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DUALITY IN NOETHERIAN INTEGRAL DOMAINS

H. PAT GOETERS

A premier paper on torsion-free abelian groups was published by Warfield [**25**], the duality aspect of which can be summarized as follows. Given a torsion-free abelian group A of rank one, put $\mathbf{C}_A$ equal to the class of torsion-free abelian groups M of finite rank such that M embeds as an $\mathrm{End}\,(A)$ submodule of A^n for some n. Warfield shows that a torsion-free abelian group M of finite rank satisfies $M \cong_{\mathrm{nat}} \mathrm{Hom}\,(\mathrm{Hom}\,(M, A), A)$ exactly when M belongs to $\mathbf{C}_A$. In functorial terminology, he shows that for any torsion-free rank one group A, the map $M \mapsto \mathrm{Hom}\,(M, A)$ on $\mathbf{C}_A$ defines a duality.

Reid was interested in extending Warfield's result to more general domains in an effort to classify his irreducible groups. Reid gave sufficient conditions in [**22**] for an arbitrary integral domain to support Warfield duality, conditions that will receive further attention below. Given a general torsion-free abelian group of finite rank, in order to understand when $\mathrm{Hom}\,(-A) : \mathbf{C}_A \to \mathbf{C}_A$ defines a rank preserving duality, one must know when $\mathrm{End}\,(A)$ supports Warfield duality [**11, 12**]. This enhances the desire to investigate extensions of Warfield duality to Noetherian domains.

For an integral domain R and a torsion-free module A of rank one, as above, let $\mathbf{C}_A$ represent the category of modules M isomorphic to $\mathrm{End}_R(A)$-submodules A^n for some n. Call a module M, *A-reflexive*, if $M \cong_{\mathrm{nat}} \mathrm{Hom}\,(\mathrm{Hom}\,(M, A), A)$ (the unadorned $\mathrm{Hom}\,(M, A)$ will be used when the ring R is prescribed). In [**6**], an integral domain R is called a *Warfield* domain if, for any rank one module A, $\mathrm{Hom}\,(-, A) : \mathbf{C}_A \to \mathbf{C}_A$ defines a duality. It is shown in [**9**] that, when R is a Noetherian domain whose integral closure is finitely generated over R, then R is Warfield if and only if every ideal of R is two-generated. The restriction on the integral closure in [**9**] was needed to show that, when R is a local domain such that every ideal is two generated, and A is a rank one R-module with endomorphism ring R, then A is finitely generated over R.

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A keen description of the integral closure of a local domain R whose ideals are two-generated is provided in [**18**] and was used in [**6**] to circumvent the difficulties encountered in [**9**]. Bazzoni and Salce show that a Noetherian domain is Warfield exactly when every ideal is two-generated.

In Section 2 we show that any one-dimensional Noetherian domain has the property that every rank one module is locally finitely generated over its ring of endomorphisms. This allows us to give a direct and simpler proof that a Noetherian domain is Warfield exactly when each ideal is two-generated.

An issue related to the duality problem is, under what circumstances is the functor $\mathrm{Hom}\,(-, A)$ exact on $\mathbf{C}_A$ whenever A is a rank one module? That $\mathrm{Hom}\,(-, A)$ is exact on $\mathbf{C}_A$ when $R = \mathbf{Z}$ and A is rank-1 was first proven in [**24**] but seems to be implicated in results from [**2**]. We will show, for a Noetherian integral domain R, that $\mathrm{Hom}\,(-, A)$ is exact on $\mathbf{C}_A$ for every rank one module A if and only if every ideal of R is two generated.

1. Integral domains. Throughout this section R will represent an integral domain and Q the quotient field of R. Following the notation of [**22**], given a torsion-free module A, set $E_A = \{\alpha \in Q \mid \alpha A \subseteq A\}$. In case A has rank one, $E_A = \mathrm{End}_R(A)$.

The category $\mathbf{C}_A$ consists of the modules isomorphic to E_A-submodules of A^n for some positive integer n. Reid considered two conditions on a ring R in his paper [**22**].

Condition $\mathbf{R}_1$. For any two rank-1 modules $A \leq B$ with $E_B \leq E_A$, $\mathrm{Ext}^1_R(A, B)$ is torsion-free.

