(k_1,\dots,k_p;m)$ with $k-m = \sum_1^p k_i$, $\delta q_n = n$, $n = 0,1,\dots,k$ and $B_{k-m}(x) = (x-a_1)^{k_1}\cdots(x-a_p)^{k_p}$. Let $\mathbf{V}$ be a square matrix of order k with terms*

$$v_{ij} = \Phi((x-t)r_{i-1}(x)r_{j-1}(x)), \quad i,j = 1,\dots,k;$$

let $\mathbf{u}$ and $\mathbf{w}$ be the vectors of components u_i and w_i, $i = 1,\dots,k$, respectively, where

$$w_i = \Phi\left(r_{i-1}(x)\left(1-\frac{H_k(x)}{H_k(t)}\right)\right), \quad i = 1,\dots,k$$

and

$$u_i = \Phi(r_{i-1}(x)), \quad i = 1,\dots,k,$$

respectively. Then

$$(k-m/k)_{\mathbf{L}}(t)=\langle \mathbf{u},\mathbf{V}^{-1}\mathbf{w}\rangle, \tag{5.1}$$

where $\langle\cdot,\cdot\rangle$ *denotes the scalar product of vectors and* $H_k=Q_k/B_{k-m}$, $Q_k(t)$ *is the denominator of the approximant.*

Proof. From (3.9) and Theorem 1, we can write

$$(k-m/k)_{\mathbf{L}}(t)=-\frac{1}{Q_k(t)}\Phi\left(\frac{S_{k-1}(x,t)}{B_{k-m}(x)}\right),$$

where $S_{k-1}(x,t)$ is a polynomial in x of degree at most $k-1$ and coefficients depending on t. Since $q_n(x)$ has degree precisely n, and since the set of rational functions $\{r_n(x)\}_{n=0}^{k-1}$ is a base for $\mathcal{G}^0(k_1,\dots,k_p;m)$, we can write

$$\frac{S_{k-1}(x,t)}{Q_k(t)B_{k-m}(x)}=\sum_{i=1}^{k}s_{j-1}(t)r_{j-1}(x),$$

whence

$$(k-m/k)_{\mathbf{L}}(t)=-\sum_{i=1}^{k}s_{j-1}(t)\Phi(r_{j-1}(x)). \tag{5.2}$$

On the other hand, from (3.8) and (3.9), for $i=1,\dots,k$, we have

$$\begin{aligned}\Phi\left(r_{i-1}(x)\left(1-\frac{H_k(x)}{H_k(t)}\right)\right)&=-\Phi\left(r_{i-1}(x)(x-t)\frac{S_{k-1}(x,t)}{Q_k(t)B_{k-m}(x)}\right)\\&=-\Phi\left(r_{i-1}(x)(x-t)\sum_{j=1}^{k}s_{j-1}(t)r_{j-1}(x)\right).\end{aligned}$$

Then

$$w_i=-\sum_{j=1}^{k}s_{j-1}(t)\Phi((x-t)r_{i-1}(x)r_{j-1}(x)),\quad i=1,\dots,k,$$

and hence,

$$\sum_{j=1}^{k} s_{j-1}(t)v_{i,j} = -w_i, \quad i = 1, \dots, k. \tag{5.3}$$

Let $\mathbf{s}$ denote the vector of components $s_j(t)$, $j = 0, \dots, k-1$. From (5.3) we have $\mathbf{s} = -\mathbf{V}^{-1}\mathbf{w}$, and from (5.2),

$$(k - m/k)_{\mathbf{L}}(t) = -\langle \mathbf{s}, \mathbf{u} \rangle.$$

This yields (5.1). □

By choosing the base $\mathcal{C}_2(k_1, \dots, k_p; m)$, we have

$$\begin{aligned} v_{ij} &= \Phi\bigg(\frac{x^{i+j-1}}{(B_{k-m}(x))^2}\bigg) - t\Phi\bigg(\frac{x^{i+j-2}}{(B_{k-m}(x))^2}\bigg) \\ &= -\sum_{r=0}^{2m-1} (\beta_r^{(i+j-1)} - t\beta_r^{(i+j-2)})c_{p+1,r+1} \\ &\quad + \sum_{s=1}^{p}\bigg(\sum_{r=1}^{2k_s}(\beta_{s,r}^{(i+j-1)} - t\beta_{s,r}^{(i+j-2)})\bigg)c_{s,r-1}, \quad i,j = 1, \dots, k, \end{aligned}$$

where $\beta_r^{(n)}$, $\beta_{s,r}^{(n)}$ are the components of the rational functions $x^n/(B_{k-m}(x))^2$, with respect to the base $\mathcal{C}_1(2k_1, \dots, 2k_p; 2m-1)$ of $\mathcal{G}(2k_1, \dots, 2k_p; 2m-1)$. Observe that we can take $c_{s,r} = 0$ for $r = k_s, \dots, 2k_s - 1$, $s = 1, \dots, p$ and $c_{p+1,r} = 0$ for $r = m+1, \dots, 2m-1$, since the $(k - m/k)$-ATPM does not depend on them.