Condition $\mathbf{R}_2$. For any two rank-1 modules $A \leq B$ with $E_B \leq E_A$, A is B-reflexive.

We will consider these conditions and additionally two modifications of $\mathbf{R}_1$.

Modification $\mathbf{R}_1^*$. For any rank-1 module A, $\mathrm{Hom}\,(-, A)$ is exact on $\mathbf{C}_A$.

Modification $\mathbf{R}_1^e$. For any two rank-1 modules $A \leq B$ with $E_B \leq E_A$, $\mathrm{Ext}^1_{E_B}(A, B)$ is torsion-free.

In [**6**], R is called a Warfield domain if the finite rank A-reflexive modules are precisely the modules in $\mathbf{C}_A$, for any rank one module A (it is an easy exercise in showing that any finite rank A-reflexive module generally belongs to $\mathbf{C}_A$). Reid showed that the conditions $\mathbf{R}_1$ and $\mathbf{R}_2$ on R imply that R is Warfield. His approach is to use $\mathbf{R}_1$ to establish $\mathbf{R}_1^*$.

Proposition 1 [**22**]. $\mathbf{R}_1^* + \mathbf{R}_2$ *implies R is a Warfield domain.*

Proof. Let $M \in \mathbf{C}_A$, and let A be a rank-1 module. We will show that M is A-reflexive by induction on the rank of M. The rank-1 case is the assertion $\mathbf{R}_2$. Regard M as an E_A-submodule of A^n for some n. By considering a projection π of A^n onto an appropriate component of A^n and taking $\pi' = \pi|_M$, we obtain a sequence $0 \to K \to M \overset{\pi'}{\to} B \to 0$ with B a nonzero E_A-submodule of A and $K = \operatorname{Ker} \pi$ an E_A-submodule of M.

From $\mathbf{R}_1^*$,

$$0 \longrightarrow B^* \longrightarrow M^* \longrightarrow K^* \longrightarrow 0$$

is exact, where we are using $*$ to denote $\mathrm{Hom}\,(-, A)$. Clearly each of the terms is an E_A-module, and it is easy to see that each belongs to $\mathbf{C}_A$. From condition $\mathbf{R}_1^*$ again, the bottom row of the commutative rectangle is exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K^{**} & \longrightarrow & M^{**} & \longrightarrow & B^{**} & \longrightarrow & 0. \end{array}$$

By induction, the outer vertical maps are isomorphisms, and by diagram chasing, so is the middle. $\square$

We remind the reader of the following property.

Lemma 2. *Let S be a ring extension of R in Q and M and N torsion-free modules which are S-modules. Then* $\mathrm{Hom}\,(M,N) = \mathrm{Hom}_S(M,N)$.

Submodules of a product, A^I, of a rank-1 module A enjoy the following property. The proof appears in [**10**] and also in [**22**].

Lemma 3. *Let A be a rank-1 module and M a submodule of a product of copies of A. If K is a finite rank pure submodule of M, then M/K is A-torsionless, in that*

$$\cap\{Ker\, f \mid f : M/K \longrightarrow A\} = 0.$$

A consequence of Lemma 3 is that $\mathbf{C}_A$ is closed under the formation of torsion-free images.

Proposition 4. $\mathbf{R}_1$ *implies* $\mathbf{R}_1^*$ *and* $\mathbf{R}_1^e$.

Proof. Assume that $\mathbf{R}_1$ holds for R, and let A be a rank-1 module with $S = E_A$. By Lemma 2, two short exact sequences of S-modules are equivalent, precisely when they are equivalent when viewed as R-modules. This implies that $\mathrm{Ext}_S^1(M,N)$ is an R-submodule of $\mathrm{Ext}_R^1(M,N)$ when M and N are torsion-free S-modules, so $\mathbf{R}_1$ clearly implies $\mathbf{R}_1^e$.

To show that $\mathbf{R}_1^*$ holds, we will show for each $M \in \mathbf{C}_A$ that A is injective with respect to any pure sequence $0 \to K \to M \to B \to 0$, and $\mathrm{Ext}_R^1(M,A)$ is torsion-free by induction on rank M. Because of $\mathbf{R}_1$, this is true when rank $M = 1$. In general, given a sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow B \longrightarrow 0$$

with K and B of smaller rank than M, we obtain

$$0 \longrightarrow B^* \longrightarrow M^* \longrightarrow K^* \xrightarrow{\alpha} \mathrm{Ext}_R^1(B,A)$$

where $* = \mathrm{Hom}\,(-,A)$.

Lemma 3 implies $B \in \mathbf{C}_A$. As argued in [**22**, Corollary 2.7], for any $N \in \mathbf{C}_A$, $\operatorname{rank} \operatorname{Hom}(N, A) = \operatorname{rank} N$, so by measuring ranks we find that $\operatorname{Im} \alpha$ is torsion. Because $\operatorname{Ext}^1_R(B, A)$ is torsion-free, Lemma 3 and induction,

$$0 \longrightarrow B^* \longrightarrow M^* \longrightarrow K^* \longrightarrow 0$$

is exact. This leaves

$$0 \longrightarrow \operatorname{Ext}^1_R(B, A) \longrightarrow \operatorname{Ext}^1_R(M, A) \longrightarrow \operatorname{Ext}^1_R(K, A)$$

exact. Thus, $\operatorname{Ext}^1_R(M, A)$ is torsion-free and induction is complete. $\square$

In [**9**] the condition $\mathbf{R}_1$ was shown to be equivalent to the condition that R is Dedekind, while below we will show that $\mathbf{R}_1^e$ is equivalent to the condition that every ideal of R is two-generated. For example, any subring R of a quadratic number field satisfies $\mathbf{R}_1^e$ but only satisfies $\mathbf{R}_1$ when R is integrally closed [**11**, Example 2], so in general $\mathbf{R}_1^e$ need not imply $\mathbf{R}_1$.

Theorem 5. $\mathbf{R}_1^*$ *is equivalent to* $\mathbf{R}_1^e$.

Proof. First assume that $\mathbf{R}_1^*$ holds, and let A and B denote rank-1 modules with $A \leq B$ and $E_B \leq E_A$. Suppose $0 \to B \to M \to A \to 0$ represents an element in $\operatorname{Ext}^1_{E_B}(A, B)[r]$, for some $0 \neq r \in E_B$. (Here we use the notation $T[r]$ to denote $\{x \in T | rx = 0\}$ for a given module T.) Then

$$(\delta) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & M & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & rB & \longrightarrow & M' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

is commutative with split bottom row. Diagram chasing reveals an E_B-module embedding of M into $M' \cong A \oplus B \leq B \oplus B$ so that $M \in \mathbf{C}_B$. By $\mathbf{R}_1^*$, $\operatorname{Hom}(-, B)$ is exact on the top row of (δ), so the top row must split. Consequently, $\operatorname{Ext}^1_{E_B}(A, B)$ is torsion-free.

Conversely, under condition $\mathbf{R}_1^e$, let B be a rank-1 module and $S = E_B$. Mimicking the proof of Proposition 4 allows us to conclude that

$\mathrm{Hom}_S(-, B)$ is exact on $\mathbf{C}_B$. But Lemma 2 implies that $\mathrm{Hom}(-, B) = \mathrm{Hom}_S(-, B)$ relative to $\mathbf{C}_B$. Therefore, $\mathbf{R}_1^*$ holds. □

We close this section by recalling a result from [**9**] relating the importance of Warfield domains to module theory.

Proposition 6. *The following are equivalent.*

(1) *R is a Warfield domain.*

(2) *For any rank-1 module A, the finite rank modules isomorphic to some module of the form* $\mathrm{Hom}(K, A)$ *are precisely the members of* $\mathbf{C}_A$.

2. Noetherian integral domains. In this section the ring R will represent a Noetherian integral domain. A ring R is called *reflexive* if any finitely generated torsion-free R-module is R-reflexive. An elegant summary of results obtained during the 60's relative to the Noetherian integral domains that are reflexive appears in [**16**], from which we extract the following. We will represent the set of maximal ideals of R by $\max(R)$.

Theorem [**14**]. *Let R be a Noetherian integral domain. The following conditions are equivalent.*

(1) *R is a reflexive ring.*

(2) *Every ideal of R is R-reflexive.*

(3) $\mathrm{Ext}_R^1(M, R) = 0$ *for all finitely generated torsion-free modules M.*

(4) *R has Krull dimension* 1 *and P^{-1} is two-generated for all maximal ideals P of R.*

A beautiful theorem of Matlis, [**16**, Theorem 57], which generalizes Theorem 7.7 in [**5**] is the following.