Defining now the matrices $\mathbf{A} = (\Phi(xr_{i-1}(x)r_{j-1}(x)))_{i,j=1}^{k}$ and $\mathbf{B} = (\Phi(r_{i-1}(x)r_{j-1}(x)))_{i,j=1}^{k}$, and assuming that the matrices $(\mathbf{A} - a_i\mathbf{B})$, $i = 1, \dots, p$ and $\mathbf{B}$ are nonsingular, we can give an expression for the coefficients $c_{i,j}$ of the formal series $\mathbf{L}$. In fact, from Theorem 7 it follows that $\mathbf{V} = \mathbf{A} - t\mathbf{B}$ and, therefore,

$$\begin{aligned} (k - m/k)_{\mathbf{L}}(t) &= \langle \mathbf{u}, ((\mathbf{A} - a_i\mathbf{B}) - (t - a_i)\mathbf{B})^{-1}\mathbf{w} \rangle \\ &= \langle \mathbf{u}, ((\mathbf{A} - a_i\mathbf{B})(\mathbf{I} - (\mathbf{A} - a_i\mathbf{B})^{-1}(t - a_i)\mathbf{B})^{-1}\mathbf{w} \rangle \\ &= \langle \mathbf{u}, (\mathbf{I} - (\mathbf{A} - a_i\mathbf{B})^{-1}(t - a_i)\mathbf{B})^{-1}(\mathbf{A} - a_i\mathbf{B})^{-1}\mathbf{w} \rangle \end{aligned}$$

$$= \sum_{j=0}^{\infty} \langle \mathbf{u}, ((\mathbf{A} - a_i\mathbf{B})^{-1}\mathbf{B})^j (\mathbf{A} - a_i\mathbf{B})^{-1}\mathbf{w}\rangle (t - a_i)^j,$$
$$i = 1, \ldots, p,$$
$$\begin{aligned}(k - m/k)_{\mathbf{L}}(t) &= \langle \mathbf{u}, (\mathbf{A} - t\mathbf{B})^{-1}\mathbf{w}\rangle \\ &= \langle \mathbf{u}, \mathbf{B}^{-1}(\mathbf{A}\mathbf{B}^{-1} - t\mathbf{I})^{-1}\mathbf{w}\rangle \\ &= \langle \mathbf{u}, t^{-1}\mathbf{B}^{-1}(t^{-1}\mathbf{A}\mathbf{B}^{-1} - \mathbf{I})^{-1}\mathbf{w}\rangle \\ &= \sum_{j=1}^{\infty} \langle \mathbf{u}, -\mathbf{B}^{-1}(\mathbf{A}\mathbf{B}^{-1})^{j-1}\mathbf{w}\rangle t^{-j}.\end{aligned}$$

Thus,

$$c_{i,j} = \langle \mathbf{u}, ((\mathbf{A} - a_i\mathbf{B})^{-1}\mathbf{B})^j (\mathbf{A} - a_i\mathbf{B})^{-1}\mathbf{w}\rangle,$$
$$j = 0, 1, \ldots, k_i - 1; \;\; i = 1, \ldots, p$$
$$c_{p+1,j} = -\langle \mathbf{u}, \mathbf{B}^{-1}(\mathbf{A}\mathbf{B}^{-1})^{j-1}\mathbf{w}\rangle, \quad j = 1, \ldots, m.$$

As an immediate consequence of the proof of Theorem 1, in Theorem 8 below we get a first expression for the error of approximation.

Theorem 8.

$$L_i(t) - (k - m/k)_{\mathbf{L}}(t) = \frac{1}{H_k(t)} \Phi\left(\frac{H_k(x)}{x - t}\right), \quad i = 1, \ldots, p+1.$$

Hence, these approximants can be deduced as in the one point case, replacing $(x - t)^{-1}$ by $(H_k(t) - H_k(x))/(H_k(t)(x - t))$ in $L_i(t) = \Phi((x - t)^{-1})$, $i = 1, \ldots, p+1$, see [**1**, p. 20].

Observing that

$$\Phi\left(\frac{H_k(x)}{x - t}\right) = \sum_{j=0}^{\infty} \Phi\left(\frac{H_k(x)}{(x - a_i)^{j+1}}\right)(t - a_i)^j,$$
$$(t \to a_i), \quad i = 1, \ldots, p$$
$$= -\sum_{j=1}^{\infty} \Phi(x^{j-1}H_k(x))t^{-j}, \quad (t \to \infty),$$

from Theorem 8 we obtain

Corollary 1.

$$L_i(t) - (k - m/k)_{\mathbf{L}}(t) = \frac{1}{H_k(t)} \sum_{j=0}^{\infty} d_{i,j}(t - a_i)^j,$$
$$(t \to a_i), \quad i = 1, \dots, p,$$
$$= \frac{1}{H_k(t)} \sum_{j=1}^{\infty} d_{p+1,j} t^{-j}, \quad (t \to \infty),$$

with

$$\begin{aligned} d_{i,j} &= \Phi\biggl(\frac{H_k(x)}{(x - a_i)^{j+1}}\biggr) \\ &= -\sum_{r=0}^{m-j-1} \beta_r^{(i,j)} c_{p+1,r+1} + \sum_{\substack{s=1 \\ s\neq 1}}^{p} \biggl(\sum_{r=1}^{k_s} \beta_{s,r}^{(i,j)} c_{s,r-1}\biggr) \\ &\quad + \sum_{r=1}^{k_i+j+1} \beta_{i,r}^{(i,j)} c_{i,r-1}, \quad j = 0, \dots, \ i = 1, \dots, p \\ d_{p+1,j} &= -\Phi(x^{j-1} H_k(x)) \\ &= \sum_{r=0}^{m+j-1} \eta_r^{(j)} c_{p+1,r+1} \\ &\quad - \sum_{s=1}^{p} \biggl(\sum_{r=1}^{k_s} \eta_{s,r}^{(j)} c_{s,r-1}\biggr), \quad j = 1, 2, \dots, \end{aligned}$$

where $\beta_r^{(i,j)}$ *are the coefficients of the quotient polynomial of* $Q_k(x)/B_{k-m}(x)(x - a_i)^{j+1}$, $\beta_{s,r}^{(i,j)}$ *are the coefficients of the partial fraction decomposition of the remainder polynomial, and* $\eta_r^{(j)}$ *and* $\eta_{s,r}^{(j)}$ *are the analogous coefficients for* $Q_k(x)x^{j-1}/B_{k-m}(x)$.

Furthermore, as a consequence of Theorem 8, we have the following error formula for (r/s)-MPTA when infinity is not considered.

Corollary 2.

$$L_i(t) - (r/s)_{\mathbf{L}}(t) = \prod_{j=1}^{p} (t - a_j)^{n_j} \frac{1}{H_s(t)} \Phi^{(r-s+1)}\biggl(\frac{H_s(x)}{x - t}\biggr),$$
$$i = 1, \dots, p,$$

where

$$H_s(x) = \frac{Q_s(t)}{(x-a_1)^{s_1}\cdots(x-a_p)^{s_p}},$$

$s_i = r_i - n_i$, $\sum_{i=1}^p r_i = r+1$, $\sum_{i=1}^p n_i = r-s+1$, $n_i \geq 0$ *for all* i *if* $r \geq s$ *and* $n_i \leq 0$ *for all* i *if* $r < s$, *and the functional* $\Phi^{(r-s+1)}$ *given as in* (4.9) *or* (4.15).

Proof. From (4.16) and Theorem 8, one has

$$\begin{aligned} L_i(t) - (r/s)_{\mathbf{L}}(t) &= \prod_{j=1}^p (t-a_j)^{n_j}(\tilde{L}_i(t) - (s-1/s)_{\tilde{\mathbf{L}}}(t)) \\ &= \prod_{j=1}^p (t-a_j)^{n_j}\frac{1}{H_s(t)}\Phi^{(r-s+1)}\left(\frac{H_s(x)}{x-t}\right), \\ & i = 1,\dots,p. \qquad \square \end{aligned}$$

Finally, if $f(t)$ is a function of one complex variable t, holomorphic in certain simply connected domains D_i, $i = 1,\dots,p+1$, of the extended complex plane $\overline{\mathbf{C}}$, such that $a_i \in \dot{D}_i$, $i = 1,\dots,p$, $\infty \in \dot{D}_{p+1}$ and $D_i \cap D_j = \varnothing$ for $i \neq j$ and $\mathbf{L}$ are the Taylor expansions of f in those points, then the preceding formulas give an integral representation of the error. Indeed, let C_i, $i = 1,\dots,p+1$ be $p+1$ pairwise disjoint closed Jordan curves such that a_i is interior to C_i and C_i is interior to D_i, for $i = 1,\dots,p+1$. By the Cauchy integral formula and Lemma 1, we can write

$$\Phi\left(\frac{1}{x-t}\right) = \frac{1}{2\pi i}\int_C \frac{f(x)}{x-t}\,dx, \quad C = \bigcup_{i=1}^{p+1} C_i.$$

Then, at least formally, from this expression and Theorem 8, we get

$$\begin{aligned} f(t) - (k-m/k)_{\mathbf{L}}(t) &= \frac{1}{2\pi i}\frac{1}{H_k(t)}\int_C \frac{f(x)H_k(x)}{x-t}\,dx \\ &= \frac{1}{2\pi i}\left(\frac{1}{H_k(t)}\left(\sum_{i=1}^{p+1}\int_{C_i}\frac{f(x)H_k(x)}{x-t}\,dx\right)\right). \end{aligned}$$

This formula is actually the error expression given in [**14**, p. 186] concerning rational interpolants with prescribed poles.

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