Theorem [**3**]. *Every ideal of an integral domain R can be generated by two elements if and only if R is Noetherian and any finitely generated ring extension of R in Q is reflexive.*

Let $\overline{R}$ denote the integral closure of R in its quotient field. When R is one-dimensional and local, $\overline{R}$ fails to be finitely generated over R precisely when $\overline{R}/R$ has a nonzero divisible submodule [**19**]. Using Matlis' development of divisible modules we can derive a result fundamental to the study of rank one modules over one-dimensional Noetherian domains, Theorem 8. First we need a lemma whose proof appears in [**9**, page 245]. Recall that an integral domain is *h-local* if

(i) every nonzero element is contained in only finitely many maximal ideals and

(ii) any nonzero prime ideal is contained in a unique maximal ideal.

In particular, every one-dimensional Noetherian domain is h-local.

Lemma 7. *Let R be an h-local domain, A a rank one module, and $M \leq A^n$ for some n. Then, for any $P \in \max(R)$, $\mathrm{Hom}_R(M,A)_P = \mathrm{Hom}_{R_P}(M_P, A_P)$.*

Theorem 8. *Let R be a Noetherian domain of dimension one. If A is a rank one R-module, then A is locally finitely generated over its ring of endomorphisms.*

Proof. By the Krull-Akizuki theorem, E_A is one-dimensional Noetherian, so there is no loss in generality in assuming that $E_A = R$. By Lemma 7, $R_P = E_{A_P}$ for any $P \in \max(R)$, so we may assume that R is local with $E_A = R$ in order to show that A is finitely generated.

We may assume (up to isomorphism) that A contains R. In [**20**], Matlis shows that, for a one-dimensional local domain R, the artinian modules are precisely the submodules of $\mathcal{E}^n$ for some n where $\mathcal{E}$ is the injective envelope of P^{-1}/R. Any finitely generated submodule T of Q/R has finite length [**21**], hence by induction on the length we may show that T contains a nonzero submodule T' with $PT' = 0$. Then $T' \leq P^{-1}/R$ and Q/R is an essential extension of P^{-1}/R. Thus Q/R embeds in $\mathcal{E}$ and thus A/R is artinian.

Let S/R denote the maximal divisible submodule of A/R. As expected, A/S is reduced, [**20**, Theorem 1.9]. Because A/R is artinian, Theorem 5.1 in [**19**] applies to show that A/S is finitely generated. In as much as $E_A = R$, the proof is complete once we argue that S is a ring and A is an S-module, for then $S = R$.

If $0 \neq s = t/r \in S$, with $t, r \in R$, then $rS + R = S$ since S/R is divisible. But then, $sS = tS + sR \subseteq S$ and S is a ring. Likewise, if $a = u/v \in A$ with $u, v \in R$, then $vS + R = S$ implying that $aS = aR + uS \subseteq A$. Thus, A is an S-module. □

Corollary 9. *If R is local with every ideal two-generated and A is a rank one module with $E_A = R$, then $A \cong R$.*

Proof. By the Bass-Matlis theorem, R is reflexive so, by the Jans-Bass-Matlis theorem, R is one-dimensional. Also, Proposition 7.2 in [**5**] states that any ideal of R is projective over its endomorphism ring. Assuming, without loss of generality, that $R \subseteq A$, we have shown in the proof of Theorem 8 that A/R is finitely generated. Since A/R is torsion, we conclude that A is isomorphic to an ideal of R. Since $E_A = R$, A is projective over the local ring R. This yields $A \cong R$. □

The equivalence of (1) and (2) in the theorem below was first established in [**9**], under the assumption that the integral closure of R is a finite ring extension of R, using the Bass-Matlis result. Due to Theorem 8, the arguments used in [**9**] can go through to remove the restriction on the integral closure of R.

Theorem 10. *Let R be a Noetherian integral domain. The following are equivalent.*

(1) *R is a Warfield domain.*

(2) *Every ideal of R is two generated.*

(3) *Each rank one module A is injective in the category $\mathbf{C}_A$.*

(4) *For any rank one module A, $\operatorname{Ext}^1_{E_A}(B, A)$ is torsion-free for any E_A-submodule B of A.*

(5) *Every ring extension of R in Q is a reflexive ring.*

Proof. (1) $\rightarrow$ (5). If S is a ring extension of R in Q, then every member of $\mathbf{C}_S$ is S-reflexive. Consequently, S is a reflexive ring.

(5) $\rightarrow$ (3). Since E_A inherits property (5), we may assume that $E_A = R$. In order to establish that $\operatorname{Hom}(-, A)$ is exact relative to a sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0,$$

we may consider the functor $\operatorname{Hom}(-, A)_P$ where P belongs to $\max(R)$. By the Jans-Bass-Matlis theorem, because R is reflexive, R is one-dimensional. Lemma 7 applies and $\operatorname{Hom}(N, A)_P = \operatorname{Hom}_{R_P}(N_P, A_P)$ for any $N \in \mathbf{C}_A$. So, establishing (3) for A is the same as checking that $\operatorname{Hom}_{R_P}(-, A_P)$ is exact on

$$(\varepsilon) \qquad 0 \longrightarrow K_P \longrightarrow L_P \longrightarrow M_P \longrightarrow 0.$$

Since every extension of R_P in Q is necessarily reflexive, every ideal of R_P is two-generated by the Bass-Matlis theorem. Because of this, Corollary 9 implies that $A_P \cong R_P$. Also, as $M \in \mathbf{C}_A$, M embeds in A^n for some n and, consequently, M_P embeds in A_P^n. Therefore, M_P is finitely generated torsion-free and, by the Jans-Bass-Matlis theorem, $\operatorname{Ext}^1_{R_P}(M_P, A_P) = 0$. This shows that A_P is injective relative to (ε).

(3) $\leftrightarrow$ (4). This is Theorem 5 for Noetherian domains.

(3) $\rightarrow$ (2). If S is any ring extension of R in Q, and M is a finitely generated S-module, then select a resolution

$$(\sigma) \qquad 0 \longrightarrow K \longrightarrow S^n \longrightarrow M \longrightarrow 0.$$

By (3) and Lemma 2, $\operatorname{Hom}_S(-, S)$ is exact relative to (σ) from which we deduce that $0 \rightarrow \operatorname{Ext}^1_S(M, S) \rightarrow 0$ is exact. By the Jans-Bass-Matlis theorem, S is a reflexive ring. The Bass-Matlis theorem then affords that every ideal of R is two-generated.

(2) $\rightarrow$ (1). Let A be a rank one R-module, and let $M \in \mathbf{C}_A$. By the theorems of Bass-Matlis and Jans-Bass-Matlis, R is one-dimensional. It is enough to show by Lemma 7 that $M_P \cong \operatorname{Hom}(\operatorname{Hom}(M_P, A_P), A_P)$ and, since every ideal of R_P is two-generated, we may assume that R is local.

As in the proof of Theorem 8, assuming that $R \subseteq A$, let S be the ring extension of R in Q for which S/R is the maximal divisible submodule of A/R. Recall that A is isomorphic to an ideal of S. We claim that S is local and every ideal of S is two-generated.

Let J be an ideal of S, and consider $I = J \cap R$ and $J' = SI$. There is a $0 \neq r \in R$ for which $rS \subseteq J'$. Then $S = rS + R \subseteq J' + R \subseteq S$, so $J = J' + I \subseteq J'$. Therefore, $J = SI$ is two-generated, and if P is the maximal ideal of R, then PS is the maximal ideal of S. As mentioned in the proof of Corollary 9, since every ideal of S is two-generated, Proposition 7.2 in [**5**] shows that A is projective over its endomorphism ring E. But E is semi-local, so $A \cong E$. By the Bass-Matlis theorem, E is reflexive, implying that M is A-reflexive as desired. □

Corollary 11. *A Noetherian integral domain R is a Warfield domain if and only if $\mathbf{R}_2$ holds for R.*

Proof. Suppose $\mathbf{R}_2$ holds for R. Given any ring extension S of R in Q, every ideal of S is S-reflexive. By the Jans-Bass-Matlis theorem, S is a reflexive ring. Therefore, R is Warfield by Theorem 10. □

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Department of Mathematics, Auburn University, Auburn, AL 36849-5310
E-mail address: `goetehp@mail.auburn.edu`